APPLIED ELASTICITY

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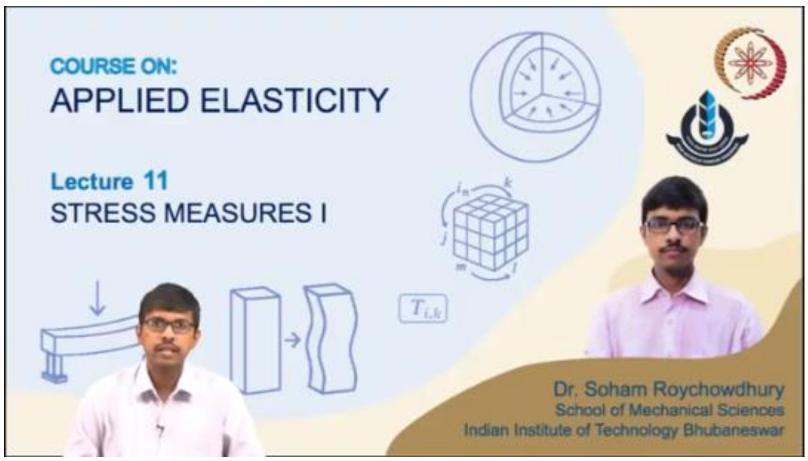
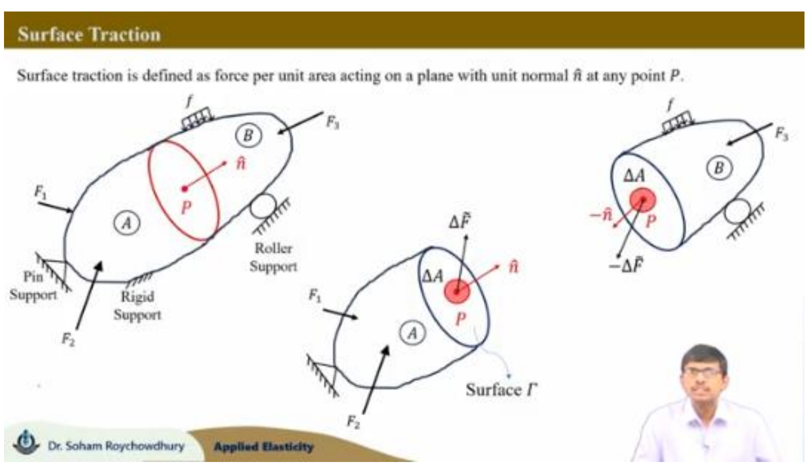
WEEK: 03
Lecture- 11



Welcome back to the course of applied elasticity. In the third week of this lecture, we are going to first discuss about the different stress measures. In the previous week, we had talked about the strain and displacement measures kinematics of deformation. Now, we are moving forward to the stress measures within a body.



First, we are going to define the surface traction and based on that the stress components would be defined.

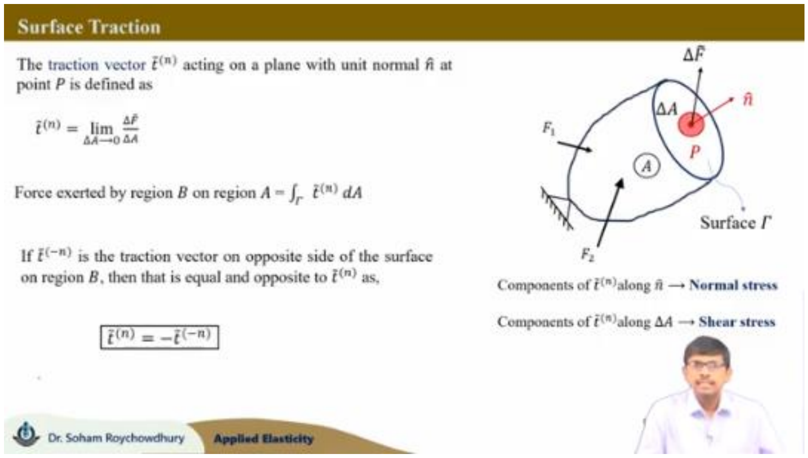
So, surface traction is defined to be the force acting per unit area on a plane at any point P and $\hat{n}$ being the unit normal to the plane. So, for defining surface traction, we need a plane defined by its unit normal direction n vector and a point on the plane. If you are just given with a single point without any plane, it is not possible to define the surface traction and thus it is not possible to define the stress components as well. So, for defining surface traction or stress component we need a point and the plane on which we are going to define the surface traction vectors.

