APPLIED ELASTICITY

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SCHOOL OF MECHANICAL SCIENCES

INDIAN INSTITUTE OF TECHNOLOGY, BHUBANESWAR

WEEK: 01
Lecture- 01

Welcome to the course on Applied Elasticity. Myself Dr. Soham Roychowdhury, faculty member from the School of Mechanical Sciences, IIT Bhubaneswar. In this course, we will discuss the theory of elasticity followed by its various applications in the mechanical engineering domain. Elasticity is the property of a material due to which the material is getting restored back

to its original or initial configuration upon removal of the load or the external forces acting on it which was causing its deformation. So, once the load is removed, the material returns to its original configuration. This particular property is called elasticity. Now in the mechanical engineering domain there are various applications, various problems

which can be solved with the help of theory of elasticity. Now what are some such problems which we will be discussing in this course. So design and analysis of different load bearing machine components. Different thermo mechanical problems in which along with the mechanical stresses, thermal stresses are also present.

Contact stress problems for different indented geometries. The fracture mechanics problem which deals with the initiation and propagation of the cracks in different machine components. So these are different set of problems which can be solved with the help of theory of elasticity. Now solution of any of this problem basically involves the formulation of field equations of elasticity. There exist 15 field equations of elasticity which are required to be solved with the help of suitable avialable boundary conditions that we will be discussing

Now before going into the theory, we will have first some discussion on the mathematical preliminaries related to tensor algebra and tensor calculus which will be used often in this particular course So, in the elasticity formulation various quantities are involved which are either scalars or vectors or tensors of different orders such as if you talk about the scalar quantities we will come across the quantities like potential energy, the strain energy stored within the elastic body due to its deformation, the total work done by all the external forces acting on the body. These are all possible scalar quantities with which we will be coming across during this course. Similarly, there are different vector quantities mostly related to the kinematics of deformation such as position vector, velocity vector, acceleration vector, angular velocity vector, angular acceleration vector or displacement vector or we can also have the force vector or surface traction vectors.

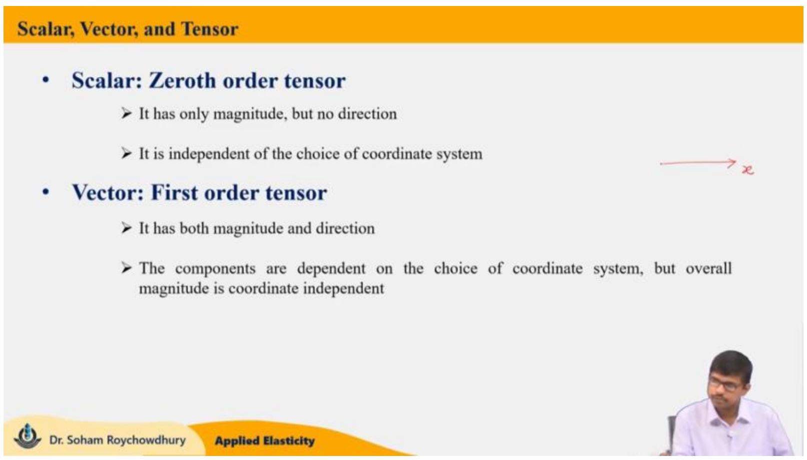
These are the vector quantities related to the elasticity problems. We may also have some tensor quantities like stress and strains which are second order tensors they cannot be defined like a vector because these are associated with a plane and on the plane there are different directions available along which we will be defining the stress and strain components. Thus, they cannot be expressed as vector quantities. So, these are second-order tensor quantities.

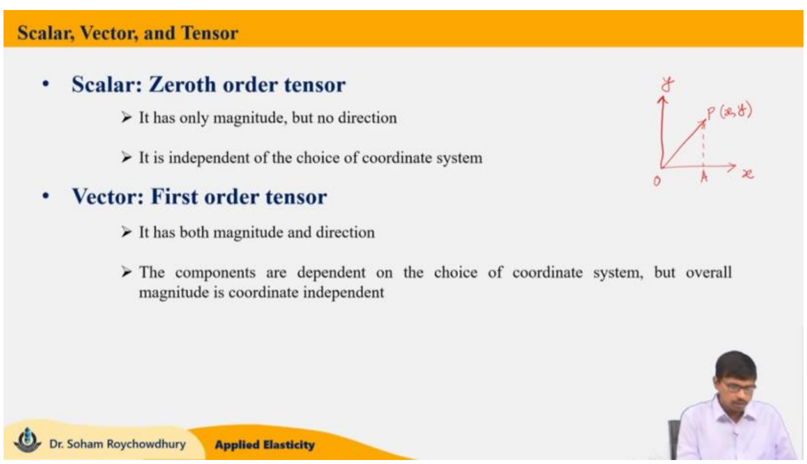
Now, if we want to relate the stress and strain tensors, that can be done by a fourth-order tensor, which is called the elastic stiffness tensor, which describes the material behavior. The material constitutive behavior is described by this fourth-order elastic stiffness tensor. So, we can see there are various types of quantities which we will be coming across within our discussion on elasticity which are either scalar or vector or tensors of different orders. Thus, it is important to have an idea regarding this different order tensors and the various operations that we need to perform on this tensor, tensor quantities during this particular course. And also it is important or it is convenient to use the tensor indicial notations for expressing all the field equations of elasticity because the indicial notation should be independent of the coordinate systems whether we are using rectangular Cartesian coordinate or polar coordinate irrespective of that.

The field equations will remain same if we are writing them in terms of the tensor indicial notation. So, first, we are going to talk about the basic introduction to scalar, vector, and tensors. Now, what is a scalar quantity? Scalar quantities are just having a magnitude not having any direction. These are also termed as zeroth-order tensors.

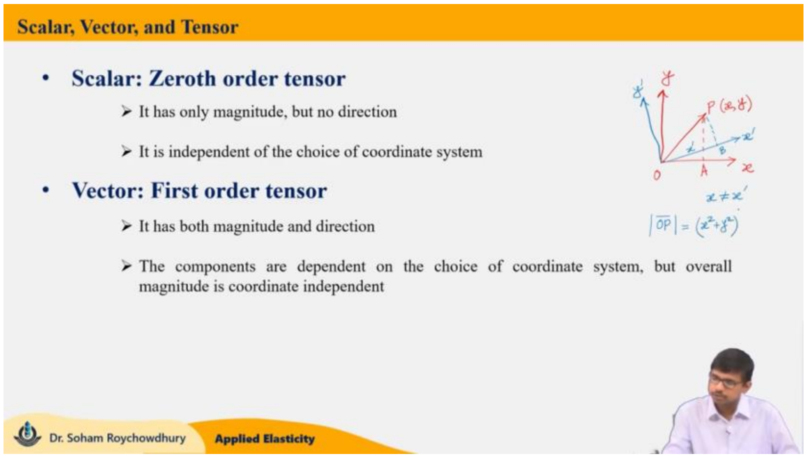
And it is independent the magnitude of the scalar quantity is independent to the choice of the reference frame or the coordinate system. now after scalar if we proceed for vector that is known as first order tensor which is having magnitude as well as direction now Any vector can be written in terms of its components. All the components of a vector depend on the choice of the reference frame. However, the overall magnitude of the vector is independent of the choice of the coordinate system. So, you can consider a vector.

So let us say x, y is one choice of coordinate frame, and O to P is a vector with components x and y. So, o to a this much is equals to x the x component of this vector. Now, instead of $x - y$ if you go for another choice of the reference frame let us say $x' - y'$. In that O to B this is the x prime component of the same vector. Now, here you can see x is not equals to x prime. So the component of same vector is different in two different coordinate systems.



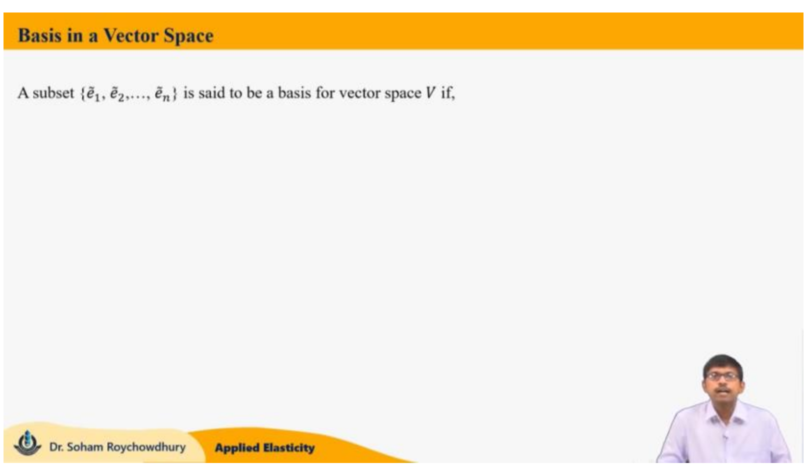


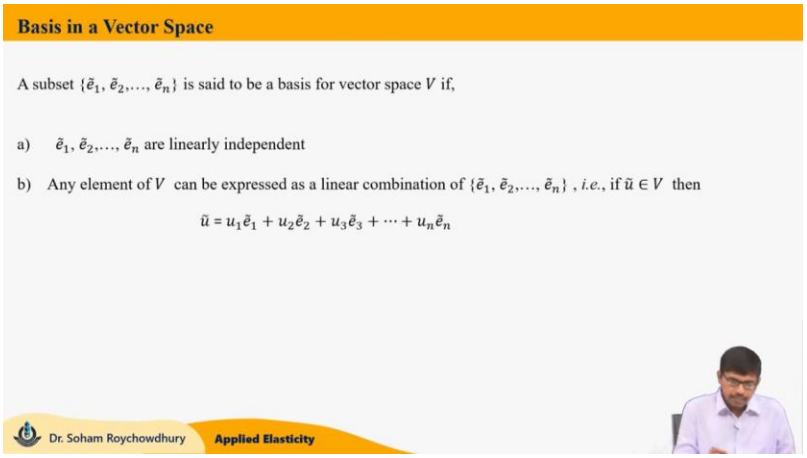
However, if you consider the overall magnitude of this vector this should be independent to the choice of the reference frame. So, in the first $x - y$ frame $\sqrt{x^2 + y^2}$ is the overall magnitude of the vector. In the second frame This should be $\sqrt{x'^2 + y'^2}$ this entire quantity and both of them must be same. So, overall magnitude of the vector is independent to the choice of the reference frame whereas, the individual components are dependent.

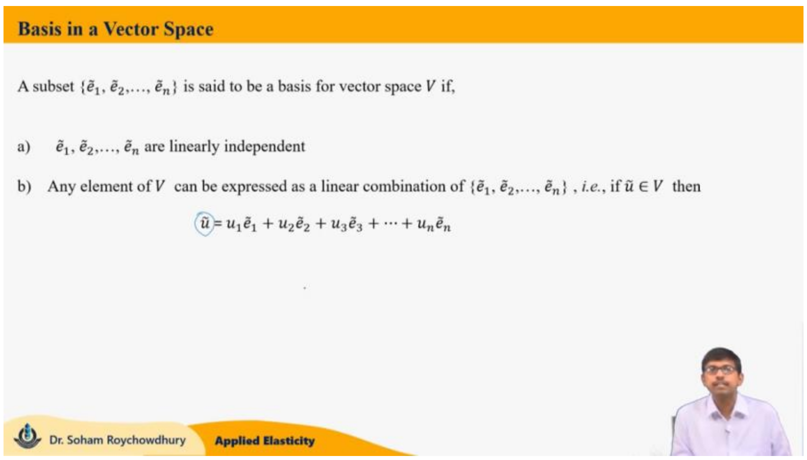


Now, if we go for tensors, these are defined as a transformation. We will come to the formal definition later. And you should understand now that tensors are having order 2, 3 or more. And the same property as discussed for the vector components is valid for the tensor components. Tensor components are also dependent on the choice of the coordinate system that is choice of x, y, and z axis.

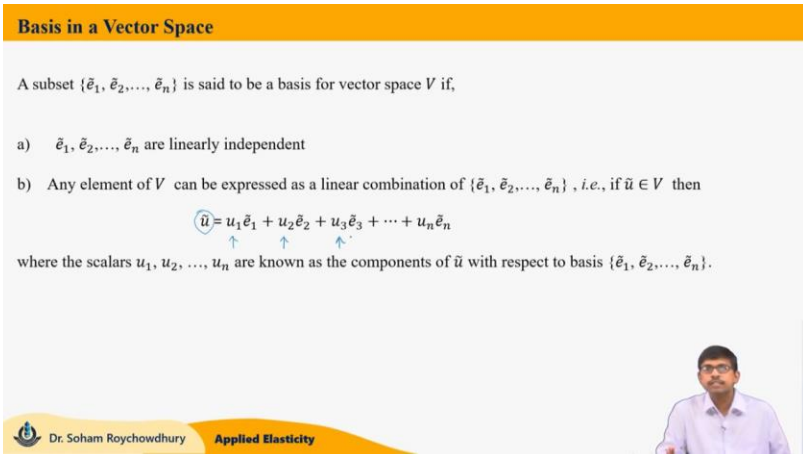
Now, first we are going to define the basis in a vector space. Let us consider a subset involving n vectors $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ till $\tilde{e}_n$ vector. any element of this vector space V can be expressed as a linear combination of all these vectors. So, let us consider any arbitrary vector $\tilde{u}$, any vector within this vector space V. If we can express u as the linear combination of all these $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ till $\tilde{e}_n$ these vectors then we can call this $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ this set is forming the basis for this vector space where the u_1, u_2,u_3 these are the scalar factors weighting factor attached with individual vectors $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ forming the basis of the vector space.

