APPLIED ELASTICITY

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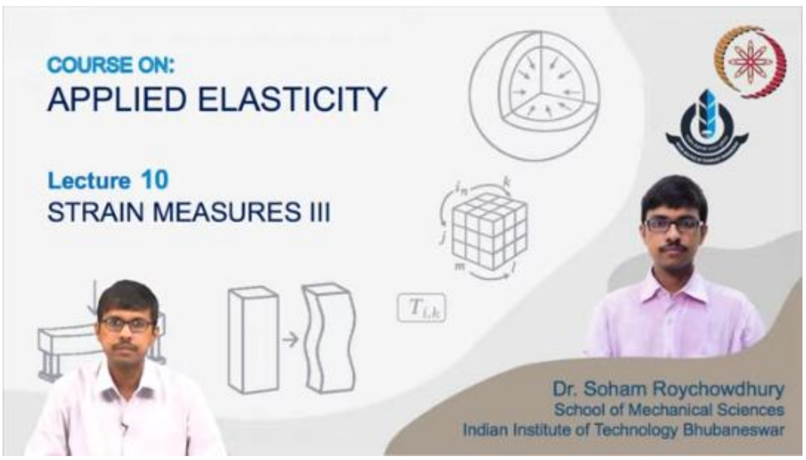
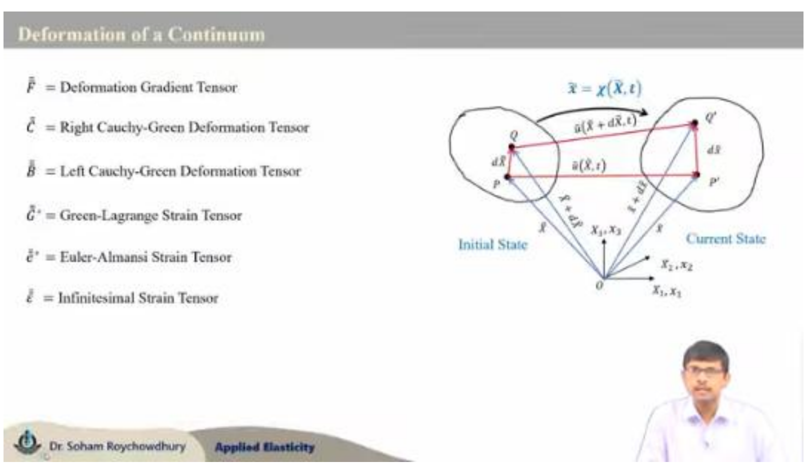
Week 2

Lecture 10: Strain Measures III

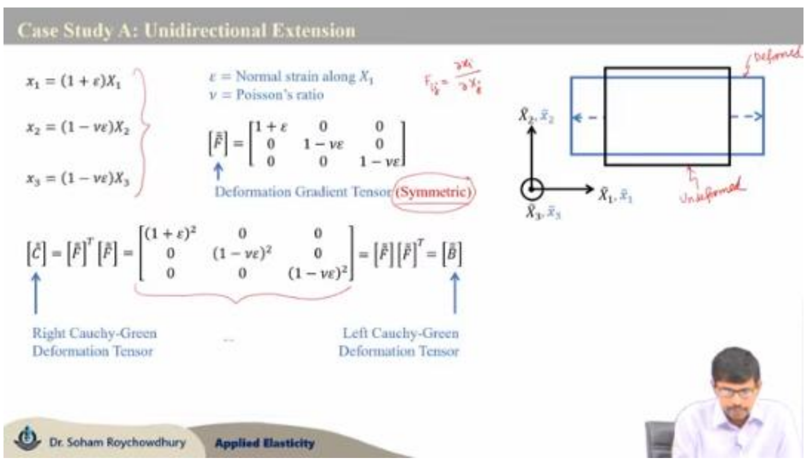


In the previous two lectures, we discussed the different strain measures, and today we will continue with the same topic on strain measures.



So, just to have a quick recap, we considered one body in the initial state defined by the PQ line element of length dX, which was deformed to $P'Q'$. The deformed line element was denoted by $\tilde{x}$. This deformation was mapped using the deformation gradient tensor $\tilde{\tilde{F}}$.

We defined several other deformation tensors or strain tensors, such as the left and right Cauchy-Green deformation tensors, $\tilde{\tilde{B}}$ and $\tilde{\tilde{C}}$. In the undeformed state or initial state, $\tilde{\tilde{G}}^*$, the Green-Lagrange strain tensor, was defined. In the current or deformed state, $\tilde{\tilde{e}}^*$, or Euler-Almansi strain tensor, was defined, and thus, for small deformation problems, epsilon, which is called linear infinitesimal strain, was also defined by neglecting the nonlinear terms of $\tilde{\tilde{G}}^*$ and $\tilde{\tilde{e}}^*$.



Now, here, we are going to take a few case studies, which are very commonly available problems. So, we are going to first consider the problem of unidirectional extension. So, let us consider a body that is subjected to unidirectional loading along the X_1 direction. So, this is the body that is subjected to loading along the X_1 direction, causing stretching along the X_1 direction, and due to that, the other two dimensions along X_2 and X_3 will contract, and thus, the shape of the body will look like this.

So, this black line denotes the undeformed body, whereas the blue one refers to the deformed state. So, you can see the length along the X_1 direction has increased, whereas the length along X_2 and X_3 directions has reduced. So, if you try to relate the deformed coordinates x_1, x_2, x_3 with the help of ϵ, where ϵ is the normal strain along the x_1 direction.

