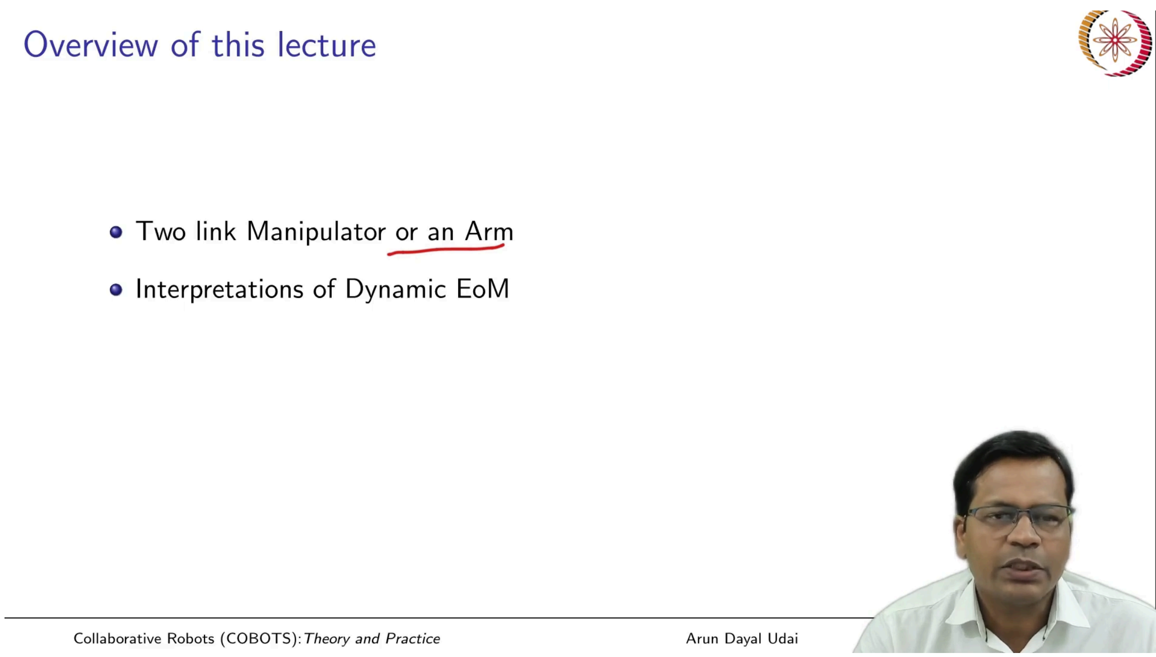
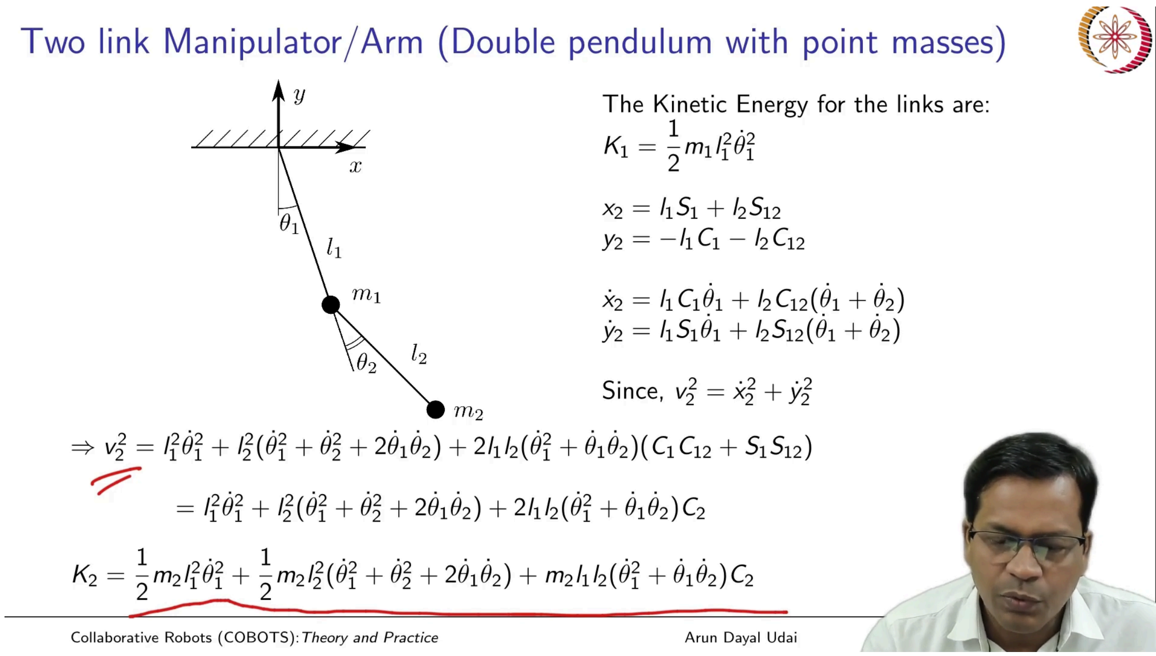
NPTEL Online Certification Courses
COLLABORATIVE ROBOTS (COBOTS): THEORY AND PRACTICE
Dr Arun Dayal Udai
Department of Mechanical Engineering
Indian Institute of Technology (ISM) Dhanbad
Week: 06
Lecture: 25