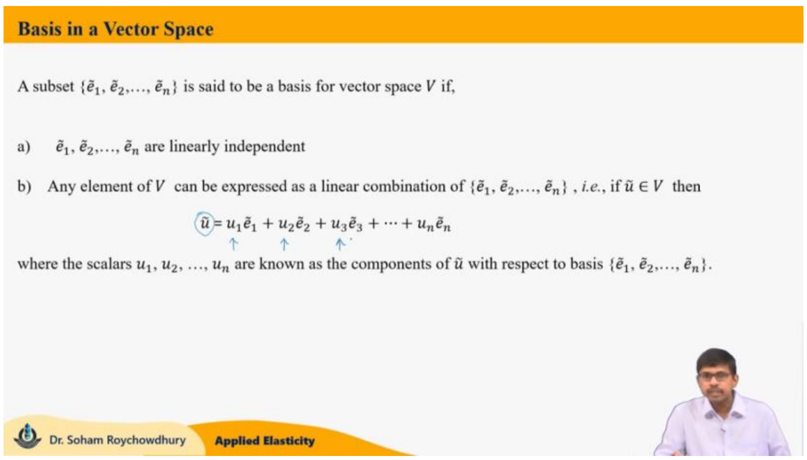




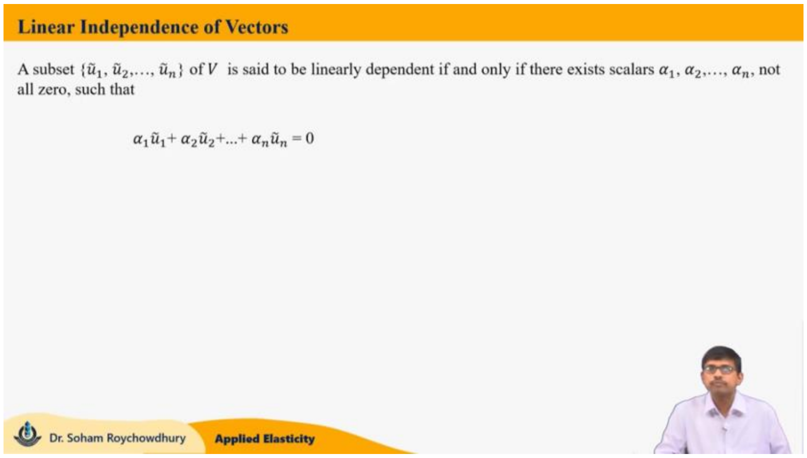


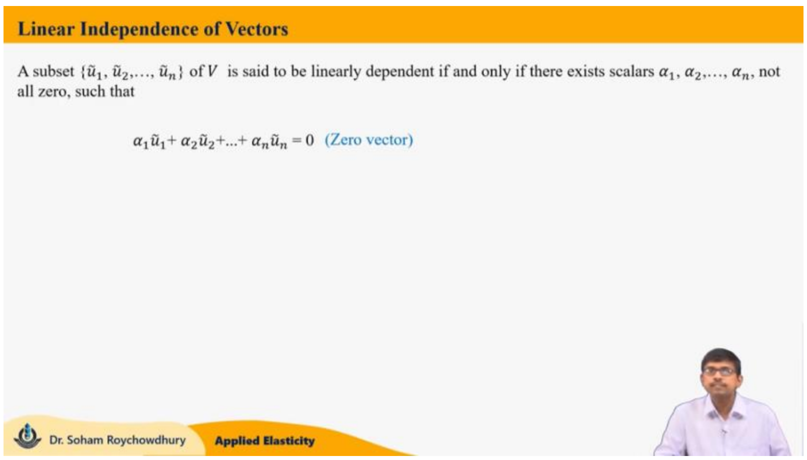
So, if this set of $\tilde{e}_1$, $\tilde{e}_2, \tilde{e}_3$ are linearly independent and With the help of them if it is possible to express any vector u as a linear combination of those then we call $\tilde{e}_1$, $\tilde{e}_2, \tilde{e}_3$ till $\tilde{e}_n$ is the basis for a vector space. So, rectangular Cartesian coordinate system unit vectors along x, y, z let us say $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ they forms the basis for the vector space in rectangular Cartesian coordinate system. And, we can go for different choice of basis or base vectors for a vector space but the number of elements, number of components present in a base vector system should be same for all possible choice.



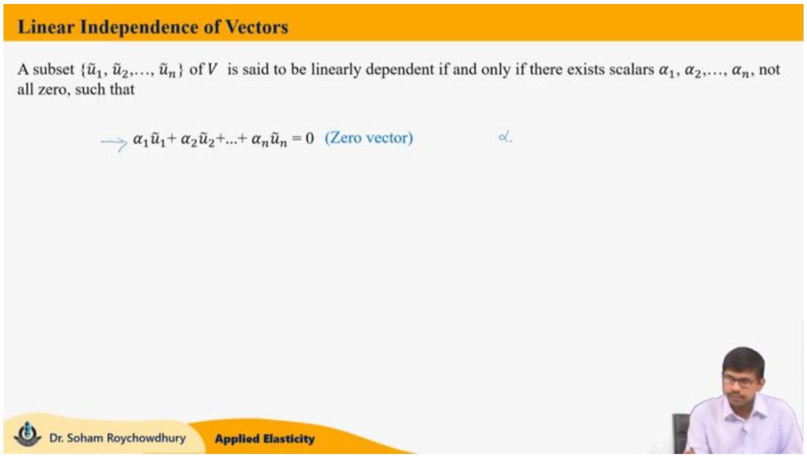


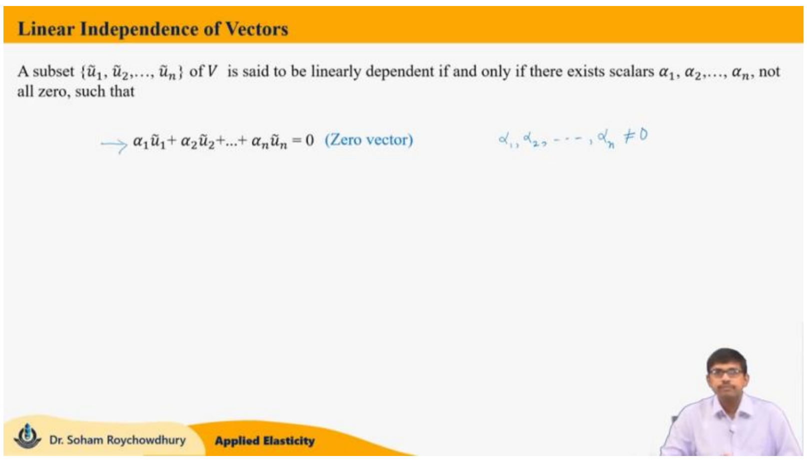
So if we are having three components in one base vector choice, then for all other choices, three elements will only be there. Now coming to linear independence of vectors. Let us consider $\tilde{u}_1$, $\tilde{u}_2$,$\tilde{u}_3$ till $\tilde{u}_n$, a set of vectors in vector space V and if you consider this equation $\alpha_1\tilde{u}_1$, $\alpha_2\tilde{u}_2$ continuing till $\alpha_n\tilde{u}_n$. So, $\sum_{i=1}^{n}\alpha_i\tilde{u}_i$ equals to 0 that is a zero vector. Now, if a set of vector $\tilde{u}_1$, $\tilde{u}_2$,$\tilde{u}_3$ till $\tilde{u}_n$ is able to satisfy this equation then they will call linearly independent only if there exist a set of scalar α_1, α_2, ..., α_n which are not all 0s.





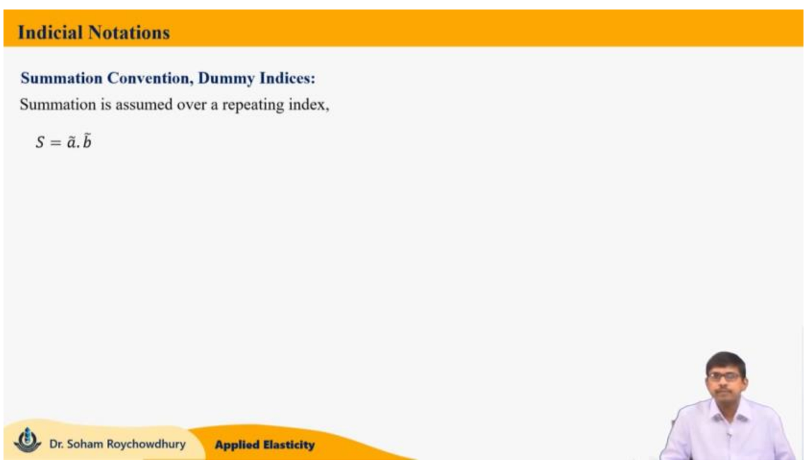
So, if it is possible to satisfy this particular equation with α_1, α_2, ..., α_n. All of them not being 0, some of them 1 or 2 may be 0, but all of them cannot be 0. If we can find any such combination, then we call those set of vectors $\tilde{u}_1$, $\tilde{u}_2$ to be linearly dependent. On the other hand, if this equation can only be satisfied with the help of α_1, α_2, ..., α_n, all these scalars being 0, then in such condition we call those vectors to be linearly independent.

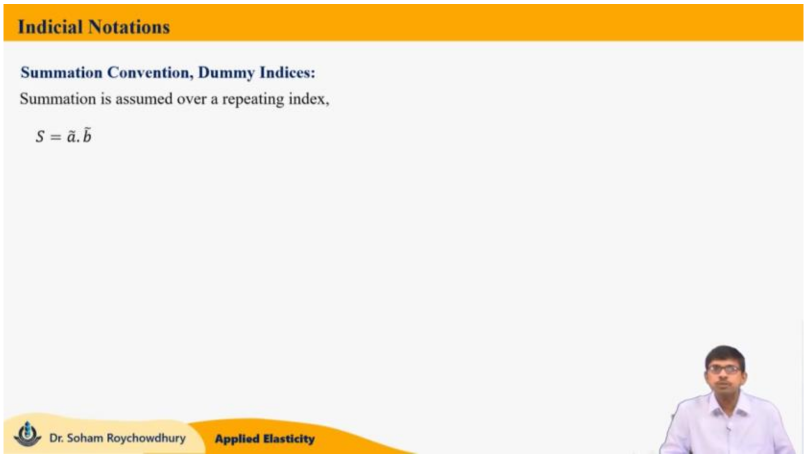


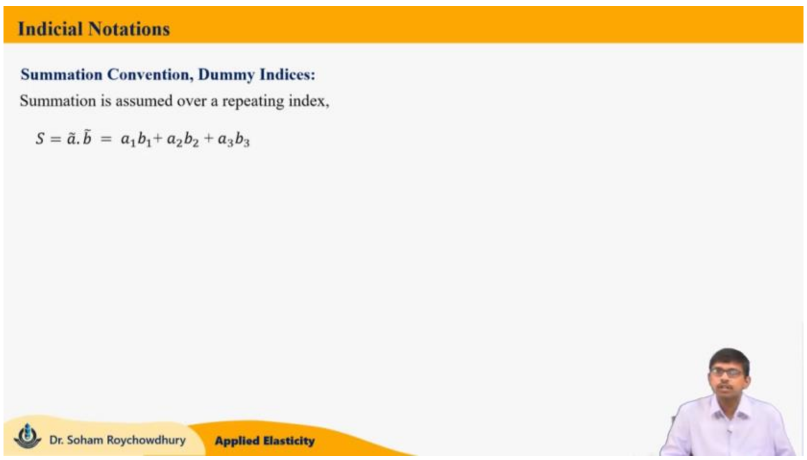


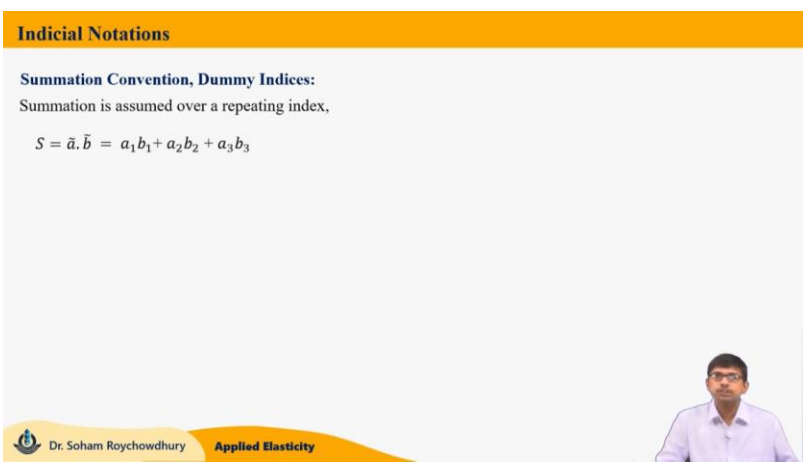
A set of vectors to be linearly independent this equation can be satisfied with all 0 values of α_i coefficients for the case of linearly independent vectors. However, for the linearly dependent vectors this equation can be satisfied for a set of α_i values where at least one of the α_i is non-zero. This is called the linear independence of vectors in a vector space. Now, coming to the indicial notation, so normally the previous expressions what we have written that is $\tilde{u}$ vector is $u_1\tilde{e}_1 + u_2\tilde{e}_2$ and so on that can be represented with the help of a summation sign.

And summation is always assumed over a repeating index. So, if you consider this equation dot product of two vectors $\tilde{a}$ and $\tilde{b}$. So, dot product of two vector will be a scalar which is defined as S here. So, S equals to $\tilde{a}.\tilde{b}$. So, we can expand this with the help of vector components of vector $\tilde{a}$ and $\tilde{b}$. So, let us say a_1, a_2, a_3 are the components of $\tilde{a}$ vector b_1, b_2, b_3 are the components of $\tilde{b}$ vector. So, we can expand this dot product as $a_1 b_1 + a_2 b_2 + a_3 b_3$.