Let us consider a body which is supported by different boundary conditions, pin support, roller support, rigid support all different types of supports may be there within this continuum. And this is subjected to different types of external loadings F_1, F_2, F_3 are point loads or concentrated loads acting at different points and small f is a distributed load acting for a specific region. Now, P is one point of interest within the body and we are interested to find out the surface traction at point P.

By its definition, for writing the surface traction expressions or defining it, we first need to define a plane. So, let us cut this body using a plane as shown by this red curve which passes through point P. and this divides the body into two halves part A and part B. So, using this particular plane passing through point P we are dividing the body into part A and part B and then we are going to draw the free body diagram of both the parts separately and this plane is defined by its unit outward normal vector small $\hat{n}$.

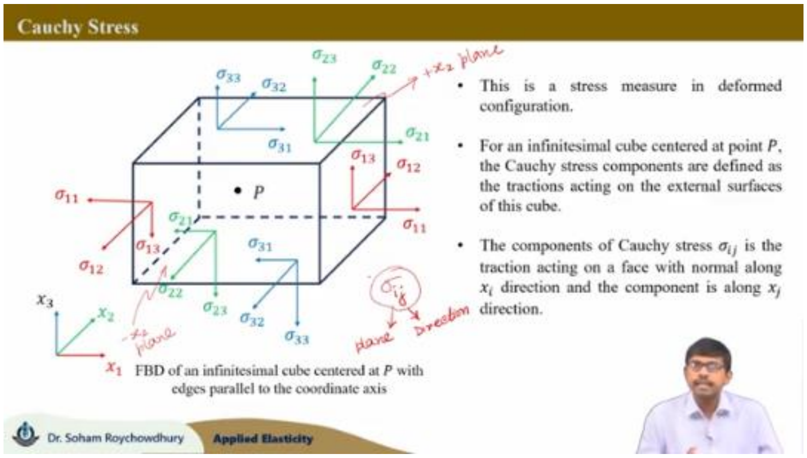
Now, first starting with the free body diagram of part A The supports and forces are shown whatever were attached to part A. Now, on this cutting plane, we are considering a small elemental area ΔA around that point P and ΔA is the small elemental area. This plane, the open surface of this plane is denoted by surface Γ, capital Γ. $\hat{n}$ is the unit normal vector to this surface Γ at point P and let us say f $\Delta\tilde{F}$ this is the elementary force acting at this particular point due to cutting of the body with this particular plane defined through n vector.

Now, similarly on the other part, part B, if you draw the free body diagram, it would be looking like this. There will be an identical P point on part B around which we are taking elemental area D A or ΔA and that is having unit outward normal of $-\hat{z}$. Opposite to the outward normal of part A surface, we will be having equal magnitude, but opposite direction unit outward normal vector on the open surface or cutting plane of part B of the body. On the point P of part B minus of $\Delta\tilde{F}$ amount of force is acting which is negative of the same amplitude force as it was acting on the part A point P on the small elemental area ΔA.