Dynamic Equation of Motion of a Two-Link Manipulator Using Lagrange-Euler (LE) Approach



Welcome to the second lecture on the Dynamic Equation of Motion of a Two-Link Manipulator using the Lagrange-Euler (LE) Approach. So, in this lecture, I will continue with the Two-Link Manipulator or an Arm using the Lagrange-Euler approach. We will obtain the dynamic equation of motion for this. We will interpret the dynamic equation of motion and various terms which are there at the end.



So, let us continue with the two-link manipulator. It is defined here as a massless link; that is, link 1 ends with a bob of mass m1, and it is attached with a joint here. There is a second link of mass zero, ending with the mass m2, which is a bob here—a concentrated mass m2. So, both m1 and m2 are concentrated masses, whereas the links are of zero mass. Theta 1 and theta 2 are the relative joint angles. So, this is theta 1, and the relative joint angle theta 2 is here. It is suspended from the top, where you have the reference frame: x is directed along the positive x-axis, and y is like this. This is your reference frame zero. So, this is how it is defined.

So, now let us continue using the Lagrange-Euler approach. The first thing that we will do is try to find out the kinetic energy and potential energy for both. That is, for joint 1that is, for mass 1, and similarly for mass 2, we will obtain that.

So, kinetic energy 1 as we have done earlier for the one link manipulator. So, we will do it exactly in the same way. We can directly obtain using 1 by 2 m1 x dot square and y1 dot square. So, that should give me 1 by 2 m1 v square or, alternately, 1 by 2 m1. L1 square that is I and similarly theta dot square that is omega square theta 1 dot square. So, this should be there. So, using that, I obtained my kinetic energy for Bob 1.

$$k_1 = \frac{1}{2} m_1 l_1^2 \dot{\Theta}_1^2$$

Now, moving to the kinetic energy for the Bob 2. This is for m2. So, again, I have obtained x 2 and y 2 first as we did it earlier, so this is my x 2 x 2 is equal to this plus this got it, so it is equal to l 1 sine theta 1 plus l 2 sine theta 1 plus theta 2, and similarly, You can obtain l1 cos theta 1 and l2 cos theta 1 plus theta 2. right? Summing them together, you can directly get the displacements along y and along x. Mind it, this displacement along y is considered negative because it is below this point; that is the reference point, okay? So, these are x and y, okay? That can directly be obtained using projections as we have done.

$$x_2 = l_1 S_1 + l_2 S_{12}$$
$$y_2 = -l_1 C_1 - l_2 C_{12}$$

So, taking the derivative of this, I get the velocity along x, x2 dot and y2 dot. Got it? So, I have obtained this.

$$\dot{x}_2 = l_1 C_1 \dot{\theta}_1 + l_2 C_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$
$$\dot{y}_2 = l_1 S_1 \dot{\theta}_1 + l_2 S_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$

And squaring and summing would give me the resultant velocity v2 due to the x2 dot and y2 dot, and that is squaring and adding. I can directly get this.