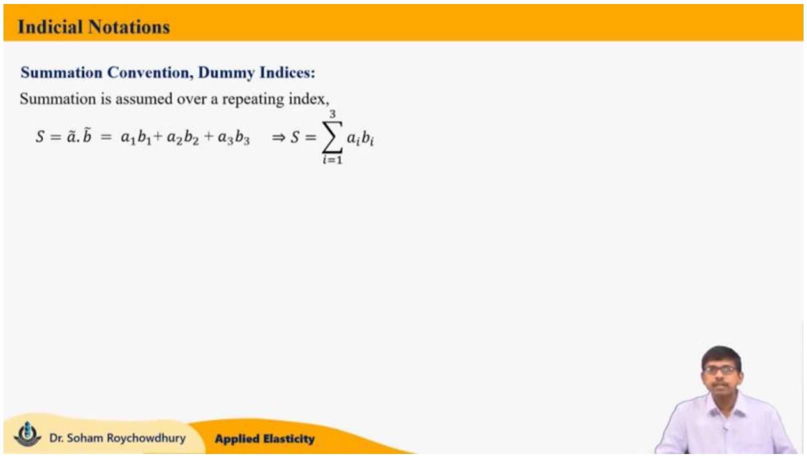




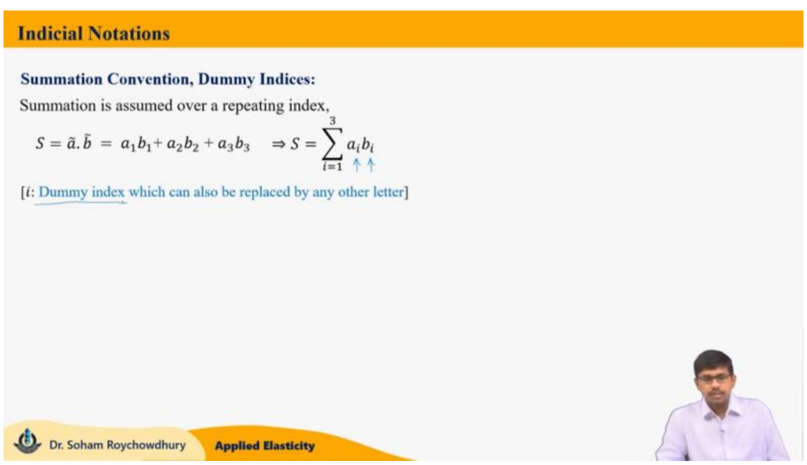


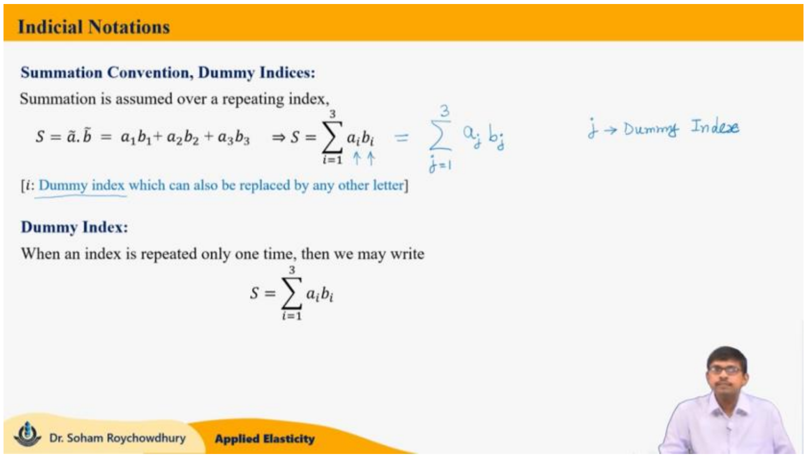


Now, it can be written as a $\sum_{i=1}^{3} a_i b_i$ where i varies from 1 to 3. So, with the help of this summation convention, instead of writing all those components individually with this plus sign, we can write it using a single term with the help of this summation sign. Now, here you can see i is a repeated index; it is appearing twice. So, summation is always assumed over repeated index, this is named as a dummy index. So, i is called a dummy index which is

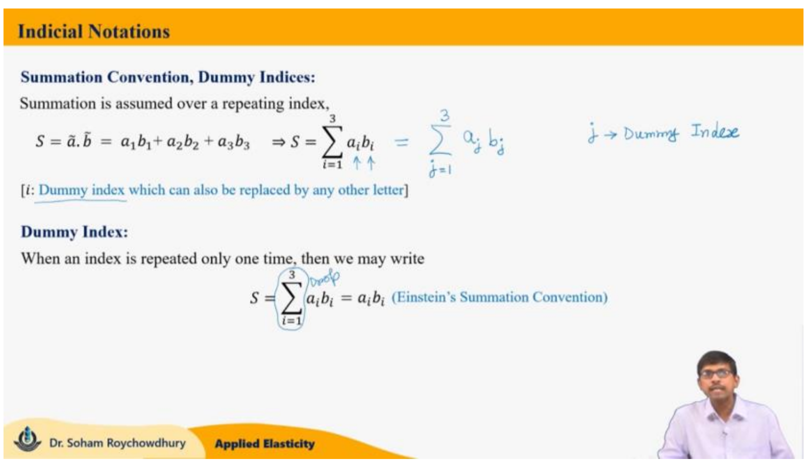


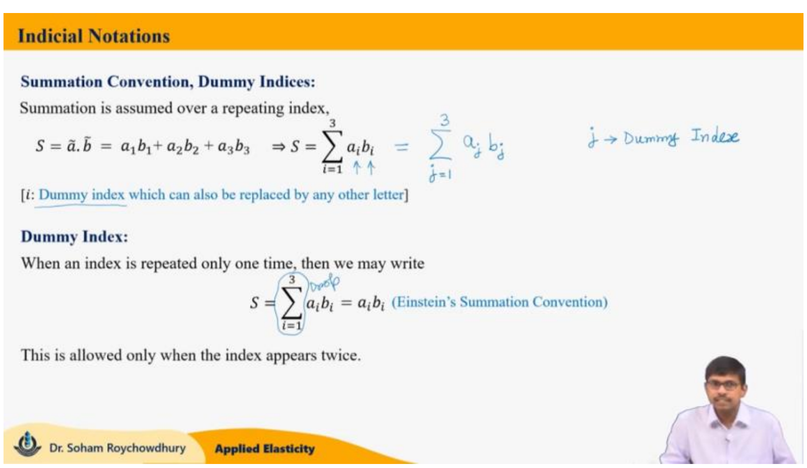
repeated ones appearing twice in the equation or repeated ones and this number this name i is just a dummy one instead of writing this in terms of i we can write in terms of some other index let us say j so we can write this as summation j from j equals to 1 to 3 $a_j b_j$ so in that case j is the dummy index summation is assumed over j So, this name is arbitrary; if it is repeated once, then we call that a dummy index. So, any index which is repeated once is called a dummy index, and using the summation convention $S = \sum_{i=1}^{3} a_i b_i$ can be just written as $a_i b_i$, we can drop this summation sign for the case of repeated dummy index So, this is called the Einstein summation convention.



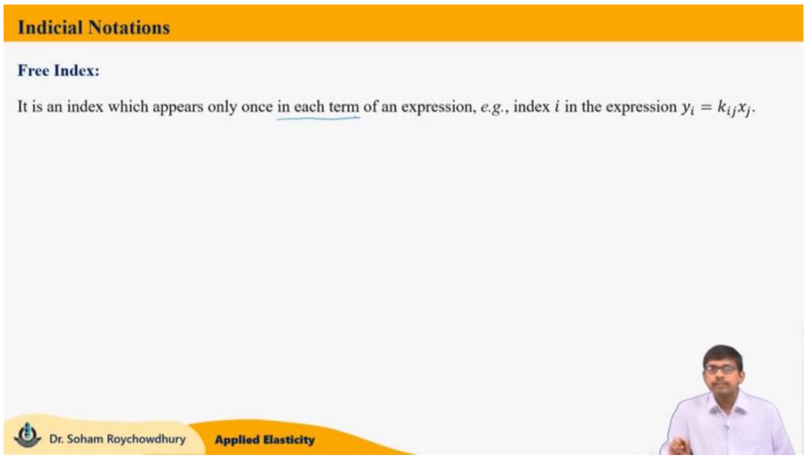


So, S equals a_ib_i basically refers to the summation of a_ib_i over the dummy index i and this index i can appear only twice. It can be repeated once, not more than that. In any tensorial expression, we cannot have an index appearing more than twice. So, that cannot define the summation convention.





Now apart from the dummy index we also have something called free index. So free index is defined as the index which is appearing only once in each and every term of the expression. So this is important. In all the terms if one index is appearing once for every term then we call that to be a free index. So if you consider this expression right hand side expression y_i equals to $k_{ij}x_j$here left hand side is having one term in which i is present right hand side is also having one term $k_{ij}x_j$ where i is present once whereas j is present twice in this particular term so j is repeated



appearing twice whereas i is appearing once in all the terms both on lhs left hand side as well as right hand side so i is a free index here j is a dummy index here so now we are going to consider few examples so let us say in this one a_i plus b_i equals to c_i so i is appearing once in all the terms Both the terms on left hand side and on the right hand side c_i terms so thus i is a free index. For the second example we are writing a vector $\tilde{a}$ as $a_i\tilde{e}_i$ vector, $\tilde{e}_i$ are the base vector or basis whereas a_i are the components. So here We can write any component a_i of this vector as a dot product of the vector and the corresponding base vector $\tilde{e}_i$. So, a_i equals to $\tilde{a}.\tilde{e}_i$. Here also you can see i is appearing once on every term.

Indicial Notations

Free Index:

It is an index which appears only once in each term of an expression, *e.g.*, index i in the expression $y_i = k_{ij}x_j$.

❖ $a_i + b_i = c_i$

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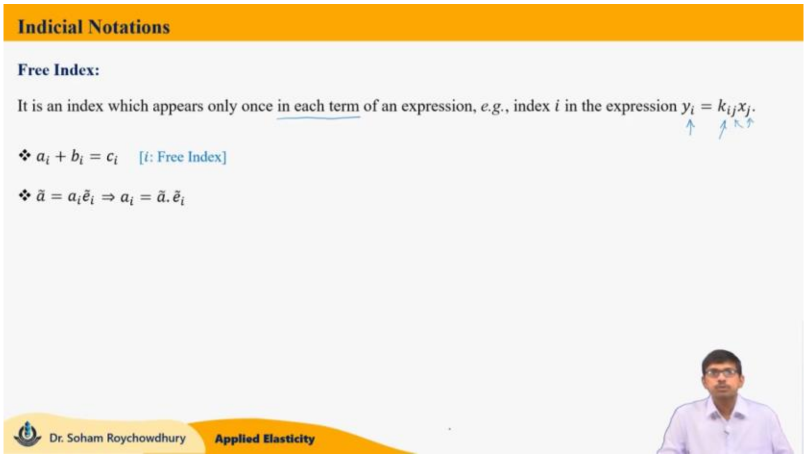

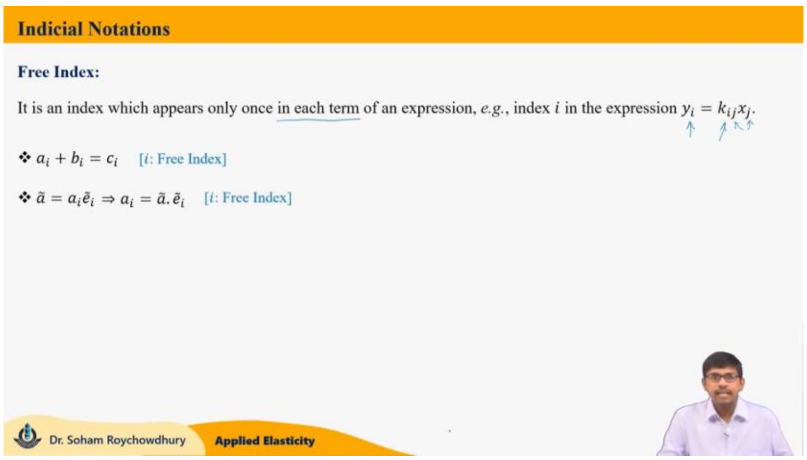

So, both left hand side and right hand side they are having single term and i is appearing once in each term. So, that's why i is a free index here. Now, in the next example u_i equals to $v_i k_j m_j = w_i$ where i is appearing once in every term and j is appearing twice in one particular term. So, as j is appearing twice or repeated that is dummy index whereas, i appearing once in every term makes i to be a free index. If you consider next example $T_{ij} = A_{im}B_{mj}$. So, here T_{ij} is a second order tensor.

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- $a_i + b_i = c_i$ [i: Free Index]
- $\tilde{a} = a_i\tilde{e}_i \Rightarrow a_i = \tilde{a}.\tilde{e}_i$ [i: Free Index]

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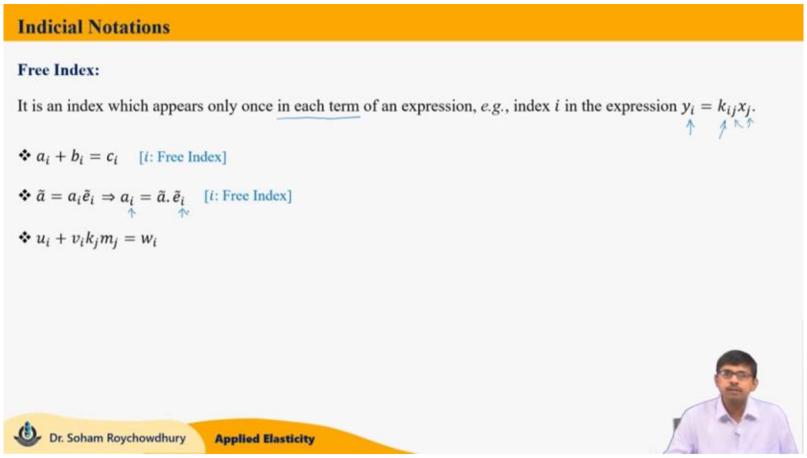
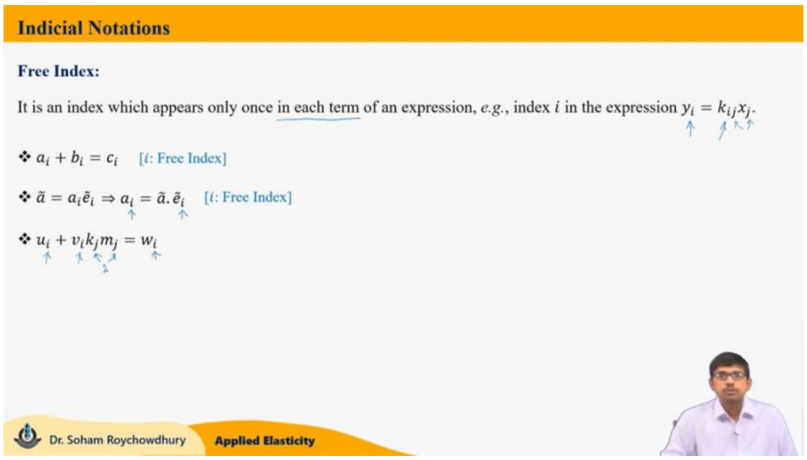
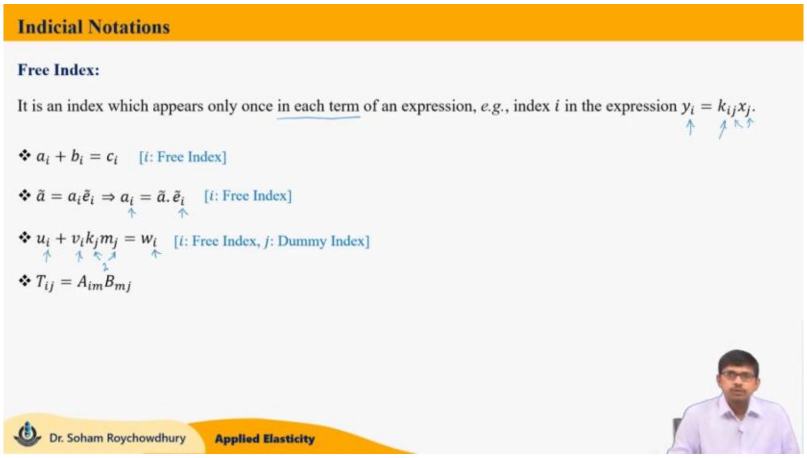
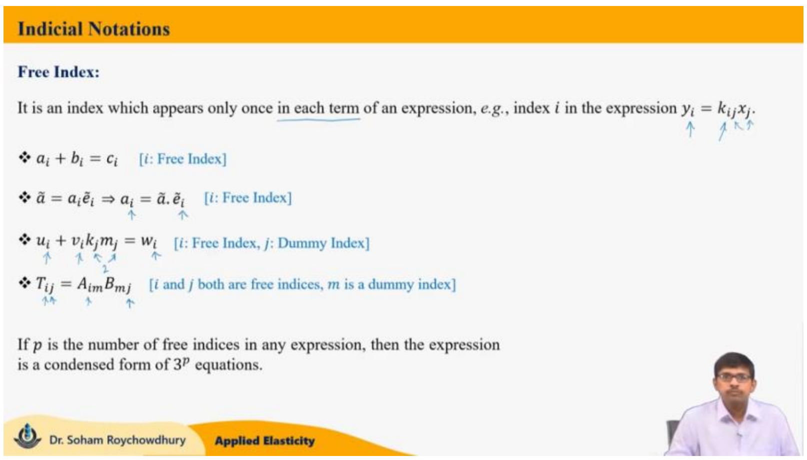