Now coming to the definition of the surface traction that surface traction vector $\tilde{t}^{(n)}$ this superscript n denotes the normal normal to the plane on which this surface traction is defined so $\tilde{t}^{(n)}$ is acting on a plane with unit normal $\hat{n}$ at point P and that is defined as limit ΔA tending to 0 $\Delta\tilde{F}$ by ΔA. So, elementally elemental force acting at point P divided by elemental area around point P with limit this elemental area ΔA tending to 0 is defined to be the surface traction $\tilde{t}$ on this particular plane defined by unit normal $\hat{n}$. if you take the components of this surface traction vector $\tilde{t}^{(n)}$. So, surface traction or $\tilde{t}^{(n)}$ is a vector quantity because this is limit ΔA $\Delta\tilde{F}$ vector by ΔA $\Delta\tilde{F}$ being a force vector that force vector per unit area is defined to be surface traction vector and thus this is a vector quantity. So, component of surface traction vector along the normal direction is called the normal stress. Component of the surface traction vector along the surface Γ parallel to this area ΔA that is called the shear stress. So, total surface traction can be can be obtained by taking the taking the resultant of the normal component and the shear component. Now, if you integrate this surface fraction over the entire surface Γ, so $\tilde{t}^{(n)}$ dA integrated over entire Γ gives us the total force exerted on surface Γ of part A due to part B or region B. So, this much amount of force the region B exerts on region A part A once we are cutting it with this plane with unit normal n that total force is integral Γ $\tilde{t}^{(n)}$ dA.

Now, $\tilde{t}^{(-n)}$ being the traction vector on opposite side of the surface of part B, that should be equal and opposite to the surface traction acting on part A. So, as already discussed $\tilde{t}^{(n)}$ will be minus of $\tilde{t}^{(-n)}$, the magnitude wise these two surface tractions are identical, but their directions would be just opposite and thus Once we are combining these two cut sections part A and part B together once again this $\tilde{t}^{(n)}$ would cancel minus of $\tilde{t}^{(-n)}$ and thus in the total body there is no force no traction acting at particular internal point P. Once you are cutting it then only that will become open surface and for both part A and part B equal and opposite surface traction forces would be acting.



This particular equation is called Cauchy's lemma. Now, moving forward to the definition of Cauchy stress which is based on the definition of surface fractions we are considering a small infinitesimal cube and this Cauchy stress is the stress measure in the deformed configuration. So, we will be looking into the stress measures in undeformed configuration later. First, we are discussing the Cauchy stress components which are the stress components defined in the deformed configuration.

For this small cube, infinitesimal cube centered at point P, the Cauchy stress components are defined as the surface tractions acting on all six external surfaces of the cube. Now, Cauchy stress tensor or its component is denoted with two subscripts σ_{ij} and thus stress is a tensor quantity. As two subscripts are required to define stress, we must define this as a tensor quantity. We cannot use vector quantity for defining stress. As long as it is traction, that is a vector.

Using traction, once we are going to define stress, it must be a second order tensor. Now, what is the significance of these two subscript i and j in any Cauchy stress tensor component σ_{ij}? Now, two subscript out of which the first one is i. which refers to the traction acting on a phase with normal along x_i. So, along x_i direction if the normal is aligned that defines the plane on which this Cauchy stress component σ_{ij} is defined and second subscript j refers to the direction.

So, first subscript refers to the plane whose normal is x_i, second subscript refer to the direction of the Cauchy stress component which is along x_j direction. Now, let us draw the Cauchy stress components for this cube. So, considering x_1, x_2 and x_3 direction, note that I have used three different colours for three directions x_1, x_2, x_3 which are deformed coordinates because Cauchy stress is defined in the deformed coordinate frame.

Now, considering the x_1 planes, So, if you consider this particular plane this is having unit normal along the x_1 direction. If you consider the back plane the left hand side plane of this ah cube that is also x_1 plane its normal is along the minus x_1 direction. For these two planes we are defining the Cauchy stress components like this. On the right hand side plane

As this plane is defined as x_1 plane, the first subscript would be 1 for all the components because i, the first subscripts refer to the plane. For this particular plane, we are having it to be a x_1 plane with unit normal along x_1. So, first subscript must be 1. So, for all these, first subscript is equals to 1. Now, second subscript refers to the direction.

This component is along one direction. So, that is why we are having σ_{11}. This component is along x_2 direction. That is why name is σ_{12} and third component is along x_3 direction. That is why name is σ_{13}.