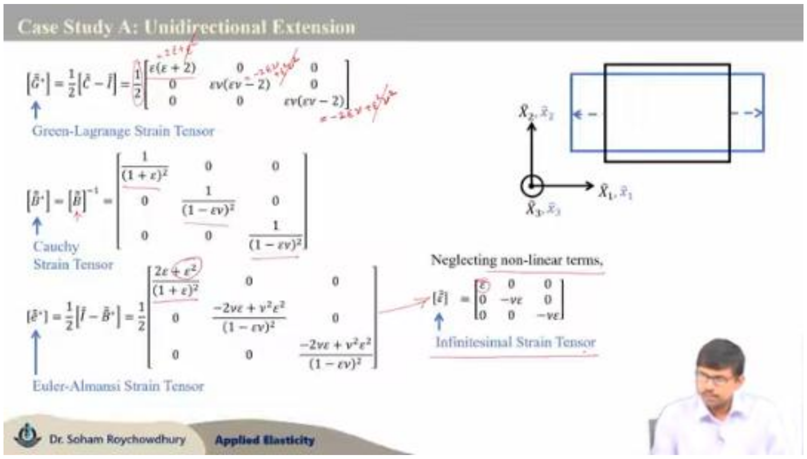
So, we are considering a normal strain, which is small, along the x_1 direction and ν, that is Poisson's ratio. So, in the x_1 direction, the deformed length x_1 would be $(1+\epsilon)X_1$. In the other two lateral directions, these are defined with the help of Poisson's ratio as: $x_2 = (1-\nu\epsilon)X_2$, and $x_3 = (1-\nu\epsilon)X_3$. So, these define the deformation mapping for the present unidirectional extension along the X_1 direction.

From that, we can define $\tilde{\tilde{F}}$; F_{ij} was defined as $\frac{\partial x_i}{\partial X_j}$. Using this definition, you can obtain the components of the deformation gradient tensor $\tilde{\tilde{F}}$, and that would come out to be a diagonal symmetric tensor with diagonal terms being: $1+\epsilon$, $1-\nu\epsilon$, $1-\nu\epsilon$, and all non-diagonal terms being 0. So, note that this particular deformation gradient tensor is symmetric.

Now, you can find out $\tilde{\tilde{C}}$ and $\tilde{\tilde{B}}$, the right and left Cauchy-Green deformation tensors. So, $\tilde{\tilde{C}}$ or the right Cauchy-Green deformation tensor is $\tilde{\tilde{F}}^T\tilde{\tilde{F}}$. And that would come out to be

$$\begin{bmatrix} (1+\epsilon)^2 & 0 & 0 \\ 0 & (1-\nu\epsilon)^2 & 0 \\ 0 & 0 & (1-\nu\epsilon)^2 \end{bmatrix}.$$

Now, if you obtain $\tilde{\tilde{B}}$, which is defined to be $\tilde{\tilde{F}}\tilde{\tilde{F}}^T$, that would also come out to be the same matrix. So, for this particular problem, both the right and left Cauchy-Green deformation tensors are the same, which is given here. Now, this is the property of the symmetric deformation gradient tensor. If $\tilde{\tilde{F}}$, the deformation gradient tensor, is symmetric, then, the right Cauchy-Green deformation tensor would come out to be equal to the left Cauchy-Green deformation tensor.



Now, coming to $\tilde{\tilde{G}}^*$, the Green-Lagrange strain tensor; this is for finite strain without the assumption of small strains. So, this is defined as $\frac{1}{2}\left(\tilde{\tilde{C}} - \tilde{\tilde{I}}\right)$; $\tilde{\tilde{C}}$ is known to us; we had already derived it in the last slide, and $\tilde{\tilde{I}}$ is the identity tensor. So, expanding $\tilde{\tilde{C}}$ minus $\tilde{\tilde{I}}$ and dividing by 2, you can get $\tilde{\tilde{G}}^*$ as half multiplied with a diagonal matrix, with diagonal terms being $\epsilon(\epsilon + 2)$, $\epsilon\nu(\epsilon\nu - 2)$, $\epsilon\nu(\epsilon\nu - 2)$.

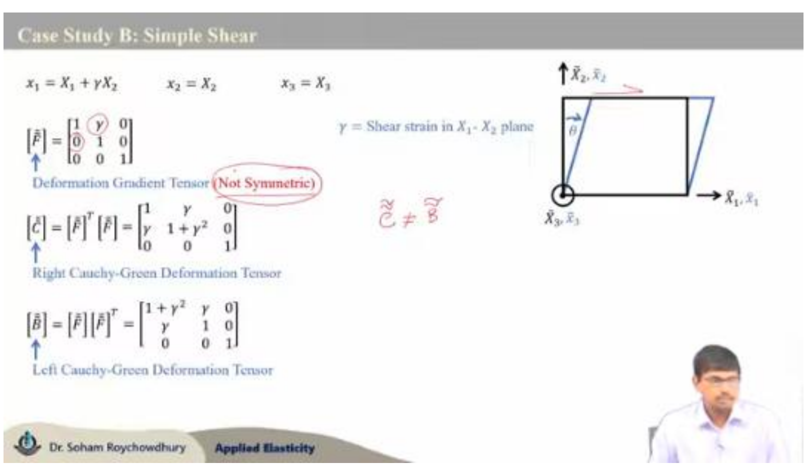
Now, after the Green-Lagrange strain tensor, which is defined in the initial state, if you move further to the deformed state $\tilde{\tilde{B}}^*$, the Cauchy strain tensor; that is nothing but $\tilde{\tilde{B}}^{-1}$ or the inverse of the left Cauchy-Green deformation tensor. So, as $\tilde{\tilde{B}}$ was a diagonal tensor with only three diagonal terms being non-zero and the rest of the non-diagonal terms being 0, $\tilde{\tilde{B}}^{-1}$ is nothing but 1 divided by those diagonal terms. So, in all three locations, these three diagonal terms are present, which are 1 divided by the corresponding diagonal term of the $\tilde{\tilde{B}}$. So, $\tilde{\tilde{B}}^*$, the Cauchy strain tensor, can be obtained like this.