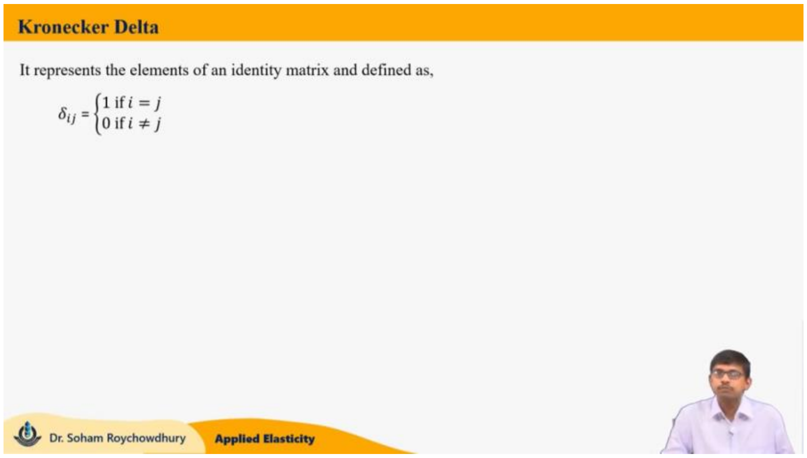


So, i is appearing once on the left-hand side and once on the right-hand side. Similarly, j is also appearing once on the left-hand side and once on the right-hand side term. So, both i and j are free indices here. However, m is a dummy index because m is repeated twice on only one term of the right-hand side, and it's not appearing on the left-hand side term. If we have p number of free indices, then that particular indicial expression is the condensed form of 3^p equations. For example, if you take the first one here, we have only one free index i, so $p = 1$. The number of free indices is 1, so this is a condensed form of 3 equations like

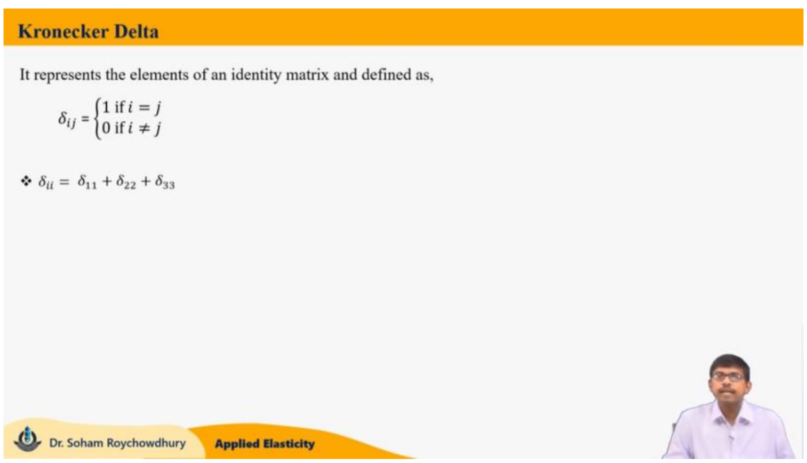


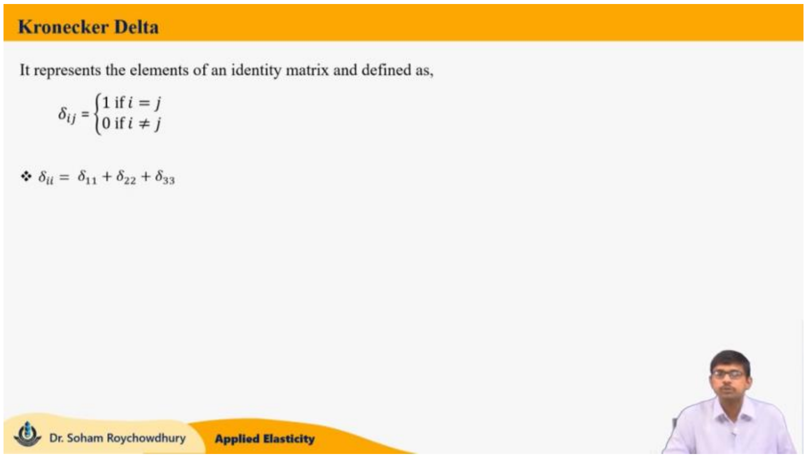


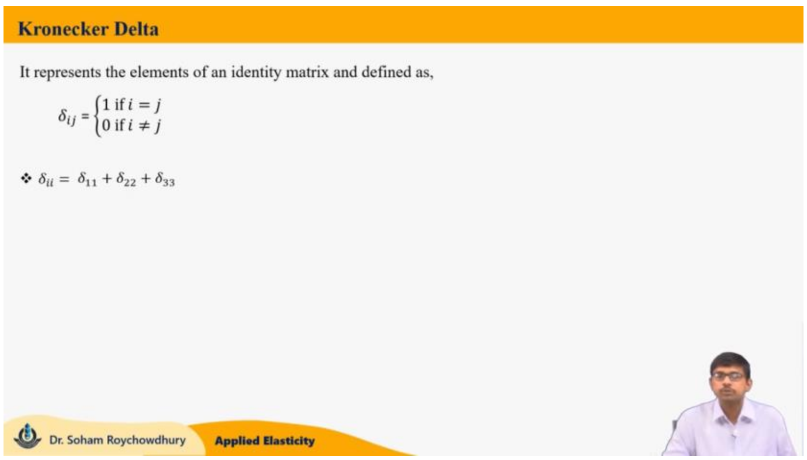
$a_1 + b_1 = c_1$, $a_2 + b_2 = c_2$, $a_3 + b_3 = c_3$. Whereas, for the last example here, we have i and j as 2 free indices, and thus this will be a condensed form of 9 equations. So, depending on the number of free indices present in the expression, we can expand and find out the number of total number of expressions which are represented by using this single indicial expression. Now, coming to the concept of Kronecker delta. This is basically the elements of an identity matrix, a matrix which has 0 at all non-diagonal terms and 1 at diagonal locations. So, $\delta_{ij} = 1$, if $i = j$, that is at the diagonal locations, and $\delta_{ij} = 0$ if $i \neq j$, that is at the non-diagonal locations.

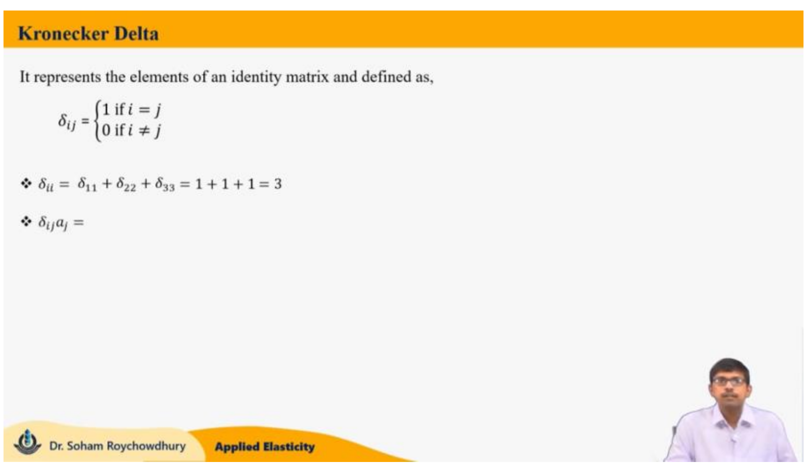


Now, if we are having both the indices to be same δ_{ii} this term. Now here i is a repeated (a dummy) index and summation is always assumed over the repeated index. δ_{ii} refers to the summation of δ_{ii} with i varying from 1 to 3, meaning $\delta_{11} + \delta_{22} + \delta_{33}$, and what will this be? This equals 3, as each of them All these individual terms equal 1, the 3 different diagonal terms. So, δ_{ii}, this quantity, equals 3. Now, if you multiply this Kronecker delta δ_{ij} with any vector component a_j, that will result in a_i, the ith component of the vector.









So, here what happens is whenever you are multiplying δ_{ij} with any vector then i is replaced with j in this vector indices. Same happens for the case of multiplication of Kronecker delta with a tensor quantity. δ_{ik} multiplied with T_{kj}. So, for this case k is replaced with i and this results a tensor with i, j subscripts. So, this i and k these two

Kronecker Delta

It represents the elements of an identity matrix and defined as,

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

- $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$
- $\delta_{ij} a_j = a_i$

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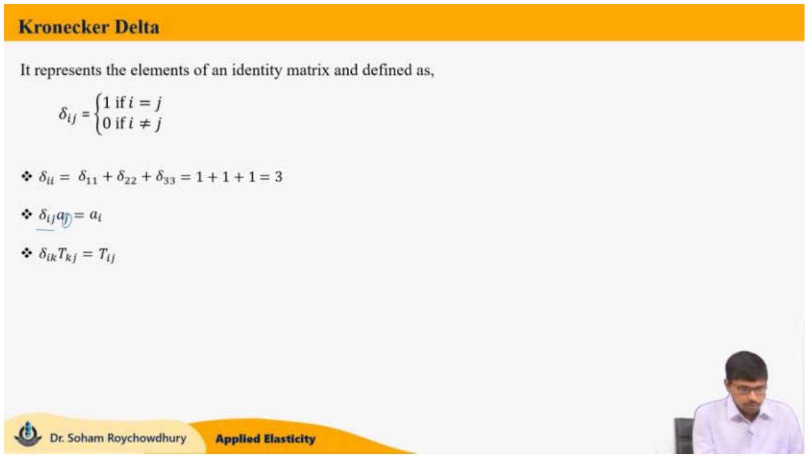
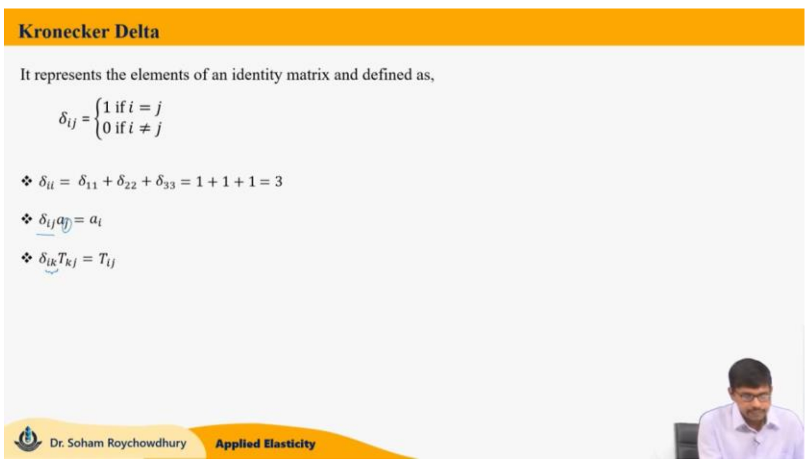
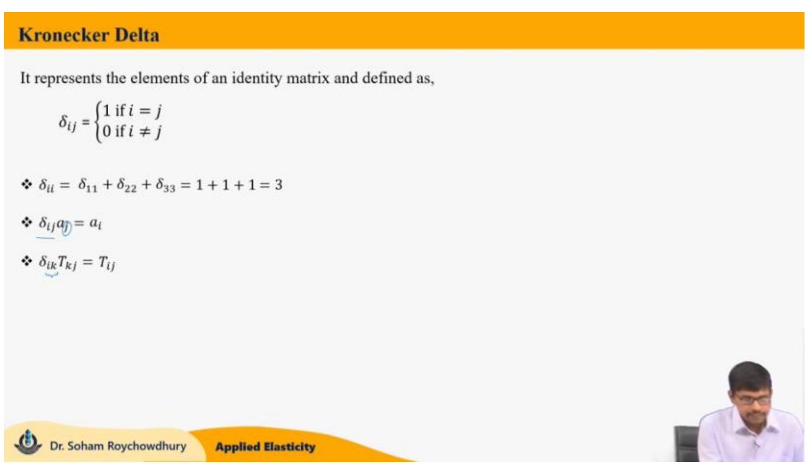


are same then only δ_{ij} will be unity 1 for rest of the cases this is 0. So, thus we are having T_{ij} here k is replaced with the help of i in the tensor component. Now, if you multiply different Kronecker delta with different indices $\delta_{ij}\delta_{jk}\delta_{kl}$. So, if you first multiply first two that will give you delta δ_{ik}. j is replaced with i now