On the negative x_1 plane that is on the back side, this is negative x_1 plane. Similarly, σ_{11}, σ_{12}, σ_{13} can be drawn. The directions are simply negative or opposite to the directions of the corresponding stress components on the positive x_1 plane. So, like that you can define 3 stress components on each plane. For x_1 plane, we are defining 3 stress components σ_{11}, σ_{12}, σ_{13} along x_1, x_2, x_3 directions acting on x_1 plane.

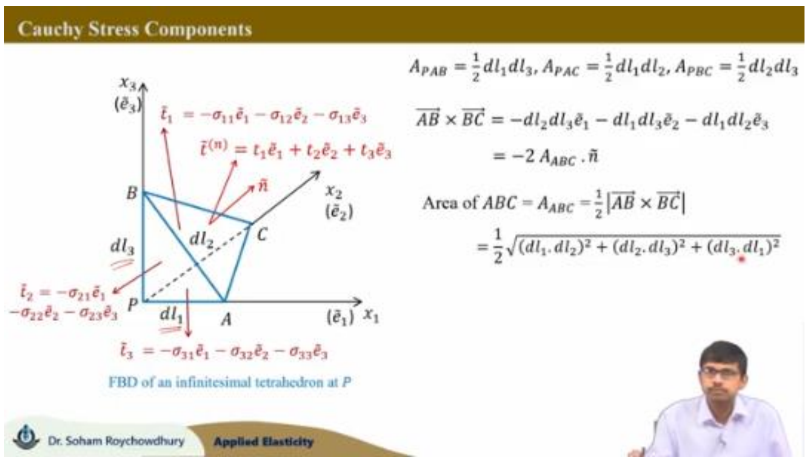
Similarly, we can move forward to rest of the planes. So, if you take the x_2 planes which are denoted by these green colours. So, that is front face and back face of the cube. The first subscript is 2 because these are x_2 plane. So, three stress components defined are σ_{21}, σ_{22}, σ_{23}.

The back face is the positive subscript. x_2 phase. This is positive x_2 plane whereas, the front is negative x_2 plane. So, for the positive x_2 plane, σ_{21}, σ_{22}, σ_{23} are along the positive x_1, positive x_2, positive x_3 direction.

For the front face that is negative x_2 plane, we are having σ_{21}, σ_{22}, σ_{23} along the negative x_1, x_2, x_3 directions. Now, coming to the x_3 plane that is the top plane and the bottom plane denoted by the blue colour stress components here the components are σ_{31} σ_{32} and

σ_{33} on both top and bottom positive x_3 and negative x_3 plane we can define the stress components. So, for σ_{ij} the first i refers to the direct to the plane with normal x_i second subscript j refers to the direction along x_j.

So, on the positive planes positive plane means planes whose normal is aligned along the positive x_i direction those are called positive planes. for positive planes along the positive direction σ_{ij} the cauchy stress component is taken to be positive On the negative planes along the negative direction we we are also taking σ_{ij} scotty stress components to be positive now with this we will move forward.



Now to relate the traction vector with the corresponding Cauchy stress components, we are going to draw the Cauchy tetrahedron. So, this is a tetrahedron, where 3 sides are along x_1, x_2, x_3 directions with unit vector $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$. The tetrahedron is named with $PABC$. The side lengths PA equals to dl_1, PB equals to dl_3, PC equals to dl_2. dl_1, dl_2, dl_3 are aligned along $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ direction. Now, if you consider the inclined face ABC, That plane is inclined through all three axes whereas PAB plane is lying on x_1,x_3 plane, PAC is lying on x_1,x_2 plane, PBC is lying on x_2,x_3 plane whereas ABC is one of the inclined plane.