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2$$

So, v2 becomes equal to this,

$$\Rightarrow v_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) + 2l_1 l_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)(C_1 C_{12} + S_1 S_{12})$$

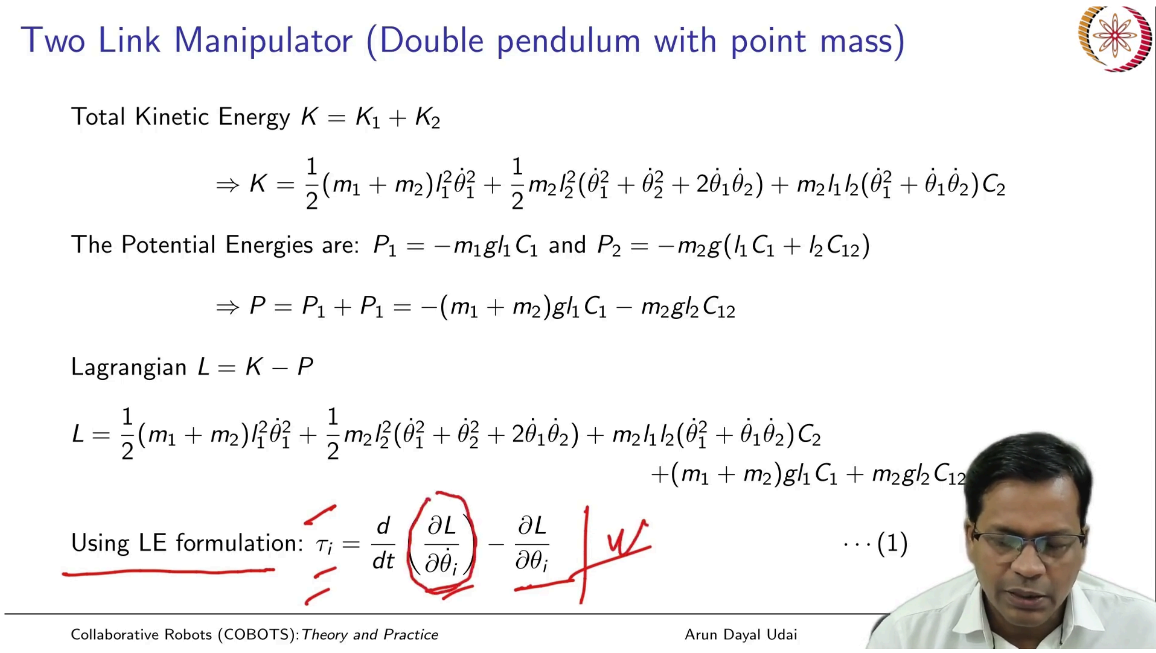
I have just eliminated the terms sin square theta plus cos square theta is equal to 1 from this. This can be reduced to this,

$$= l_1^2 \dot{\theta}_1^2 + l_2^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) + 2l_1 l_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) C_2$$

I have also used cos A cos B plus sin A sin B. As you know is a trigonometrical identity. Using that, you can reduce it to this. So, these are a few identities that are used here to reduce it, and I got the v2, which is the velocity for the bob 2. For kinetic energy, this is equal to 1 by 2 mv2 square. So, you can directly use this term here: 1 by 2 m2 v2 square. That gives me this kinetic energy. Got it?

$$K_2 = \frac{1}{2}m_2 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) + m_2 l_1 l_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)C_2$$

So, the kinetic energy for the first bob and the second bob are both obtained.