Another strain measure in the deformed state is called the Euler-Almansi strain tensor, which is $\tilde{\tilde{e}}^*$, and defined as $\frac{1}{2}\left(\tilde{\tilde{I}} - \tilde{\tilde{B}}^*\right)$, the identity tensor minus the Cauchy strain tensor. So, substituting $\tilde{\tilde{I}}$ and $\tilde{\tilde{B}}^*$ and simplifying, this $\tilde{\tilde{e}}^*$, the Euler-Almansi strain tensor, would come out to be like this. Now, if you carefully look at the expressions for $\tilde{\tilde{G}}^*$, $\tilde{\tilde{B}}^*$, or $\tilde{\tilde{e}}^*$, all of them have nonlinear terms due to the presence of square-order terms of ϵ. So, you can see here, the first term is $\epsilon(\epsilon + 2)$.

So, this is equal to $2\epsilon + \epsilon^2$. This is equal to $-2\epsilon\nu + \epsilon^2\nu^2$. This is also equal to the same: $-2\epsilon\nu + \epsilon^2\nu^2$. So, in all the non-zero terms, we have non-linear terms of ϵ present. Now, if you are neglecting these non-linear terms under the small strain assumption from both $\tilde{\tilde{G}}^*$ and $\tilde{\tilde{e}}^*$, we can define the small strain tensor $\tilde{\tilde{\varepsilon}}$ to be this, just by neglecting the nonlinear term. So, these nonlinear terms of ϵ^2 are now neglected, and with that, $\tilde{\tilde{G}}^*$ reduces to $\tilde{\tilde{\varepsilon}}$. And that becomes $\begin{bmatrix} \epsilon & 0 & 0 \\ 0 & -\nu\epsilon & 0 \\ 0 & 0 & -\nu\epsilon \end{bmatrix}$. You can check that all these '2's sitting in the diagonal terms are canceled with the half which is sitting outside the matrix. So, this is the infinitesimally small linear strain tensor for the present problem, obtained by dropping the nonlinear terms of $\tilde{\tilde{G}}^*$.

Similarly, if you start from $\tilde{\tilde{e}}^*$, the Euler-Almansi strain tensor, and drop the nonlinear terms, it is possible to obtain the same $\tilde{\tilde{\varepsilon}}$, the small linear strain tensor. For example, here in the first term, ϵ^2 would be dropped. And if you expand the bottom one, $(1+\epsilon)^2$, with the help of binomial expansion, taking it in the numerator as $(1+\epsilon)^{-2}$, and then expand it, and just consider the first-order ϵ terms, you will get back this $\tilde{\tilde{\varepsilon}}$.

In the same fashion, by dropping all the nonlinear terms of the Euler-Almansi strain tensor, we can get back this $\tilde{\tilde{\varepsilon}}$, or infinitesimal strain tensor. So, for unidirectional extension with normal strain, ϵ, and material Poisson's ratio, ν, this is the small strain tensor, the infinitesimal linear strain tensor, $\tilde{\tilde{\varepsilon}}$, for unidirectional extension.

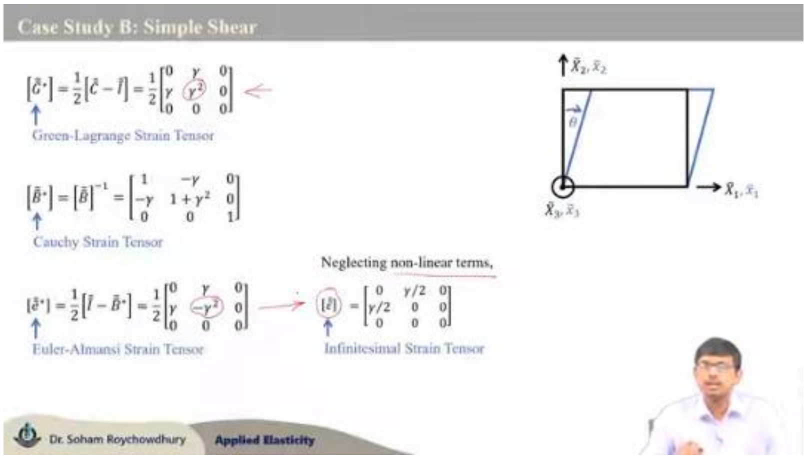


Now, moving to the second case study of simple shear in the X_1 - X_2 plane. So, we are considering a rectangle in the X_1 - X_2 plane. So, X_3 is constant here. This is subjected to pure shear or simple shear due to a shear loading acting here, on this side, causing the theta amount of deformation of the X_2 edge.

So, the blue one is the deformed state, and the black one is the undeformed state. Now, γ being the shear strain, the change in angle, we can write the deformation mapping as: $x_1 = X_1 + \gamma X_2$, $x_2 = X_2$, and $x_3 = X_3$. So, γ is the shear strain, i.e., the change in the angle between two small orthogonal line elements. The way engineering shear strain was defined in the last lecture, here also, this γ is defined in the same fashion.