Kronecker Delta

It represents the elements of an identity matrix and defined as,

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

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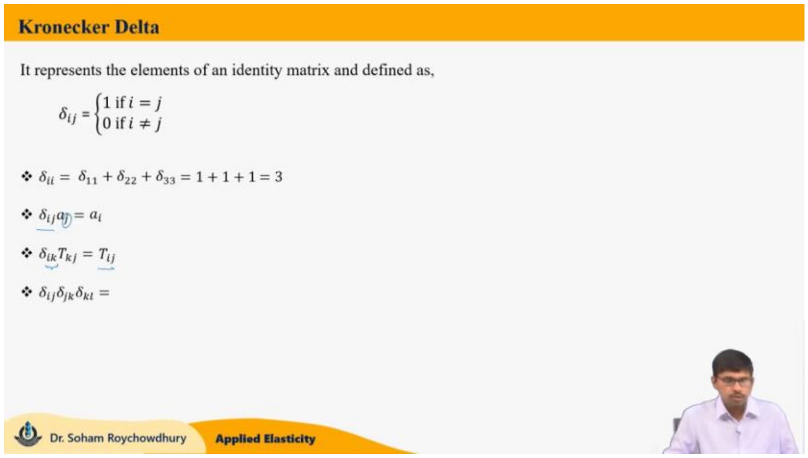
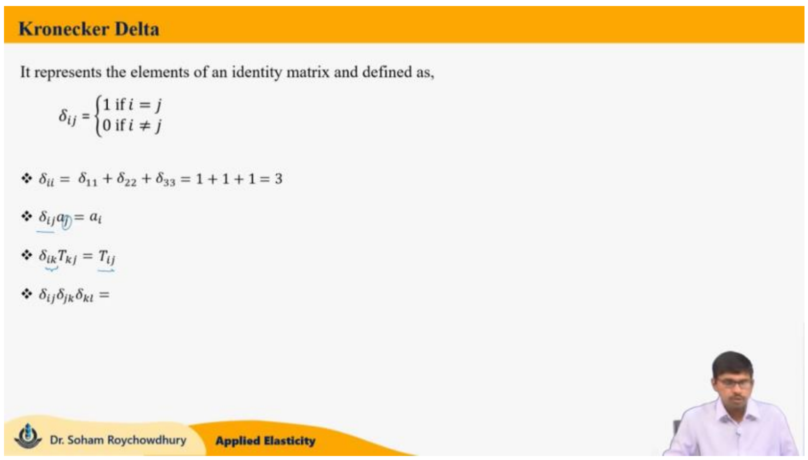
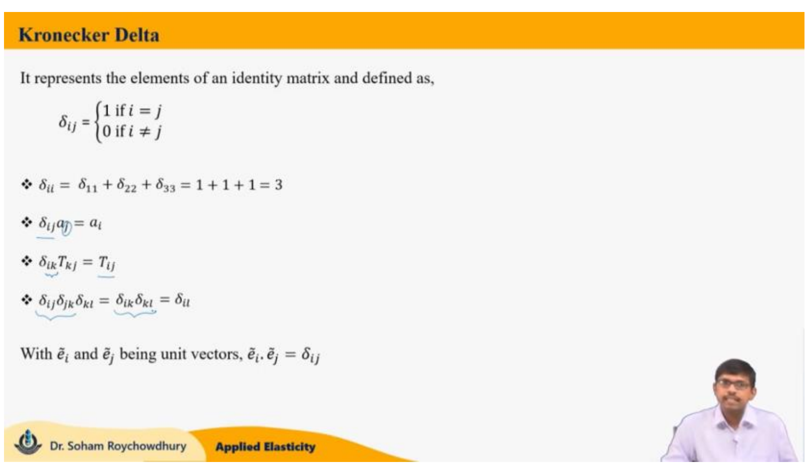




their dot product can also be written with the help of δ_{ij}. If $\tilde{e}_i$ is dotted with $\tilde{e}_j$ then only we will get a unity when $i = j$. If we are taking dot product of two different unit vectors $\tilde{e}_1$ and $\tilde{e}_2$ then that will be equals to zero. So, if i not equals to j this vector will be zero. Thus, dot product of any two base vectors or unit vectors can be written in terms of δ_{ij}.



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With $\tilde{e}_i$ and $\tilde{e}_j$ being unit vectors, $\tilde{e}_i . \tilde{e}_j = \delta_{ij}$

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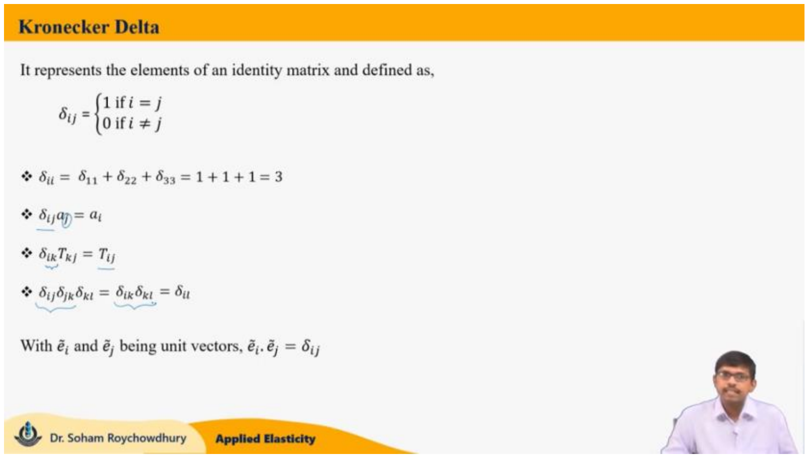
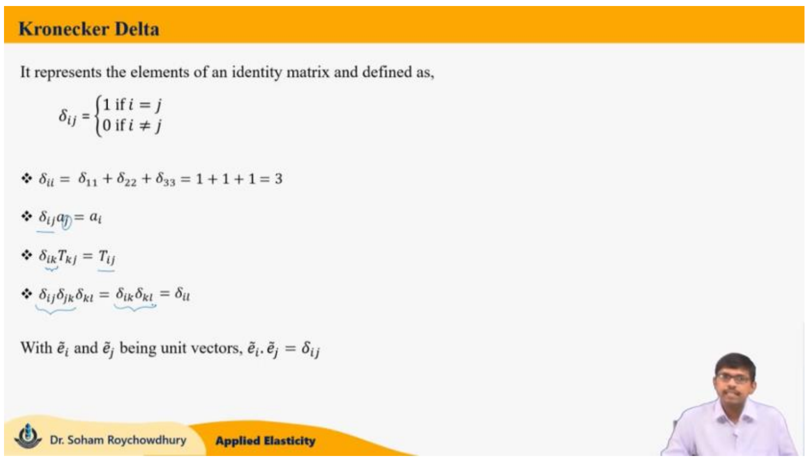
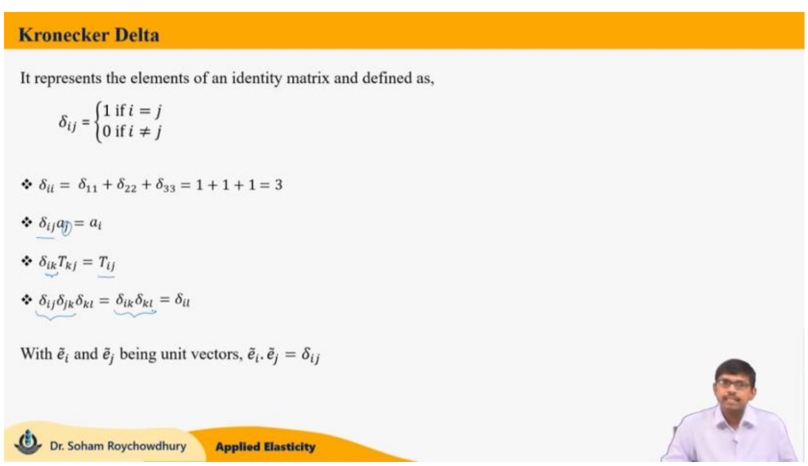
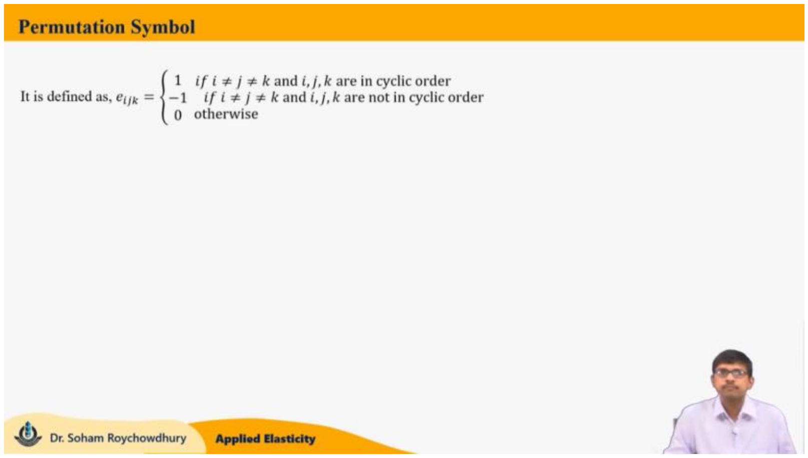
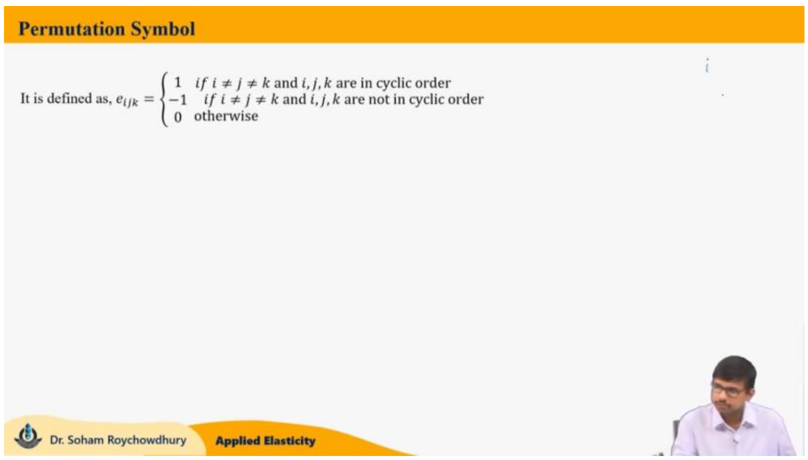


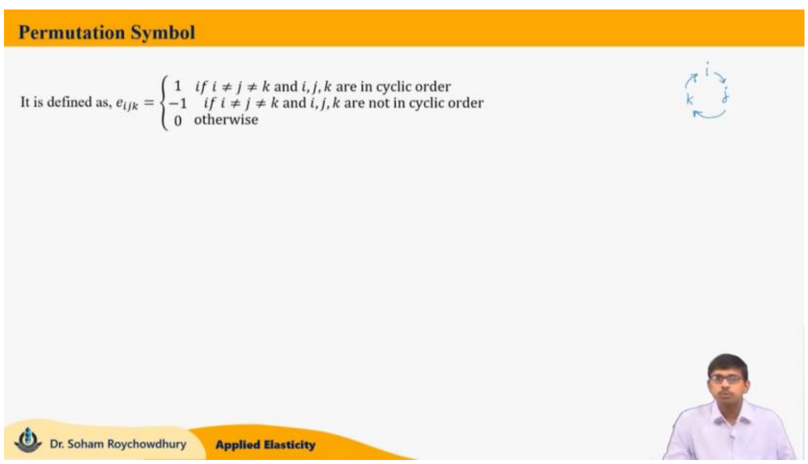


Now, coming to the definition of permutation symbol. Permutation symbol is defined with e_{ijk}having 3 indices, 3 subscripts, and this is equals to 1 when $i, j,$ and k are not equals to each other and they are in cyclic order. So, let's say we are considering this i, j, k to be in cyclic order, then it will be equals to 1. Permutation symbol equals to 1.

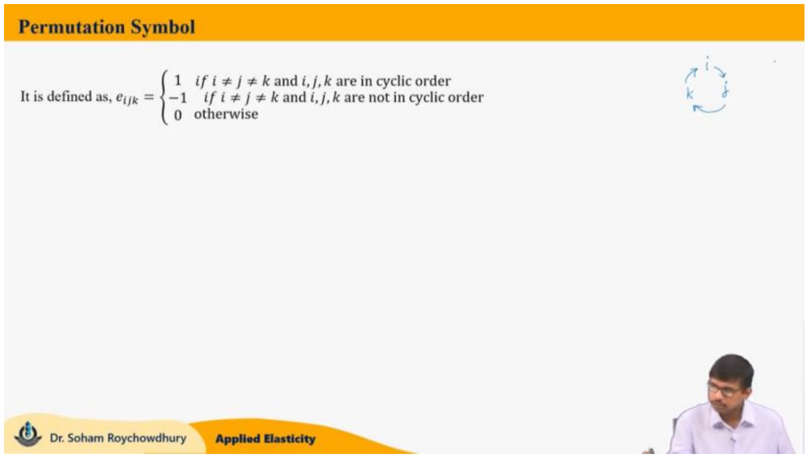




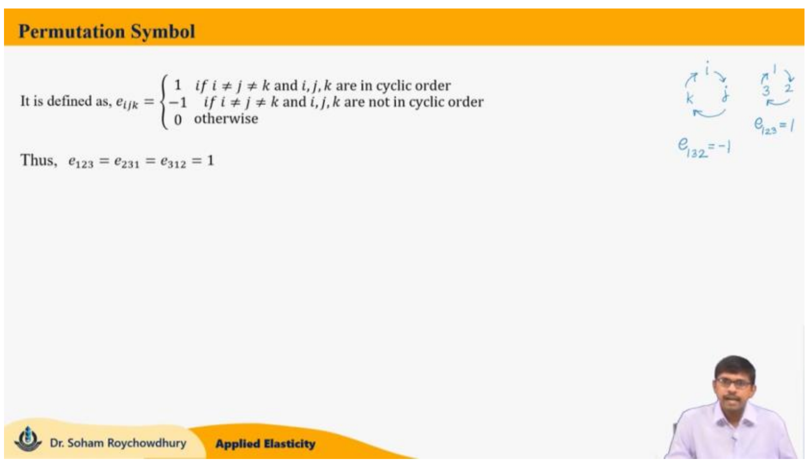




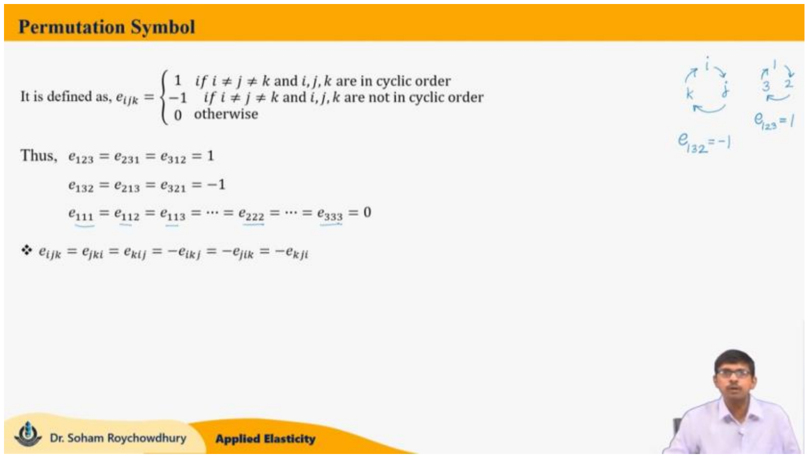
So, let's say i equals to 1 j equals to 2 k equals to 3, and these subscripts are appearing in this order 1, 2, and 3 then it will be equals to 1. If it is not appearing in the cyclic order, so first is 1 which is followed by 3 then followed by 2. This should be equals to minus 1.