Now, this for small infinitesimal tetrahedron defined as Cauchy tetrahedron around point P. Let us take Ai are the acceleration components along x_i direction, b_i are the body force component per unit mass along the xi direction, $\tilde{t}^{(n)}$ is the traction vector acting on the inclined surface ABC. So, on the inclined surface, which is having n to be its unit normal vector, $\tilde{t}^{(n)}$ is defined that is surface traction on this inclined surface ABC defined by unit normal n and components of this $\tilde{t}^{(n)}$ traction vector are $\tilde{t}_1$, $\tilde{t}_2$, $\tilde{t}_3$. $\tilde{t}_n$ vector becomes $\tilde{t}_1\tilde{e}_1$ vector plus $\tilde{t}_2\tilde{e}_2$ vector plus $\tilde{t}_3\tilde{e}_3$ vector.

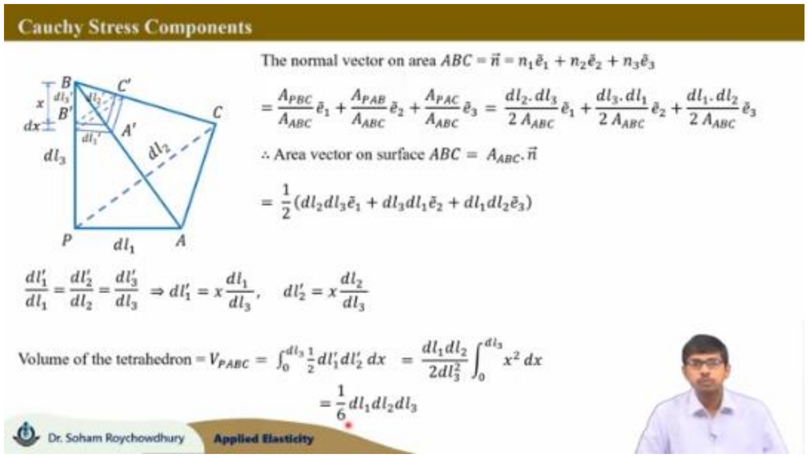
The traction components on other three phases PBC, PAB, PAC. PBC is nothing but x_1 plane, $PBAB$ is nothing but x_2 plane and PAC is x_3 plane. On all those three planes, the traction components are defined with the help of corresponding Cauchy stress tensor components defined on those particular planes. So, $\tilde{t}_1$ vector is that is the fraction component on PBC plane is equals to $-\sigma_{11}\tilde{e}_1 - \sigma_{12}\tilde{e}_2 - \sigma_{13}\tilde{e}_3$.

Why minus? Because all these components are acting on the negative x_1, x_2, x_3 directions. Similarly, on plane PBC, you can define this traction vector $\tilde{t}_2$. On PAC, the bottom phase, you can define the traction vector $\tilde{t}_3$ with the help of corresponding stress components like this. And this completes the overall free body diagram of the small tetrahedron around point P.

Now, if we are writing the expression of area of all these different sides, if you consider PAB, the front face, front vertical face in x_1,x_3 plane, that is basically a triangle with base length dl_1 and with height dl_3. of this particular triangle PAB is equals to half of base dl_1 times dl_3 that is written here. Area of PAB equals to half $dl_1 dl_3$. Similarly, area of PAC can be obtained as half of $dl_1 dl_2$, area of PBC can be obtained as half of $dl_2 dl_3$.

Moving forward if you take the cross product of $\overrightarrow{AB}$ vector and $\overrightarrow{BC}$ vector that can be written as minus of 2 times A_{ABC} . $\tilde{n}$ vector where $\tilde{n}$ is the unit normal on the inclined plane $\tilde{n}$. So, by writing $\overrightarrow{AB}$ vector and $\overrightarrow{BC}$ vector as per its components as per their components and then taking the cross product you can show that $\overrightarrow{AB}$ cross $\overrightarrow{BC}$ equals to minus of two times area of inclined face ABC dot $\tilde{n}$ vector. now from this A_{ABC} can be obtained as half of modulus of this cross product $\overrightarrow{AB}$ and $\overrightarrow{BC}$

so substituting the components of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ from this equation the One component, first component of this cross product is dl_2, dl_3. Second component is dl_1, dl_3. Third component is dl_1, dl_2. Substituting all those here, the A_{ABC} inclined face can be obtained as half of square root of dl_1, dl_2 whole square plus dl_2, dl_3 whole square plus dl_3, dl_1 whole square.