So, with this given deformation mapping, the deformation gradient tensor comes out to be $\tilde{\tilde{F}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This is the deformation gradient tensor, and note that this particular deformation gradient tensor is not symmetric, because $F_{12} = \gamma$, whereas $F_{21} = 0$. So, due to this, this is an asymmetric deformation gradient tensor.

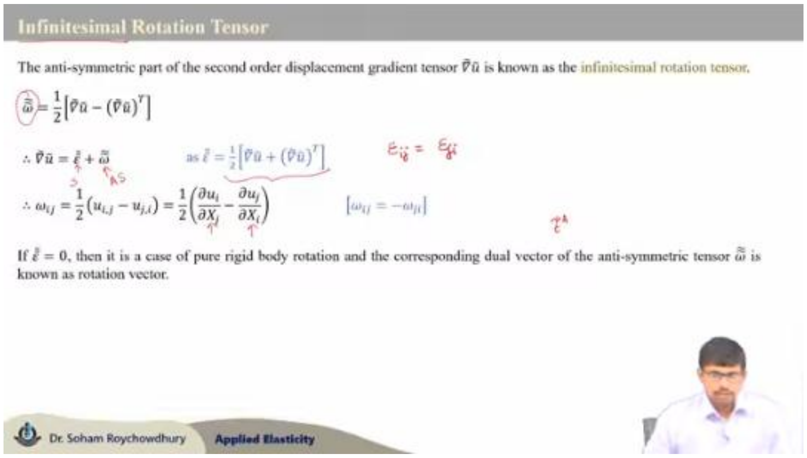
Now, moving forward to the left and right Cauchy-Green deformation tensor, $\tilde{\tilde{C}}$ can be obtained as $\tilde{\tilde{F}}^T \tilde{\tilde{F}}$ as this, and $\tilde{\tilde{B}}$, the left Cauchy-Green deformation tensor, can be obtained as $\tilde{\tilde{F}} \tilde{\tilde{F}}^T$, as this. Now, if you compare these two, they are not equal. So, here, $\tilde{\tilde{C}} \neq \tilde{\tilde{B}}$. This is because $\tilde{\tilde{F}}$, the deformation gradient tensor, is not symmetric. In the previous example, $\tilde{\tilde{F}}$ being symmetric, we obtained the same set of right Cauchy-Green and left Cauchy-Green deformation tensors, but if $\tilde{\tilde{F}}$ is not symmetric, they would not be the same. So, this point you should note.



Now, moving forward, we can find $\tilde{\tilde{G}}^*$, i.e., the Green-Lagrange strain tensor, in the undeformed state. Then, in the deformed state, we can find the Cauchy strain tensor and Euler-Almansi strain tensor, $\tilde{\tilde{e}}^*$, by using the respective formulae. So, just by manipulating all these matrices, we can obtain the $\tilde{\tilde{G}}^*$, $\tilde{\tilde{B}}^*$, and $\tilde{\tilde{e}}^*$ tensors, strain tensors in both undeformed and deformed measures. Now, these are the measures of finite strain.

The value of γ being large, this set of expressions is valid. Now, once we assume small shear strain, γ being small, we need to neglect the nonlinear terms. So, just by neglecting the nonlinear term of $\tilde{\tilde{G}}^*$, the Green-Lagrange strain tensor; $\tilde{\tilde{G}}^*$ is having only one nonlinear term, γ^2 in 2,2 location. So, by dropping that, $\tilde{\tilde{\varepsilon}}$, the infinitesimally small strain tensor, would become $\begin{bmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Same strain tensor can also be obtained from $\tilde{\tilde{e}}^*$ by dropping the non-linear term, which is $-\gamma^2$ at the same location. So, you note: in between $\tilde{\tilde{G}}^*$ and $\tilde{\tilde{e}}^*$ only difference is the change in sign of the element located at 2,2 position, and that is the non-linear element.

With the small strain assumption, that would be dropped, and both of them would come down to the same expression of $\tilde{\tilde{\varepsilon}}$, i.e., linear strain tensor. Now, you note that this linear strain tensor $\tilde{\tilde{\varepsilon}}$ is symmetric and, even both $\tilde{\tilde{G}}^*$ and $\tilde{\tilde{e}}^*$ were also symmetric. So, even if $\tilde{\tilde{C}}$ and $\tilde{\tilde{B}}$, left and right Cauchy-Green deformation tensor, or $\tilde{\tilde{F}}$, deformation gradient tensor, are not symmetric, but all these strain measures are coming out to be symmetric by their definition.