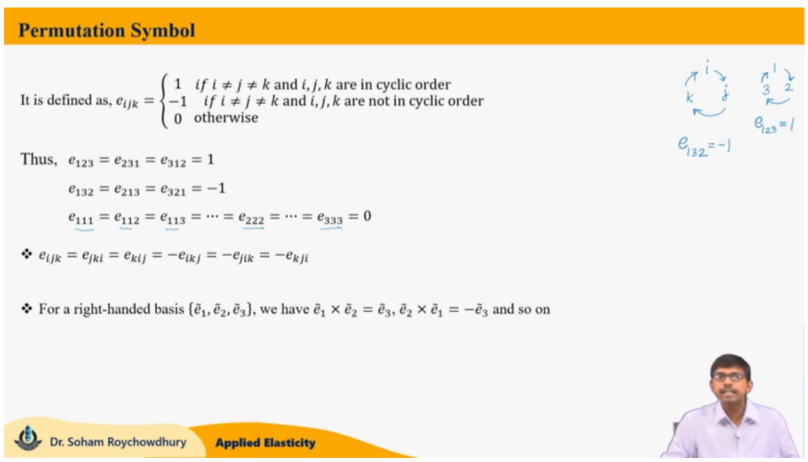
So, we are not following the cyclic order of 1 after that 2 after that 3. If this is not followed then permutation symbol is equals to minus 1 and 0 otherwise. So, if any 2 or 3 indices of the permutation symbol are same then that would be equals to 0. So, we can write this explicitly as $e_{123} = e_{231} = e_{312} = 1, e_{132} = e_{213} = e_{321} = -1$ and all rest are 0.

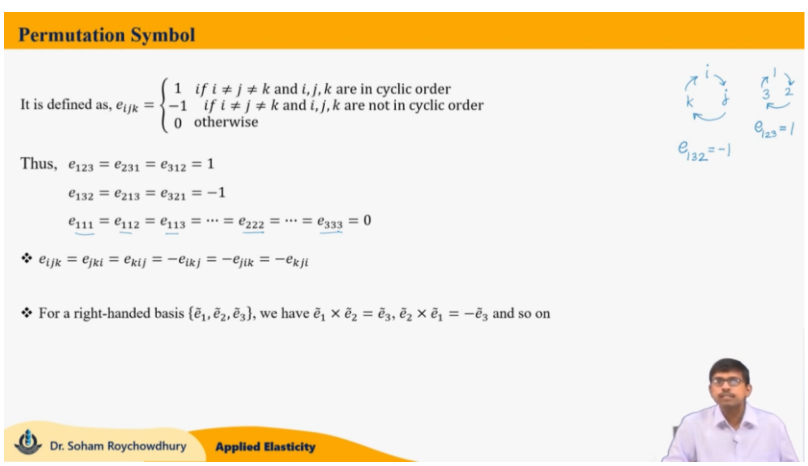


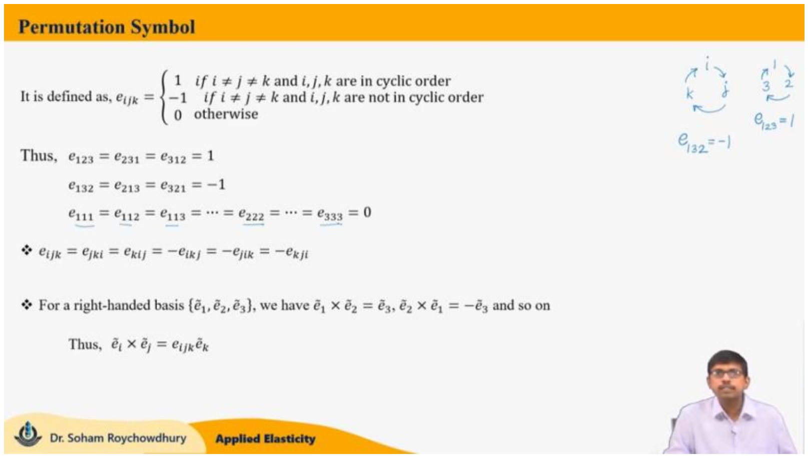
Whenever the index is repeated, you can see here 1 is repeated, here 1 is repeated, 1 is repeated, 2 is repeated, 3 is repeated. For all such repetitions of index, the permutation symbol will result in 0. So, ijk being in cyclic order, they are equal to 1; ijk being opposite to the cyclic order, all of them are equal to minus 1. Now, with the help of the permutation symbol, we are able to express the cross products.



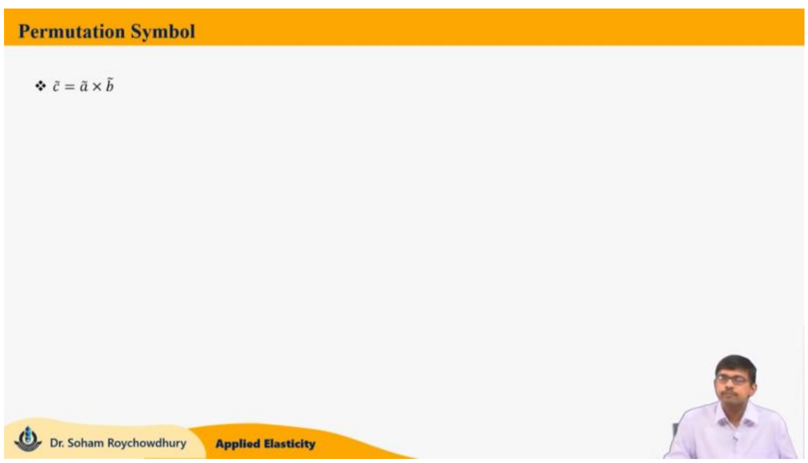
So, if you consider a right-hand basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$, these are the 3 unit vectors of a right-hand basis, then $\tilde{e}_1 \times \tilde{e}_2 = \tilde{e}_3$, $\tilde{e}_2 \times \tilde{e}_1 = -\tilde{e}_3$ and so on. These are known to you. Now, this can be combinedly written with the help of the permutation symbol as, The unit vector $\tilde{e}_i \times \tilde{e}_j$ is equal to the permutation symbol e_{ijk} into the unit vector $\tilde{e}_k$.

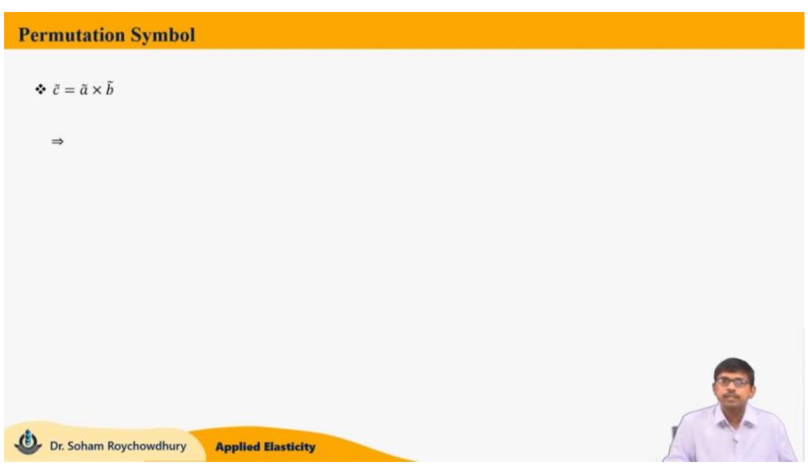




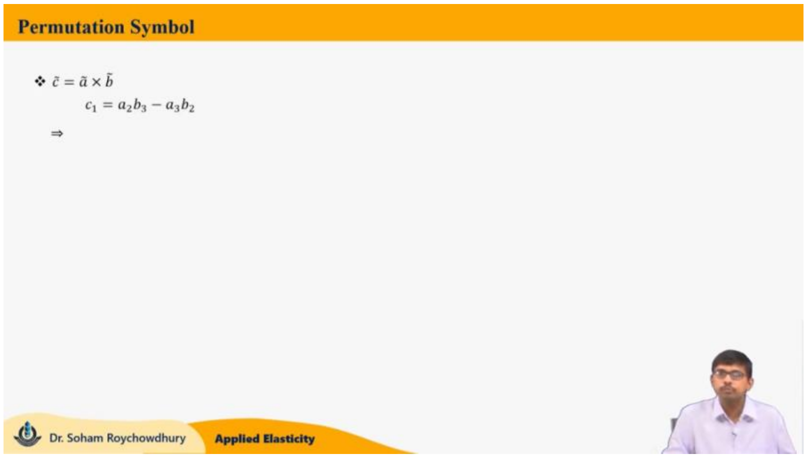


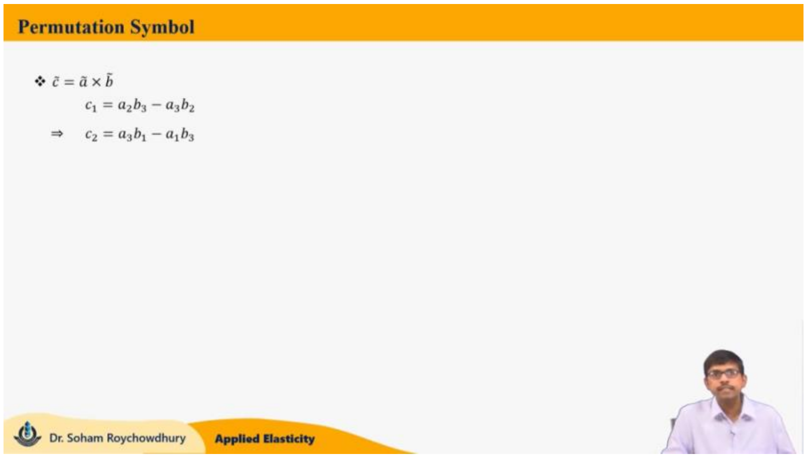
So, the cross product of any two unit vectors can be written with the help of the permutation symbol. Now, using this, we can express the cross product of any two arbitrary vectors. Let us say $\tilde{a}$ and $\tilde{b}$ are two vectors, and $\tilde{c}$ is the cross product of those two vectors. So, $\tilde{c}$ equals $\tilde{a} \times \tilde{b}$. This equation, if you expand and write in terms of their components.

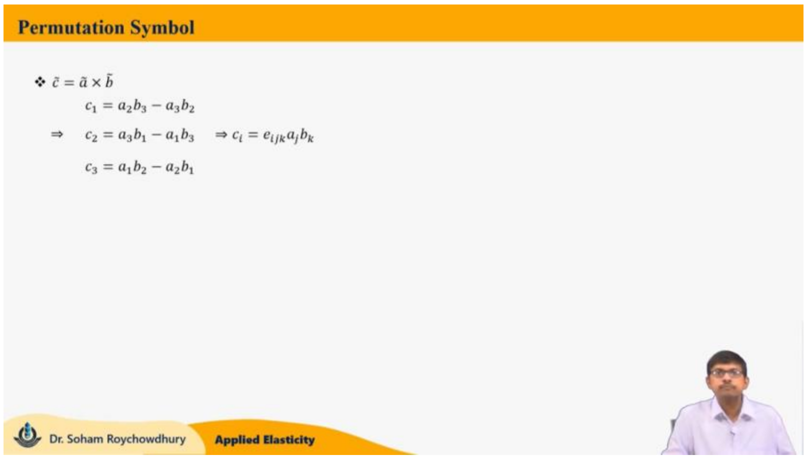




So, c_1 is $a_2b_3 - a_3b_2$, c_2 is $a_3b_1 - a_1b_3$, c_3 is $a_1b_2 - a_2b_1$. Now, if we try to condense all three of these equations into a single indicial notation (or indicial expression), that will become $c_i = e_{ijk}a_jb_k$. So, the $\tilde{c}$ vector can be written as $c_i\tilde{e}_i$.