Now, I will sum them together to get the total kinetic energy of the system.

$$k = k_1 + k_2$$

$$\Rightarrow K = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) + m_2 l_1 l_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)C_2$$

You know, for the Lagrange-Euler, you need the total kinetic energy. Now, the potential energy is very straightforward. It is mg into l1c1. Let us go back to this, and you can just see here it is l1 c1. So, this is the mass, and this is the acceleration due to gravity. Got it? So, it is directly m1gl1c1. This is your potential energy for the first bob, and similarly for the second bob. Both of them I have added here, and I got the total potential energy. Got it? For the second one, you have used what? You have used the total distance here, which is l2c12. So, it is l1c1 plus l2c12m2 into minus g. So, all those will come here, and finally, it will make it to this. So, this is the potential energy for the second bob, and both summed together, I got this. So, this is the total potential energy, P1 plus P2, which is here.

$$P = P_1 + P_1 = -(m_1 + m_2)gl_1C_1 - m_2gl_2C_{12}$$

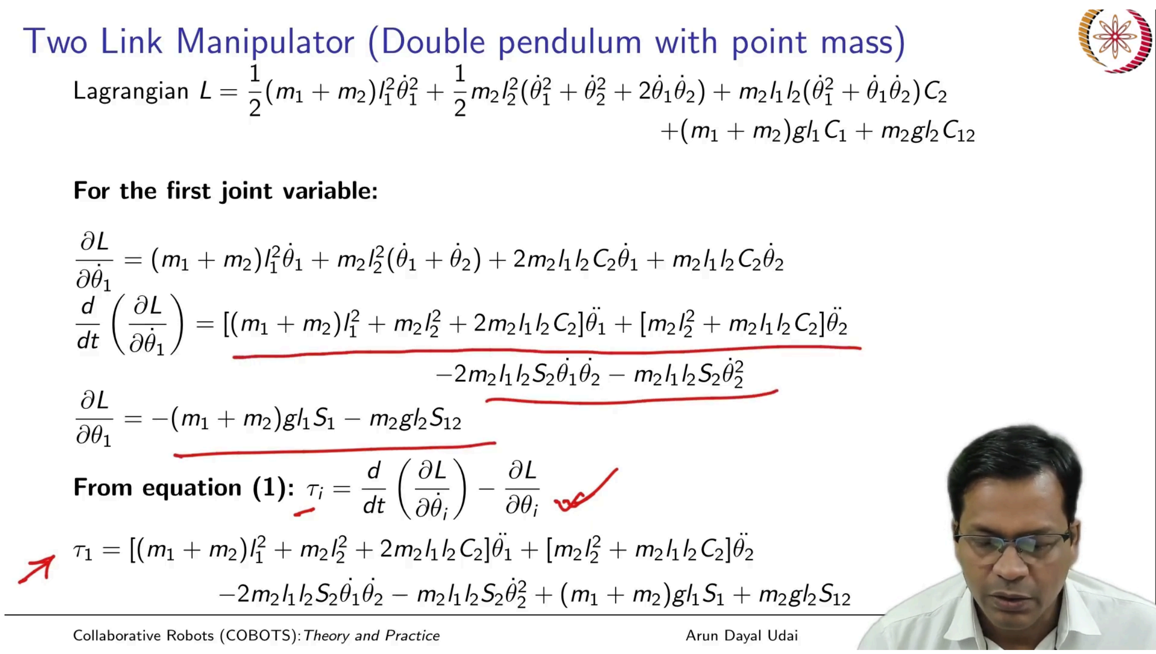
So, now I will use K and P, the total potential and total kinetic energies, and I will obtain what? The Lagrange-Euler. This is the Lagrange-Euler of the system, kinetic energy minus potential energy, K minus P.

$$L = K - P$$

So, now I will use this, that is the Lagrange-Euler formulation. It is defined as tau here because it is. Both of the joints are revolute joints. So, it will be tau, and the joint angle will be joint angular displacement. So, that is theta 1 dot and theta 1. Similarly, for the second one, it will be theta 2 dot and theta 2 here, and tau 1 and tau 2 are the joint torques. That should be the case. So, independently for both joints, we will have to use this one.

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Theta}_i}\right) - \frac{\partial L}{\partial \Theta_i}$$

So, starting with again, the first bracketed term, which is here, that you can see.