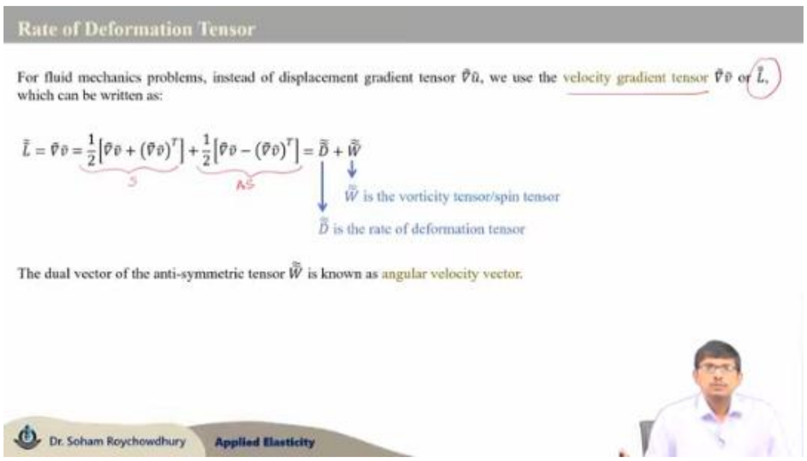
Now, moving forward, we are going to define the small rotation tensor or infinitesimally small linear rotation tensor. This is defined to be the anti-symmetric part of the second order displacement gradient tensor $\tilde{\nabla}\tilde{u}$. So, displacement gradient tensor $\tilde{\nabla}\tilde{u}$ can be divided into two parts. One is its symmetric part, another one is its anti-symmetric part. The anti-symmetric part of $\tilde{\nabla}\tilde{u}$ is defined to be the infinitesimally small rotation tensor which is denoted by this $\tilde{\tilde{\omega}}$.

So, $\tilde{\tilde{\omega}} = \frac{1}{2}\left[\tilde{\nabla}\tilde{u} - \left(\tilde{\nabla}\tilde{u}\right)^T\right]$. And we already know $\frac{1}{2}\left[\tilde{\nabla}\tilde{u} + \left(\tilde{\nabla}\tilde{u}\right)^T\right]$ is nothing but $\tilde{\tilde{\varepsilon}}$. $\tilde{\tilde{\varepsilon}}$, or the linear strain tensor, was defined as the symmetric part. This is the symmetric part of the $\tilde{\nabla}\tilde{u}$. Thus, the total $\tilde{\nabla}\tilde{u}$ is the summation of the symmetric part.

This is the symmetric part of $\tilde{\nabla}\tilde{u}$, and this is the antisymmetric part of $\tilde{\nabla}\tilde{u}$. The symmetric part of $\tilde{\nabla}\tilde{u}$ is called the linear strain tensor, and the antisymmetric part of $\tilde{\nabla}\tilde{u}$ is called the small linear rotation tensor. So, $\tilde{\tilde{\varepsilon}} + \tilde{\tilde{\omega}} = \tilde{\nabla}\tilde{u}$. In terms of its components, $\tilde{\omega}$ can be written as $\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right)$. And explicitly, by expanding this derivative, it can be written as $\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)$.

As it is infinitesimally small, for the linear rotation tensor, there is no difference between the derivative with respect to I and i, similar to the definition of $\tilde{\tilde{\varepsilon}}$. And also, you can verify that the property of the antisymmetric tensor is valid for $\tilde{\tilde{\omega}}$, as this is one antisymmetric tensor. So, $\omega_{ij} = -\omega_{ji}$, whereas $\varepsilon_{ij} = -\varepsilon_{ji}$, which was a symmetric tensor. But this rotation tensor is an antisymmetric tensor, and thus, $\omega_{ij} = -\omega_{ji}$.

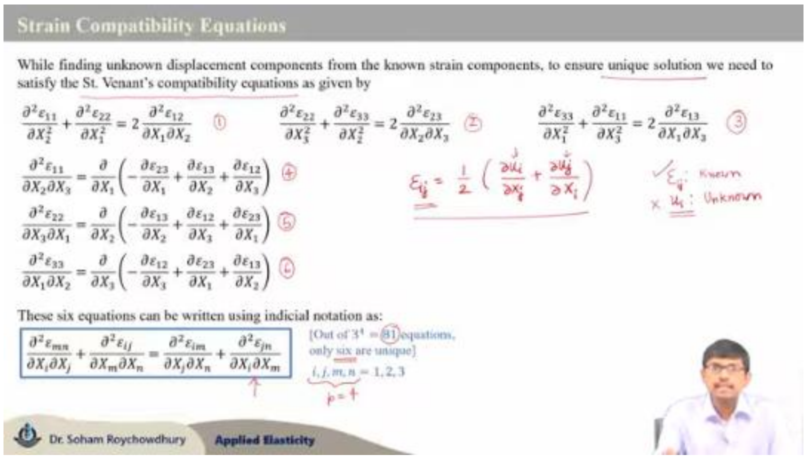
Now, if $\tilde{\tilde{\varepsilon}} = \tilde{\tilde{0}}$, for any body during transformation, if there is no strain generated, that is the case of pure rigid body rotation. With $\tilde{\tilde{\varepsilon}}$ being 0, $\tilde{\nabla}\tilde{u} = \tilde{\tilde{\omega}}$, and that is the case of pure rigid body rotation, and the corresponding dual vector is $\tilde{t}^A$ vector for that anti symmetric tensor $\tilde{\tilde{\omega}}$, and is called the rotation vector or angular velocity vector.



Now coming to the rate of deformation tensor. The previous quantities, $\tilde{\tilde{\varepsilon}}$, or $\tilde{\tilde{\omega}}$, or deformation gradient tensor, whatever we had used, all those are valid for solving the solid mechanics problems but for the problems involving the fluid mechanics domain, instead of displacement gradient tensor $\tilde{\nabla}\tilde{u}$, we define the velocity gradient tensor. So, rate of change of displacement gradient tensor, i.e., rate of change of $\tilde{\nabla}\tilde{u}$ is defined to be $\tilde{\nabla}\tilde{v}$, and it is denoted by $\tilde{\tilde{L}}$. This is called velocity gradient tensor. So, this can be written as $\tilde{\tilde{L}} = \tilde{\nabla}\tilde{v}$.