Moving further, Considering this A_{ABC} with a normal vector $\vec{n}$, where n_1, n_2, n_3 are the direction cosines for this normal vector. These direction cosines can be written as the ratio of the projected area and the inclined area. So, $\vec{n}$ is the unit normal on this inclined A_{ABC}. If you project this inclined area on x_1 plane,

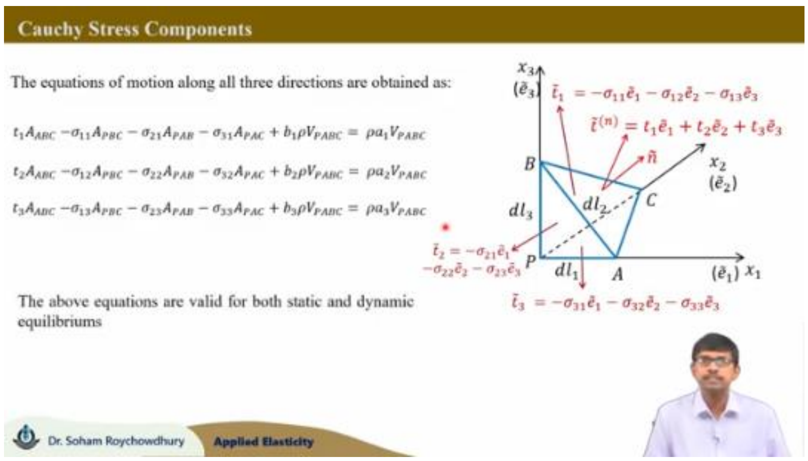
that would become PBC. This particular plane, the back face, this is basically equals to the projection of ABC on the x_1 plane. Thus, area of projected PBC divided by inclined area ABC gives us the first direction cosine n_1. Similarly, taking the projection of inclined area on x_2 plane and x_3 plane. We can write n_2 as A_{PAB} by A_{ABC} and n_3 as A_{PAC} divided by inclined A_{ABC}.

Now writing the expressions of all 3 areas of 3 x_1, x_2, x_3 planes PBC, PAB and PAC which we had already derived as half of dl_2, dl_3 and so on. Substituting those we can get the normal vector. And going back to the previous slide if you see $\overrightarrow{AB}$ cross $\overrightarrow{BC}$ was defined as minus of 2 times inclined area dot $\tilde{n}$ vector. Now, this dot $\tilde{n}$ vector we had obtained thus the surface area vector for ABC can be written as half of $dl_2dl_3\tilde{e}_1$ plus $dl_3dl_1\tilde{e}_2$ plus $dl_1dl_2\tilde{e}_3$. This gives us the surface area vector for this inclined surface ABC.

Now for finding the volume, let us take up another small slice which is shown here at a distance x from this tip B. then at a distance dx further from this x, we are taking another slice. And we will try to express the volume of this small region of the tetrahedron, which is sliced by drawing two planes at a distance x and x plus dx from the tip point B. So, considering the lengths at that particular cutting plane to be dl_1dl_2 and dl_3 along the x_1, x_2, x_3 axis, x_1, x_2, x_3 directions and using the property of the symmetry of geometric symmetry we can write dl_1' prime by dl_1 equals to dl_2' by dl_2, dl_3' prime by dl_3 and using that x quantity dl_1' is equals to x times dl_1 by dl_3 because x is nothing but dl_3'. dl_2'

can be written as x times dl_2 by dl_3 and volume of the small tetrahedron this small region elemental part of the tetrahedron can be written as half of $dl'_2 \; dl_2$. If you integrate it over the total length from B to P. So, volume of that small tetrahedron with thickness dx was half of $dl'_1 dl'_2 \; dx$ integrating it for the total length of x varying from 0 to dl_3 from point B to P we can get the total volume as x integral to be 1 by 6 $dl_1 dl_2 dl_3$.