So, if this is the Lagrange-Euler, for the first joint variable, dou L by dou theta 1 dot, I have discarded all the terms that do not have theta 1 dot, and the whole of this gets reduced. So, this term goes off, this term goes off, and even this will go off. Only the terms which have theta 1 dot will remain, and taking the partial derivative, I get to this.

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1+m_2)l_1^2\dot{\theta}_1 + m_2 l_2^2(\dot{\theta}_1+\dot{\theta}_2) + 2m_2 l_1 l_2 C_2\dot{\theta}_1 + m_2 l_1 l_2 C_2 \dot{\theta}_2$$

Now, with the time derivative, this becomes this.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = [(m_1+m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2]\ddot{\theta}_1 + [m_2 l_2^2 + m_2 l_1 l_2 C_2]\ddot{\theta}_2$$

So, again, all the terms that have theta 1 dot will go to theta 1 double dot. Taking common it leads to this. So, this is the first term of the Lagrange-Euler expression.

The second term can be derived as $\partial L/\partial\theta_1$. So, in this case, all the terms that do not have pure theta 1 can go off. So, this will go, this will go, this will go; only the term which is here would remain, and taking the partial derivative with theta 1 gives me this.

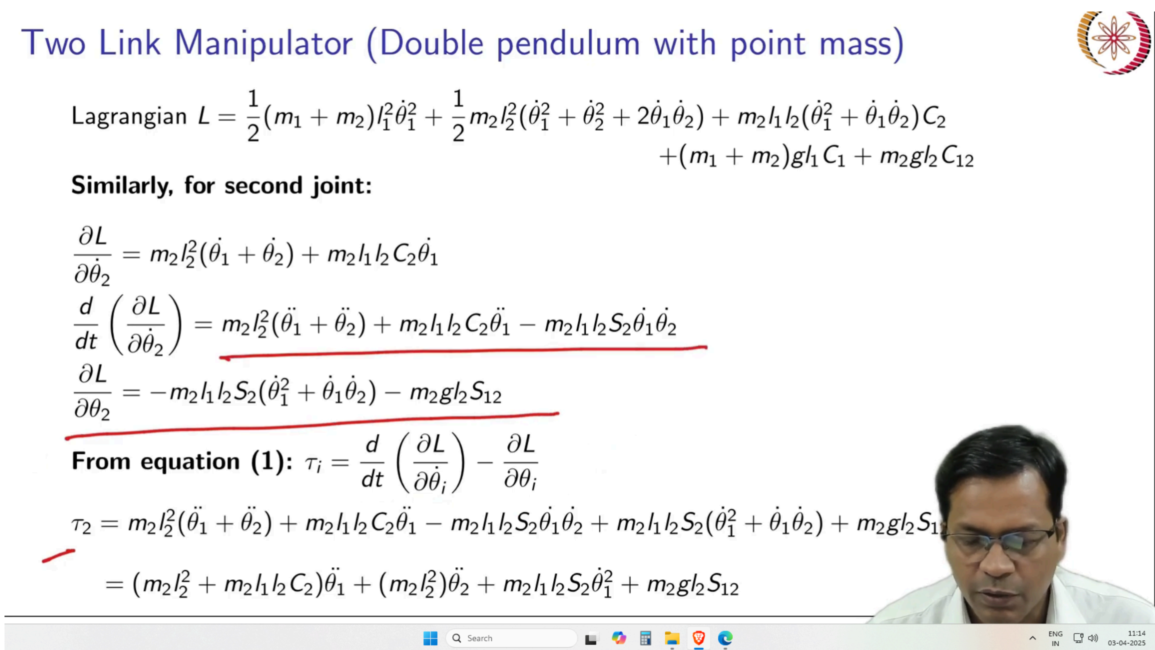
$$\frac{\partial L}{\partial \theta_1} = -(m_1+m_2)gl_1S_1 - m_2 g l_2 S_{12}$$

So, now, summing these two terms together and forming my Lagrange-Euler equation leads to this.