This is summation of the symmetric part of $\tilde{\nabla}\tilde{v}$ and anti-symmetric part of $\tilde{\nabla}\tilde{v}$. So, this is symmetric part of $\tilde{\nabla}\tilde{v}$, and this is anti-symmetric part of $\tilde{\nabla}\tilde{v}$. The symmetric part is named as $\tilde{\tilde{D}}$, whereas anti-symmetric part is named as $\tilde{\tilde{W}}$. $\tilde{\tilde{D}}$ is called the rate of deformation tensor and $\tilde{\tilde{W}}$ is called the vorticity tensor or spin tensor for the fluid problems.

So, if you just compare it with the previous slide, in that, $\tilde{\nabla}\tilde{u}$ was having its symmetric part named as the linear strain tensor, and anti symmetric part named as the rotation tensor. Whereas, the symmetric part of $\tilde{\nabla}\tilde{v}$ or $\tilde{\tilde{L}}$ is called the rate of deformation tensor, and anti symmetric part of $\tilde{\tilde{L}}$ is called $\tilde{\tilde{W}}$ or the vorticity or spin tensor. So, the dual vector of anti symmetric part of $\tilde{\tilde{W}}$ is called the angular velocity vector; this is same as the previous case.



Now, coming to an important concept of strain compatibility equations. So, these are the set of equations which relates the displacements with the strain components. So, already while checking the expression, or deriving the expression of linear strain $\tilde{\tilde{\varepsilon}}$, we had seen that $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$. Those expressions are called strain displacement relations, when strains are small or linear. Now, using those we can relate the six strain components with the three displacement components u_1, u_2, u_3.

But for all the problems it may not be feasible to use those equation and end up with unique displacement field. If you are having a solid mechanics or elasticity problem, where the displacement components are unknown and strain components are given, for such problems to ensure the unique solution, we need to use the following compatibility equations, which are known as St. Venant's strain displacement compatibility equations.

There are six such equations, and those six equations are classified into two categories. Equations 1, 2, and 3 fall under the first category of the first class of St. Venant's compatibility equations. Whereas the next three, these three look similar. So, equations 4, 5, and 6 form the second category of compatibility equations. If you recall, $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$.

Now, if $\tilde{u}$ is known—u_i, u_j, the displacement components are known— it is very easy to substitute those $\tilde{u}$ here and obtain $\tilde{\tilde{\varepsilon}}$, the strain components, by taking the partial derivatives of displacement components with respect to X, the undeformed coordinate vector components. However, if the case is just the opposite—instead of $\tilde{u}$ being known, $\tilde{u}$ is the unknown, and we know $\tilde{\tilde{\varepsilon}}$, the strain components— and we need to find out the displacement components u_i , for such cases, we cannot directly use this strain-displacement relation or the definition of $\tilde{\tilde{\varepsilon}}$; we need to integrate this to find u_i.

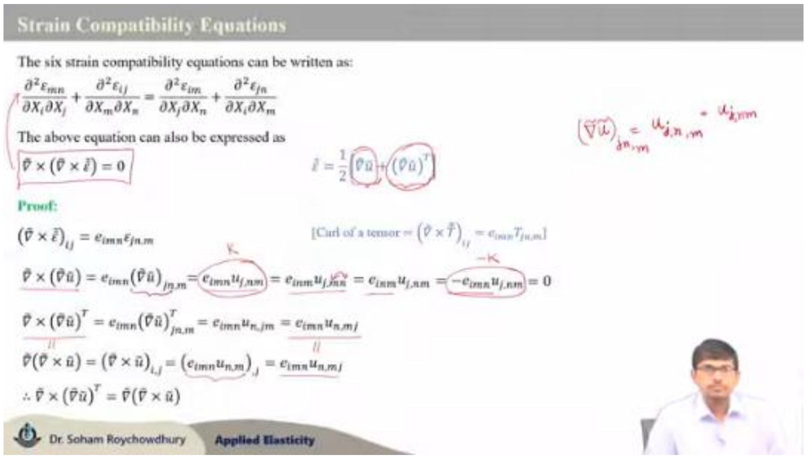
All six equations of ε_{ij} are required to be integrated to find the unknown displacement components. Now, as there are six equations, and we are integrating them to find out three displacements, we may not get a unique solution for displacement by integrating different strain equations. Now, to ensure the existence of a unique displacement field, instead of the strain-displacement relation, we use St. Venant's compatibility equations.

This ensures a unique solution for the displacement fields while integrating. So, this is the requirement of the strain-displacement compatibility equations. These are required when the strain components are known, but the displacement components are unknown, and those we need to find out by integration, and to ensure a unique solution, we must use these six strain-displacement compatibility equations.

Now, using the indicial notation, these six equations can be combined into this single indicial form: $\frac{\partial^2 \varepsilon_{mn}}{\partial X_i \partial X_j} + \frac{\partial^2 \varepsilon_{ij}}{\partial X_m \partial X_n} = \frac{\partial^2 \varepsilon_{im}}{\partial X_j \partial X_n} + \frac{\partial^2 \varepsilon_{jn}}{\partial X_i \partial X_m}$. So, in this equation, i, j, m, and n—we have four free indices—all are varying between 1, 2, and 3, and having four free indices, p being 4, the total number of equations, by expanding this particular indicial expression, should be 81.

We had discussed that if the number of free indices is p, then the total number of equations, whose condensed form is represented by that indicial notation, is 3^p. Here, p

being 4, having four free indices, we are having $3^4 = 81$ equations. But due to the symmetry of epsilon, if you write all those 81 equations and compare, only six such equations are unique. Those six equations are written here as six strain-displacement compatibility equations.