So, with this we got the volume of the tetrahedron we got the surface area vector of this inclined face ABC.



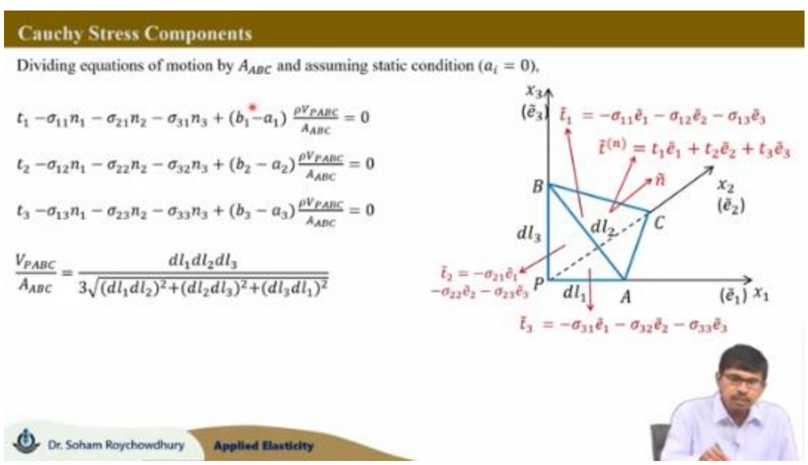
Now, moving forward to the force balance or writing the governing equation along 1, 2 and 3 directions. The equations of motion along all 3 directions are obtained. So, along the x_1 direction using the Newton's law of motion, total mass into acceleration is net force acting along a specific direction. So, considering a_1 to be acceleration along the 1 direction, multiplied with the total mass mass is density times total volume so ρ times volume of the tetrahedron is total mass multiplied with one acceleration component along x direction this is mass into acceleration along x_1 direction now left hand side is the net force along the x_1 direction now If you consider the tetrahedron, there are four planes, ABC, PBC, PAC and PAB. On all four faces, we are having four traction vectors acting $\tilde{t}^{(n)}$, t_1, t_2, t_3 and each of them are having one component along x_1, another component along x_2, another component along x_3. So, taking the respective x_1 components as this is the force balance along x_1 direction,

We are taking the x_1 component acting along all this. So, let us say ABC inclined phase x_1 component is t_1 multiplied with the corresponding area because we are writing total force. this fraction or this σ_{11} these are having these are stresses with the unit of force per unit area. So, we must multiply that with the corresponding area component to get the net force. So, $t_1 A_{ABC}$ is the net force acting on inclined face ABC along x_1 direction minus σ_{11} area of PBC is net force acting along the x_1 direction of PBC.

Similarly, σ_{21} area of PAB is force on the PAB plane along x_1 direction minus σ_{31} A_{PAC} is the force acting on PAC side PAC plane along the 1 direction. So, these 4 are the forces along 1 direction for 4 different sides of the tetrahedron, 4 different planes of the tetrahedron. Now, coming to this term, this is defined b_1 is defined to be the body force per unit mass. So, if you multiply that with the total mass ρ times volume, we will get the net body force along the 1 or x_1 direction.

So, this completes the net equation of motion. Mass into acceleration along x_1 direction is net force along x_1 direction which includes surface fractions on four surfaces of the tetrahedron plus the body force within the body within the tetrahedron along x_1 direction. Similarly, if you write for x_2 and x_3 directions, these would be the equations of motion coming for all three directions. So, these are the three equations of motion for x_1, x_2 and x_3 direction.