Permutation Symbol

- $\tilde{c} = \tilde{a} \times \tilde{b}$

$$\Rightarrow \begin{aligned} c_1 &= a_2 b_3 - a_3 b_2 \\ c_2 &= a_3 b_1 - a_1 b_3 \\ c_3 &= a_1 b_2 - a_2 b_1 \end{aligned} \quad \Rightarrow c_i = e_{ijk} a_j b_k$$

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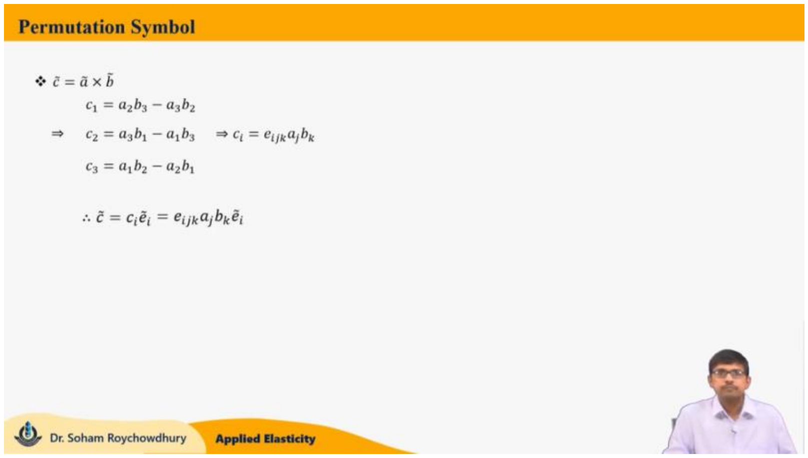
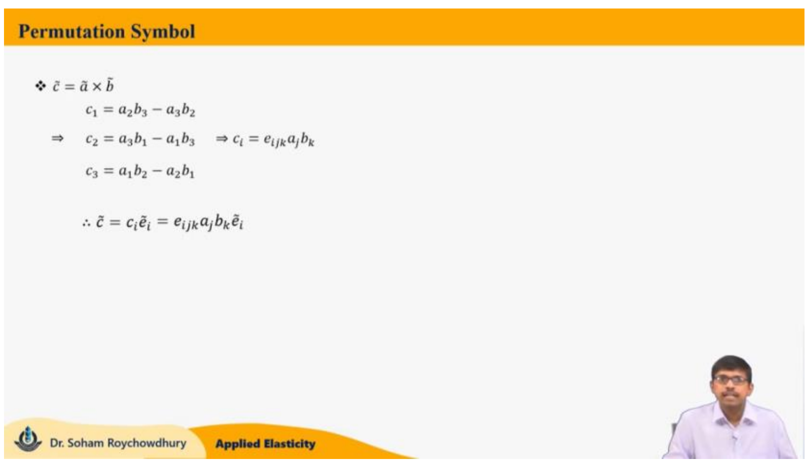


That is the unit vector along the i-direction as $e_{ijk}\ a_jb_k$into $\tilde{e}_i$. So, for this equation here, if we take i equals 1, for that case c_i will be $e_{1jk}\ a_jb_k$. Now, j and k can take two values, 2 and 3; in the rest of the cases, the permutation symbol will be 0 because the indices are the same. So, this is basically $e_{123}a_2b_3 + e_{132}a_3b_2$, and we know that the first quantity e_{123} is 1, whereas e_{132} is -1. So, c_1 can be written in terms of $a_2b_3 - a_3b_2$, which is the first equation.



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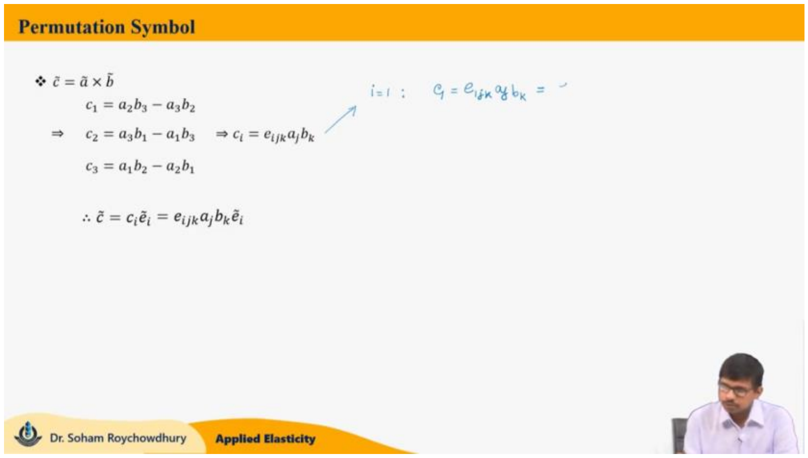
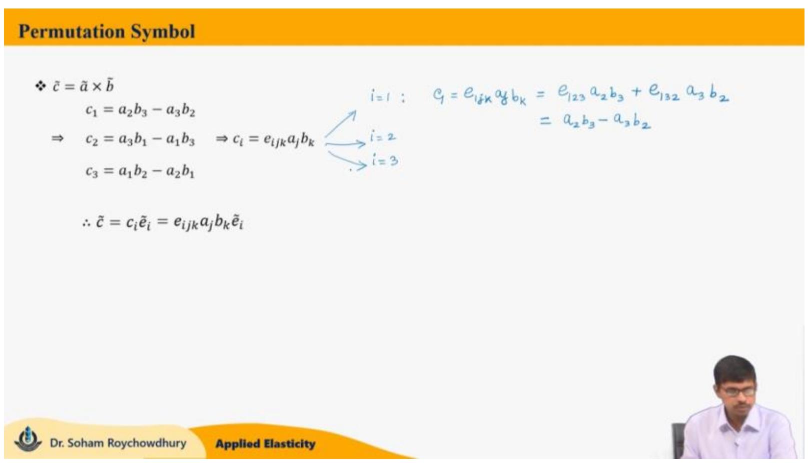
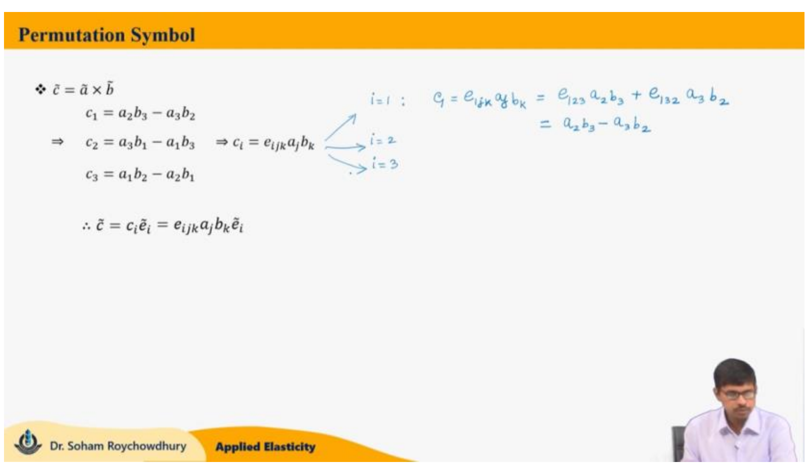
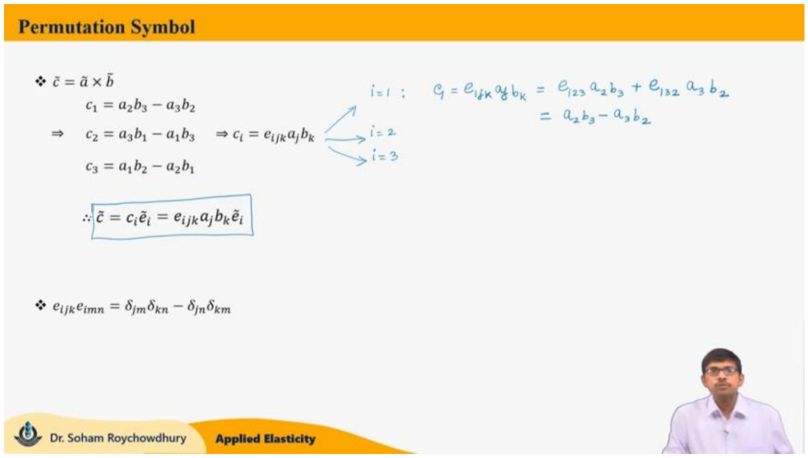


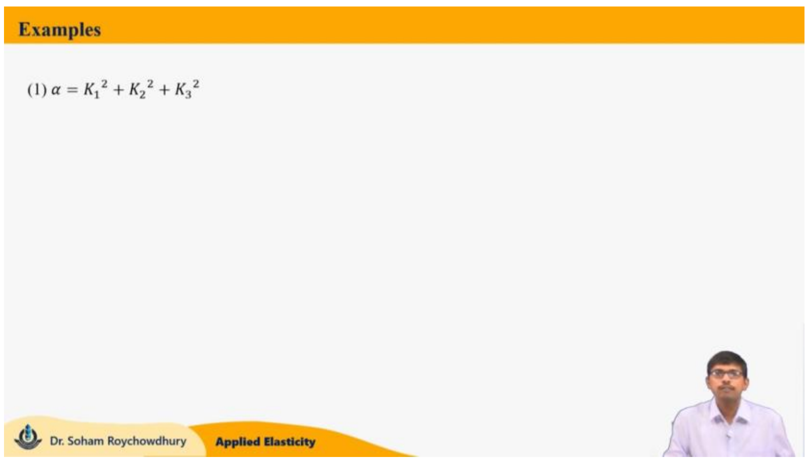
In the same fashion, using i equals 2 and 3, the next two equations can also be expanded. So, thus the $\tilde{c}$ vector, which is the cross product of two vectors $\tilde{a}$ and $\tilde{b}$, can be written as $e_{ijk} a_j b_k$into $\tilde{e}_i$, which is the unit vector along the i-direction. And it can be shown or proved that this permutation symbol e and the Kronecker delta, they can be related with the help of this particular expression: e_{ijk} times e_{imn} equals $\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$.





So, the first two subscripts for both the permutation symbol on the left-hand side are the same for such cases. The product of two permutation symbols can be expressed with the help of these two Kronecker deltas. So, this particular expression is called the e-δ identity. Now, let us consider a few example problems. α is given as ${K_1}^2 + {K_2}^2 + {K_3}^2$.





So, here in each term, we have K_i multiplied by K_i. So, this can be written as K_i times K_i, where summation is assumed over this repeated index i, with i varying from 1, 2, and 3. If you have the second example as $\frac{\partial^2\phi}{\partial {x_1}^2}+\frac{\partial^2\phi}{\partial {x_2}^2}+\frac{\partial^2\phi}{\partial {x_3}^2}$, this can be written as $\phi_{,ii}$. So, any quantity comma i refers to $\frac{\partial}{\partial x_i}$, the derivative with respect to x_i once.

Examples

(1) $\alpha = {K_1}^2 + {K_2}^2 + {K_3}^2$

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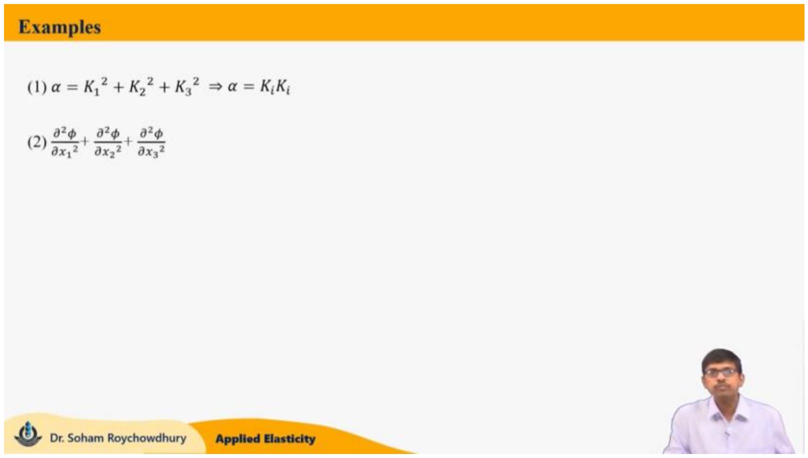
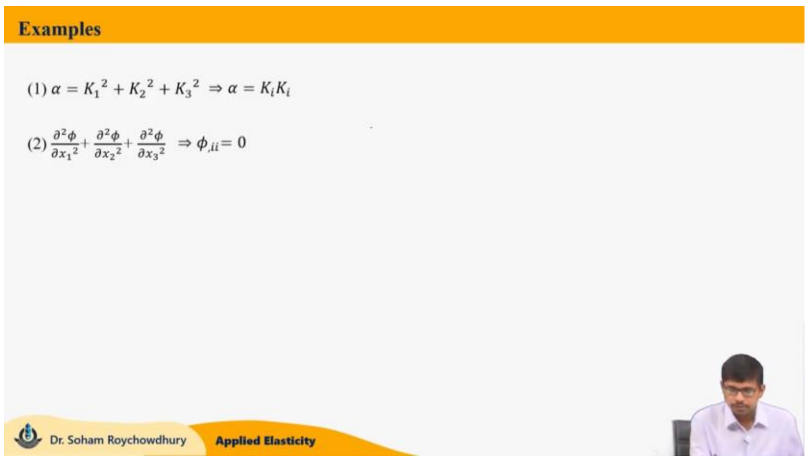
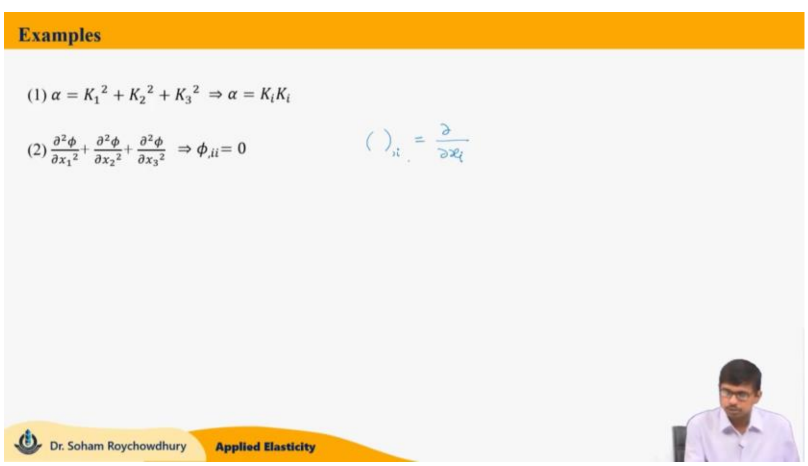




So, comma double i refers to the derivative with respect to x_i twice. So, here such examples, such expressions can be written in terms of $\phi_{,ii} = 0$. Now, coming to the third one, a scalar α is considered, which is $\tilde{a}.(\tilde{b} \times \tilde{c})$. So, for the dot products, if we write that with the help of indicial notation, this is $a_i(\tilde{b} \times \tilde{c})_i$.