Now, this particular strain-displacement compatibility equation can be written in this form, and this is very easy or convenient to express. So, $\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{\tilde{\varepsilon}}) = 0$; this is the most common form of the strain-displacement compatibility equation, and if you expand this, we can easily show that it would come down to this indicial form. $\tilde{\tilde{\varepsilon}}$ is defined as $\frac{1}{2}\left[\tilde{\nabla}\tilde{u} + (\tilde{\nabla}\tilde{u})^T\right]$.

Now, we will try to prove this identity: $\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{\tilde{\varepsilon}}) = 0$, where $\tilde{\tilde{\varepsilon}}$ is the small strain defined as $\frac{1}{2}\left[\tilde{\nabla}\tilde{u} + (\tilde{\nabla}\tilde{u})^T\right]$. Now, taking $\tilde{\nabla} \times \tilde{\tilde{\varepsilon}}$, $(\tilde{\nabla} \times \tilde{\tilde{\varepsilon}})_{ij}$ component can be written as $e_{imn}\varepsilon_{jn,m}$. By using the definition of the curl of a tensor, $\left(\tilde{\nabla} \times \tilde{\tilde{T}}\right)_{ij} = e_{imn}T_{jn,m}$. Now, here, that tensor $\tilde{\tilde{T}}$ is $\tilde{\tilde{\varepsilon}}$; thus, we can write $(\tilde{\nabla} \times \tilde{\tilde{\varepsilon}})_{ij} = e_{imn}\varepsilon_{jn,m}$.

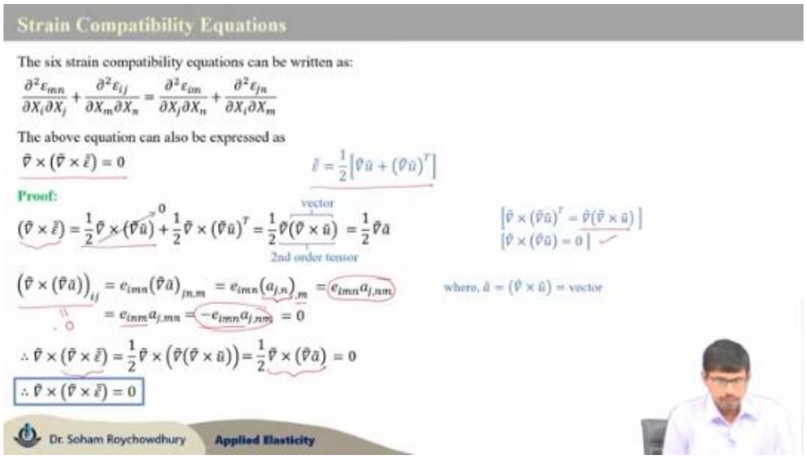
Now, $\tilde{\tilde{\varepsilon}}$ has two parts: one is $\tilde{\nabla}\tilde{u}$, and the other is $(\tilde{\nabla}\tilde{u})^T$. So, first, we are taking the curl of the first part of $\tilde{\tilde{\varepsilon}}$. So, $\tilde{\nabla} \times \tilde{\nabla}\tilde{u}$ is taken, and using this definition of the curl of a tensor, we can write that $\tilde{\nabla} \times \tilde{\nabla}\tilde{u} = e_{imn}(\tilde{\nabla}\tilde{u})_{jn,m}$. Now, this part—$(\tilde{\nabla}\tilde{u})_{jn}$—can be written as $u_{j,n}$, by the definition of $\tilde{\nabla}\tilde{u}$. So, $(\tilde{\nabla}\tilde{u})_{jn} = u_{j,n}$, and we were having another '$,n$' outside. So, writing both partial derivatives, it would be $u_{j,mn}$ or $u_{j,nm}$.

Now, interchanging the indices m and n on both terms, renaming m with n and n with m, we can write this as: $e_{inm}u_{j,mn}$. Now, I am flipping the order of the partial differentiation, which can always be done, so this would be $e_{inm}u_{j,nm}$. And now, by using the permutation symbol property $e_{inm} = -e_{imn}$, this quantity would be $-e_{imn}u_{j,nm}$. Now, if you compare this particular term and this particular term, if you call this some quantity K, this is equal to $-K$. And $K = -K$ is valid only if $K = 0$. Thus, the first term of $\tilde{\nabla} \times \tilde{\nabla}\tilde{u} = 0$.

Now, let us proceed to the second term. We are taking the curl of the second term of $\tilde{\tilde{\varepsilon}}$, $\tilde{\nabla} \times \left(\tilde{\nabla}\tilde{u}\right)^T$. This can be written using the definition of the curl of a tensor as $e_{imn}\left(\tilde{\nabla}\tilde{u}\right)^T_{jn,m}$. Following a similar approach, this would be $e_{imn}u_{n,jm}$. Because of the transpose, here j and n—these two indices are flipped, and the transpose sign is removed.

Now, this can also be written by changing or flipping the position of the partial derivatives as $e_{imn}u_{n,mj}$. And thus, this $\tilde{\nabla} \times \left(\tilde{\nabla}\tilde{u}\right)^T$, we have shown to be $e_{imn}u_{n,mj}$. Now, let us choose another identity, which is $\tilde{\nabla}\left(\tilde{\nabla} \times \tilde{u}\right)$. So, that is the gradient of curl of $\tilde{u}$, and we can write that as $\left(\tilde{\nabla} \times \tilde{u}\right)_{i,j}$.