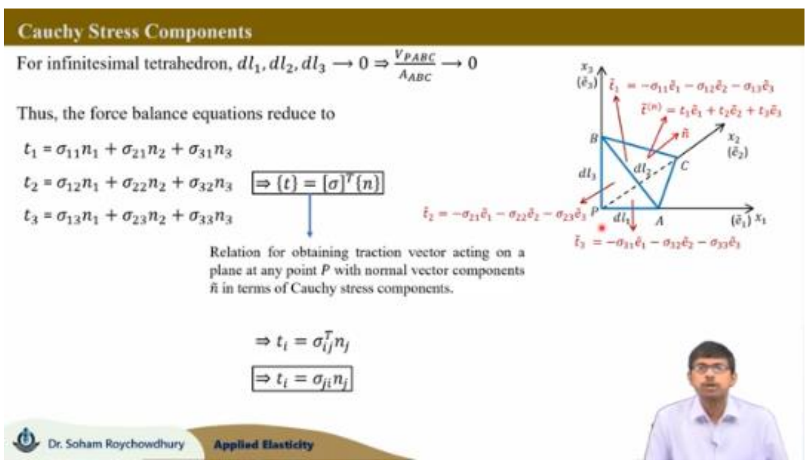
Now, these equations are valid for both static and dynamic equilibrium because we are considering the acceleration of the system as well. Acceleration components were taken to be non-zero. So, these equations are valid for dynamic cases as well.



Now, moving forward. dividing all the terms of those three equations by the inclined area ABC and assuming the static condition.

So, previous expressions were valid for both static and dynamic. Now, moving to the static case, we are forcing acceleration components to be 0, forcing them to be 0 and dividing all the terms of three equations by inclined area ABC, we will be getting these three equations. Now, in the last term of all three equations, you can see ρ into volume by inclined area ABC. This term is coming.

So, from geometry, we had already derived the volume expression and inclined area ABC expression. So, this ratio, volume of the tetrahedron divided by inclined area ABC can be obtained like this. Now, as we are considering a small elemental tetrahedron with dl_1, dl_2, dl_3 to be small, this particular quantity volume by area for the tetrahedron should also be a very small quantity. And with dl_1, dl_2, dl_3 tending to 0 for small tetrahedron, this volume by area term, this entire term will go to 0. All the last terms in all three equations will go to 0.



So, for infinitesimal small tetrahedron dl_1, dl_2, dl_3 tending to 0, volume by area term goes to 0 and thus that static force balance reduces to this. The first equation along x_1 becomes t_1 equals to $\sigma_{11}n_1$ plus $\sigma_{21}n_2$ plus $\sigma_{31}n_3$. The expression along 2 direction x_2 direction becomes t_2 is $\sigma_{12}n_1$ plus $\sigma_{22}n_2$ plus $\sigma_{32}n_3$. t_3 directional equation becomes t_3 equals to $\sigma_{13}n_1$ plus $\sigma_{23}n_2$ plus $\sigma_{33}n_3$. Now, in total, using a single matrix notation, you can write the traction vector $\tilde{t}$ to be σ transpose, transpose of the Cauchy stress tensor times n vector, where n is the unit normal vector on that particular plane. Now, as we had already taken dl_1, dl_2, dl_3 tending to 0, this ABC plane is moving towards point P and we can consider that n is the unit normal vector of the plane passing through point P because of small tetrahedron assumption with dl_1 dl_2 dl_3 tending to 0 volume by area ratio also went to 0. so this particular expression $\{t\}$ equals to $[\sigma]^T\{n\}$ this gives us the relation between the traction vector acting on a plane at a point P with a normal vector direction n in terms of its cauchy stress components so you must remember that for defining the cauchy stress component we need a point and a plane both are required

plane is defined with respect to unit normal vector $\tilde{n}$ or $\tilde{n}$ vector and then only we can define the relation between cauchy stress components and surface traction vector as t equals to $[\sigma]^T\{n\}$ In the indicial notation, we can write t_i to be $\sigma_{ij}^T n_j$ or t_i to be $\sigma_{ji} n_j$. So, this relates the stress components, Cauchy stress components with the surface fraction component t_i equals to $\sigma_{ji} n_j$. So, in this lecture, we had discussed about the introduction to the surface traction, then Cauchy stress components and their relation with the help of the concept of Cauchy tetrahedron. Thank you.