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(3) $\alpha = \bar{a}.(\bar{b} \times \bar{c})$

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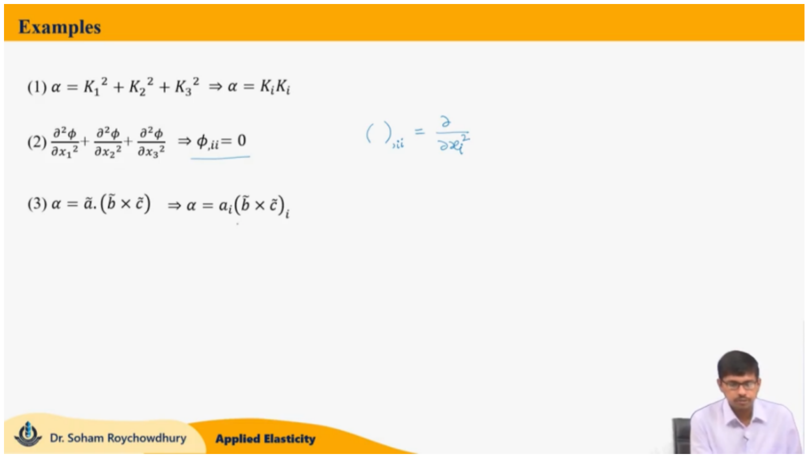
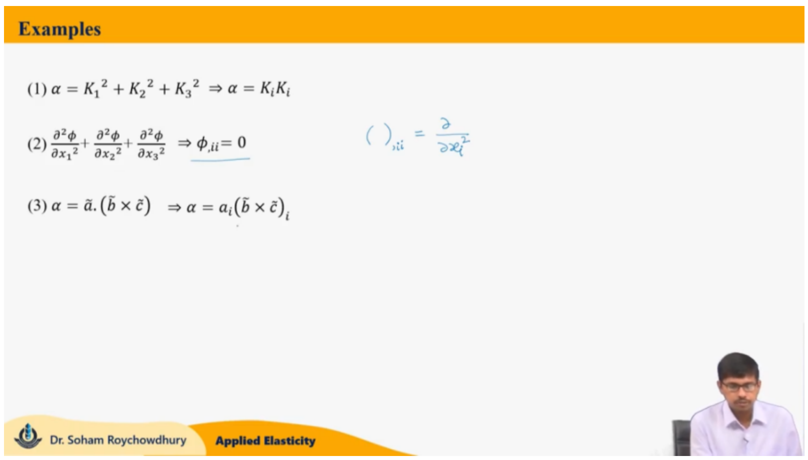
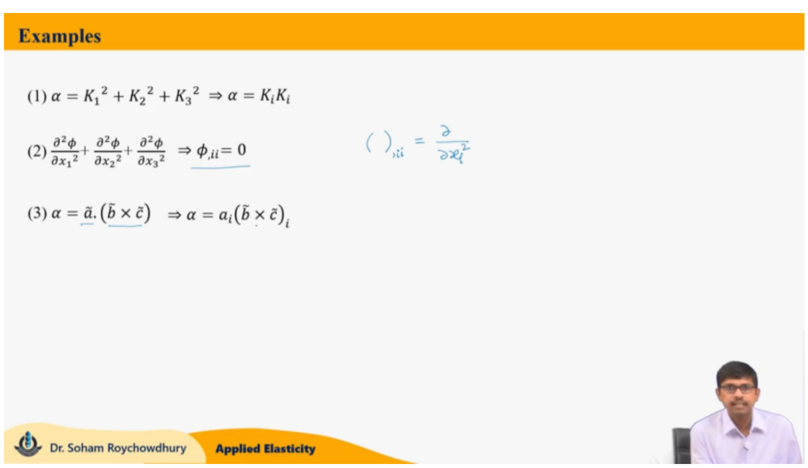




So, $\tilde{a}$ was first vector $\tilde{b} \times \tilde{c}$ is second vector we are attaching i to both the vector a_i $(\tilde{b} \times \tilde{c})_i$ and then $(\tilde{b} \times \tilde{c})_i$can be simplified with the help of permutation symbol. And thus total α will become $a_i e_{ijk} b_j c_k$. Similarly, if you consider the next example α equals to $(\tilde{a} \times \tilde{b}).(\tilde{c} \times \tilde{d})$. So, this is dot product of two vectors, this is first vector $\tilde{a} \times \tilde{b}$, this is second vector.



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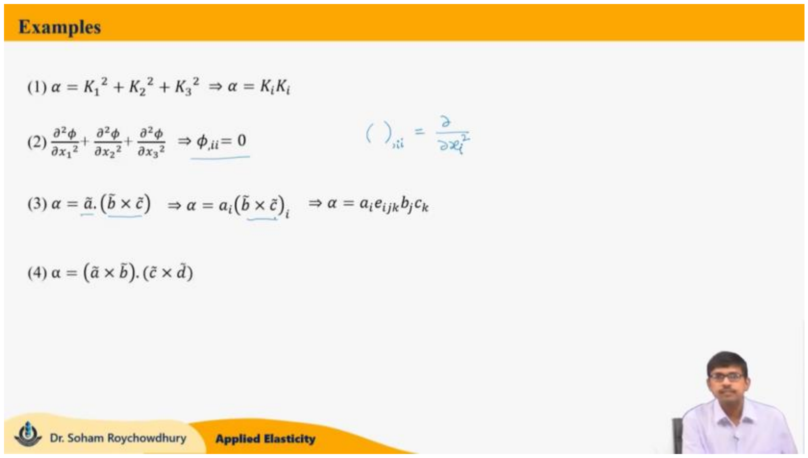

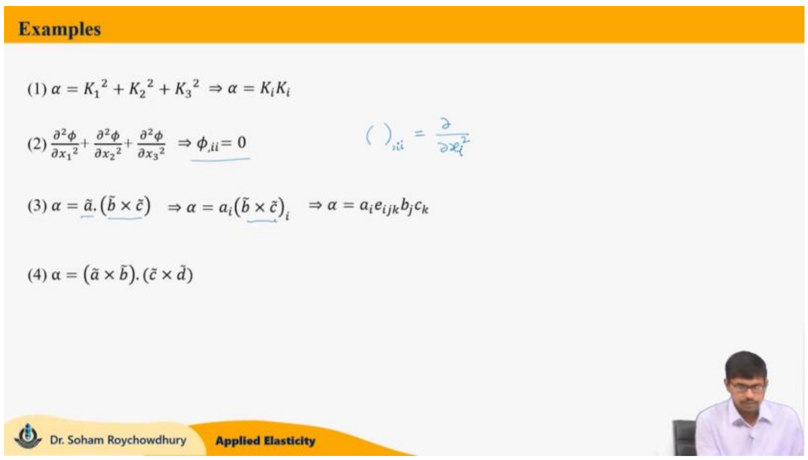

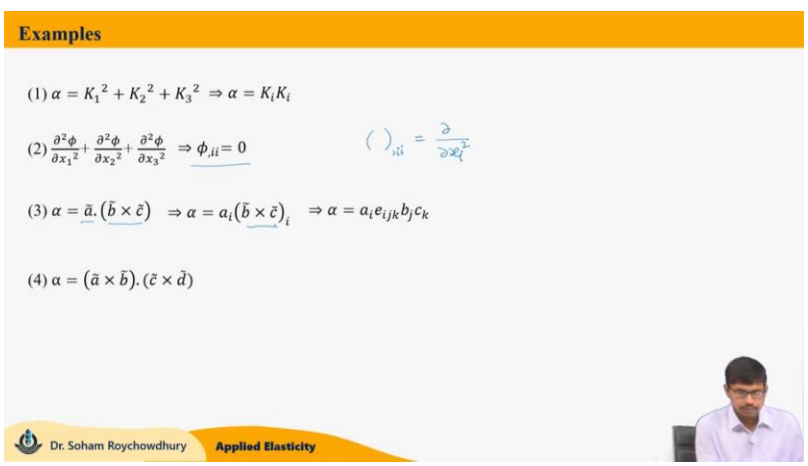

So, any dot product can be written as i component of first vector into i component of second vector. $(\tilde{a} \times \tilde{b})_i$ $(\tilde{c} \times \tilde{d})_i$ and both of them can be expanded with the help of two permutation symbols. So first one $\tilde{a} \times \tilde{b}$ is written as $e_{ijk} a_j b_k$ whereas second one $(\tilde{c} \times \tilde{d})_i$ is once again written as $e_{imn} c_m d_n$. Note that while expanding the first term, we had already used j, and k these two dummy indices.

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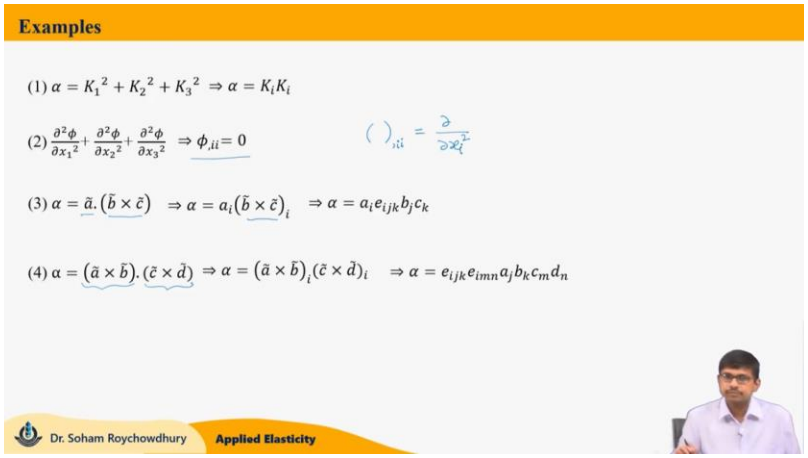
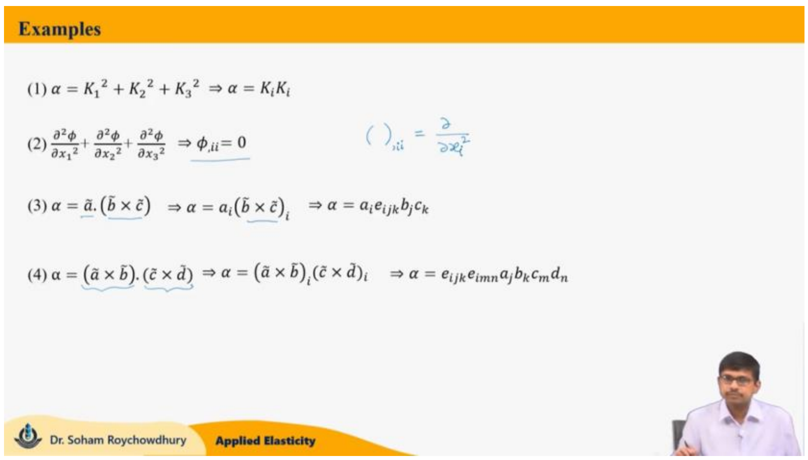
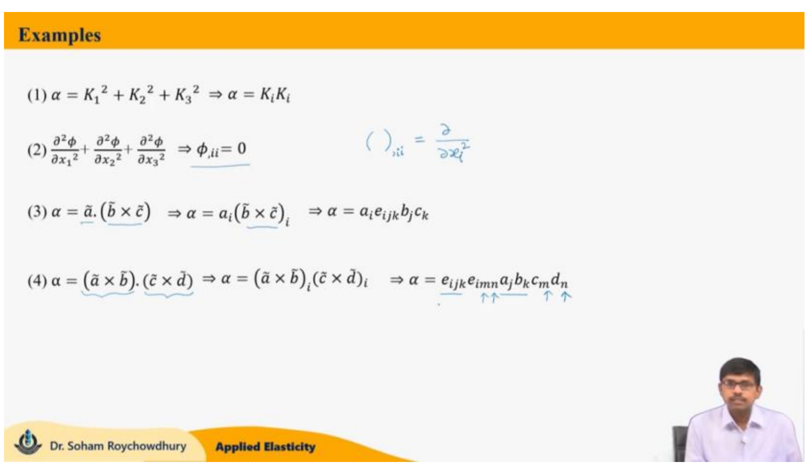
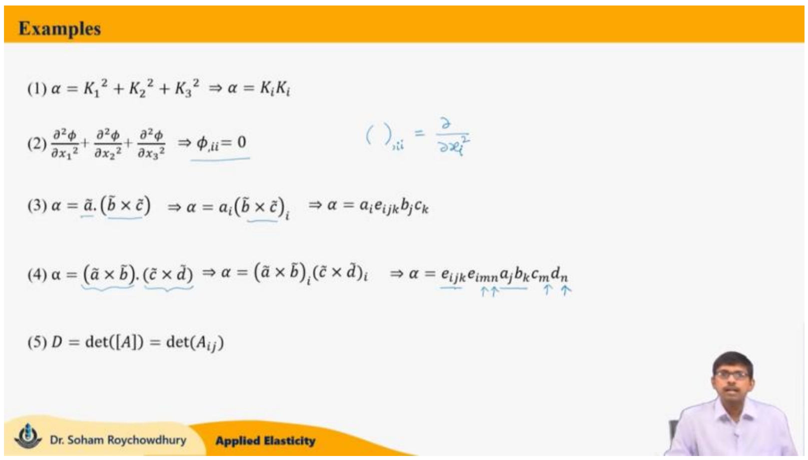


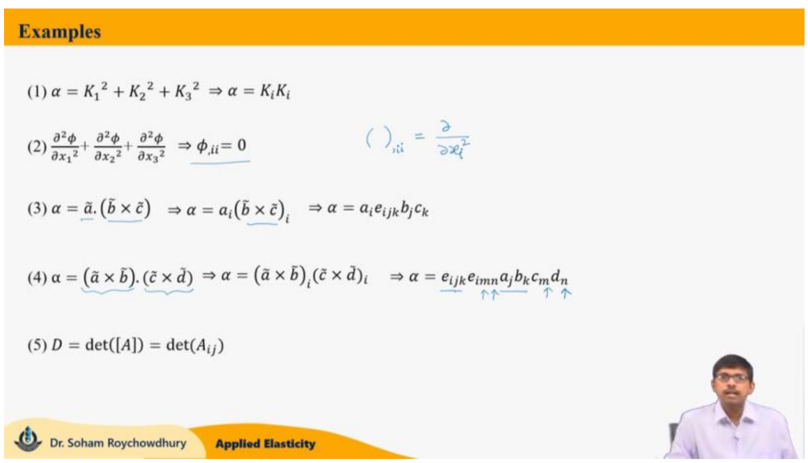


Thus j and k cannot be repeated for expanding the second cross product. So here two new indices m and n are used. first one is e_{ijk}, second one is also e_{ijk} that is not allowed. The name of dummy indices for the second term must be different than the dummy indices coming due to the first term. So, you can use any letter, but not the same which was already used.



Now, if you consider the determinant of any matrix A, the determinant of A_{ij}, where i, j are the matrix components. This can also be represented if you expand the determinant. Then, this can also be proved to be represented with the help of an expression. D equals to $e_{ijk}A_{1i}A_{2j}A_{3k}$, so with the help of the Kronecker delta and permutation symbol, it is possible to condense many of these expressions into this single indicial form with the help of summation conventions.





So, in this particular lecture, we have discussed the fundamental concepts of scalars, vectors, and tensors. We have also discussed the summation convention or indicial notation, and introduced the concepts of the Kronecker delta and permutation symbol. Thank you.