And by using the definition of the curl of a vector, $\left(e_{imn}u_{n,m}\right)_{,j}$, i.e., $\left(\tilde{\nabla} \times \tilde{u}\right)_{i,j}$. So, expanding this, there would be no derivative on the permutation symbol. So, e_{imn} remains as it is, and the second term is $u_{n,mj}$. So, if you compare, both of them are the same. So, this second term on the $\tilde{\tilde{\varepsilon}}$, $\tilde{\nabla} \times \left(\tilde{\nabla}\tilde{u}\right)^T$, we can write as $\tilde{\nabla}\left(\tilde{\nabla} \times \tilde{u}\right)$.



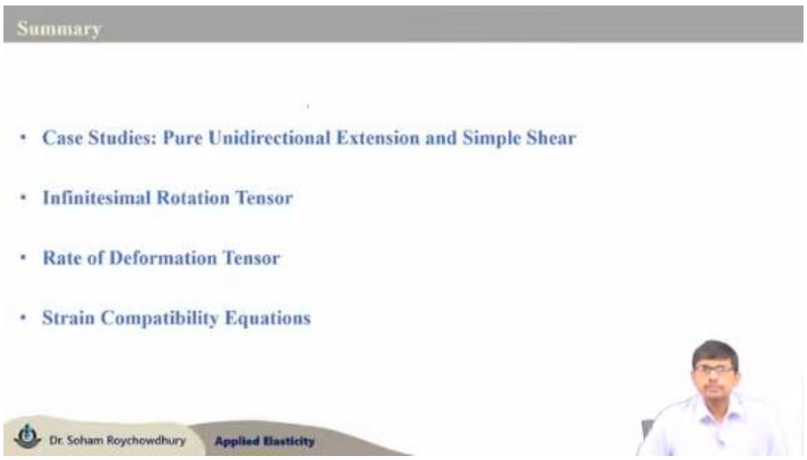
Now, moving forward, we are taking $\tilde{\nabla} \times \tilde{\tilde{\varepsilon}}$. And, using the definition of $\tilde{\tilde{\varepsilon}}$, there will be two terms: $\tilde{\nabla} \times \tilde{\nabla}\tilde{u}$ and $\tilde{\nabla} \times \left(\tilde{\nabla}\tilde{u}\right)^T$. As we had already proved that $\tilde{\nabla} \times \tilde{\nabla}\tilde{u} = 0$, the first term would go to 0. The first term in $\tilde{\nabla} \times \tilde{\tilde{\varepsilon}}$ is 0, and the second term, I am rewriting as this. So, this would be $\frac{1}{2}\tilde{\nabla}\left(\tilde{\nabla} \times \tilde{u}\right)$.

Now, choosing $\tilde{\nabla} \times \tilde{u}$ to be a vector, $\tilde{u}$ being a vector, $\tilde{\nabla} \times \tilde{u}$ is another vector, and if you add the gradient operator, $\tilde{\nabla}\left(\tilde{\nabla} \times \tilde{u}\right)$ will be a second-order tensor. So, writing this vector $\tilde{\nabla} \times \tilde{u}$ as a new vector $\tilde{a}$, this term would be $\frac{1}{2}\tilde{\nabla}\tilde{a}$. Now, if you simplify $\tilde{\nabla} \times \tilde{\nabla}\tilde{a}$, it is another second-order tensor. Its ij component would be $e_{imn}\left(\tilde{\nabla}\tilde{a}\right)_{jn,m}$.

Now, expanding this, it would be $e_{imn}\left(a_{j,n}\right)_{,m}$. So, combining both partial derivatives, it would be $e_{imn}a_{j,nm}$. Now, interchanging the names of both dummy indices, m and n, it would be $e_{inm}a_{j,mn}$. Then, by using the definition of the permutation symbol $e_{imn} = -e_{inm}$, we can show that this quantity becomes $-e_{imn}a_{j,nm}$.

Now, if you compare this term and this term, once again a quantity is equal to minus itself, which means it must be 0. So, the left-hand side $\tilde{\nabla} \times \tilde{\nabla}\tilde{a}$ must be 0. Now, coming to the strain compatibility equation, $\tilde{\nabla} \times \left(\tilde{\nabla} \times \tilde{\tilde{\varepsilon}}\right)$. So, $\tilde{\nabla} \times \tilde{\tilde{\varepsilon}}$, this inner part, we had already shown it to be $\frac{1}{2}\tilde{\nabla}\left(\tilde{\nabla} \times \tilde{u}\right)$ or $\frac{1}{2}\tilde{\nabla}\tilde{a}$. Substituting that, this external curl is still present.

So, it would be $\frac{1}{2}\tilde{\nabla} \times \tilde{\nabla}\tilde{a}$, and $\tilde{\nabla} \times \tilde{\nabla}\tilde{a}$, we had already shown to be 0. So, thus, $\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{\tilde{\varepsilon}})$ would come out to be 0. So, this proves the strain compatibility equation. So, these are required to be used if the displacement fields are unknown, which we need to find out from the known strain fields and to ensure a unique displacement field, we must use this strain-displacement compatibility equation, either in this form or in this indicial notation form.



So, in this lecture, we discussed two case studies of pure unidirectional extension and simple planar shear. We introduced the concepts of rotation tensor and rate of deformation tensor, and finally, discussed the important concept of strain-displacement compatibility equations.

Thank you.