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$$

τ_1 is equal to d by dt (∂L by $\partial\theta_1$ dot) minus ∂L by $\partial\theta_1$. So, that gives me τ_1. So, that is the whole of this and this together. So, that is clubbed together like this.

$$\tau_1 = [(m_1+m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2]\ddot{\theta}_1 + [m_2 l_2^2 + m_2 l_1 l_2 C_2]\ddot{\theta}_2$$
$$-2m_2 l_1 l_2 S_2 \dot{\theta}_1\dot{\theta}_2 - m_2 l_1 l_2 S_2 \dot{\theta}_2^2 + (m_1+m_2) g l_1 S_1 + m_2 g l_2 S_{12}$$



Similarly, for the second joint, I will use the same Lagrange-Euler here, that is, the system Lagrange-Euler. Take the partial derivative with respect to theta 2 dot. Again, this time, I will neglect all the terms that have theta 1 dot. So, all the terms which have theta 1 dot will go away. So, only a few terms are remaining. So, it will lead to this.

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2) + m_2 l_1 l_2 C_2 \dot{\theta}_1$$

Now, the time derivative of this would be this.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 C_2 \ddot{\theta}_1 - m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2$$

So, it is dou L by dou theta 2 dot d by dt. So, all the theta 2 dot terms and theta 1 dot terms would become double dots. So, it leads to this.

So, now taking dou L by dou theta 2, that is the second term of the Lagrange-Euler. This time, it is theta 2. So, all the terms that have Theta 2 only will remain. So, it is this term and this term only. Remaining all of them, they do not have theta 2 alone. So, either they have theta 1 dot, theta 2 dot or theta 1. So, all those will be ignored. So, dou L by dou theta 2 would become derivatives of this. This and this. So, both the terms of the Lagrange-Euler layer expression is obtained. From there, I can obtain the equation of motion for the second joint.

So, from equation 1 again,

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$$

I can take the sum of both of these, and I can write tau 2 is equal to the sum of this and this. So, it gives me tau 2.

Two Link Manipulator (Double pendulum with point mass)

$$\tau_1 = [(m_1+m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2]\ddot{\theta}_1 + [m_2 l_2^2 + m_2 l_1 l_2 C_2]\ddot{\theta}_2 \\ -2m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 l_1 l_2 S_2 \dot{\theta}_2^2 + (m_1+m_2) g l_1 S_1 + m_2 g l_2 S_{12}$$

$$\tau_2 = (m_2 l_2^2 + m_2 l_1 l_2 C_2)\ddot{\theta}_1 + (m_2 l_2^2)\ddot{\theta}_2 + m_2 l_1 l_2 S_2 \dot{\theta}_1^2 + m_2 g l_2 S_{12}$$

Expressing τ_1 and τ_2 in the matrix form we get:

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}(m_1+m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2 & m_2 l_2^2 + m_2 l_1 l_2 C_2\\ m_2 l_2^2 + m_2 l_1 l_2 C_2 & m_2 l_2^2\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} \\ + \begin{bmatrix}0 & -m_2 l_1 l_2 S_2\\ m_2 l_1 l_2 S_2 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1^{\,2}\\ \dot{\theta}_2^{\,2}\end{bmatrix} + \begin{bmatrix}-m_2 l_1 l_2 S_2 & -m_2 l_1 l_2 S_2\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\dot{\theta}_2\\ \dot{\theta}_2\dot{\theta}_1\end{bmatrix} \\ + \begin{bmatrix}(m_1+m_2) g l_1 S_1 + m_2 g l_2 S_{12}\\ m_2 g l_2 S_{12}\end{bmatrix}$$

General Equation of Motion (EoM) as follows:

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \overset{2\times2}{\begin{bmatrix}\text{Equivalent}\\ \text{Inertia Component}\end{bmatrix}}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \overset{2\times2}{\begin{bmatrix}\text{Centripetal}\\ \text{Component}\end{bmatrix}} + \overset{2\times2}{\begin{bmatrix}\text{Coriolis}\\ \text{Component}\end{bmatrix}} + \overset{2\times1}{\begin{bmatrix}\text{Gravity}\\ \text{Component}\end{bmatrix}}$$

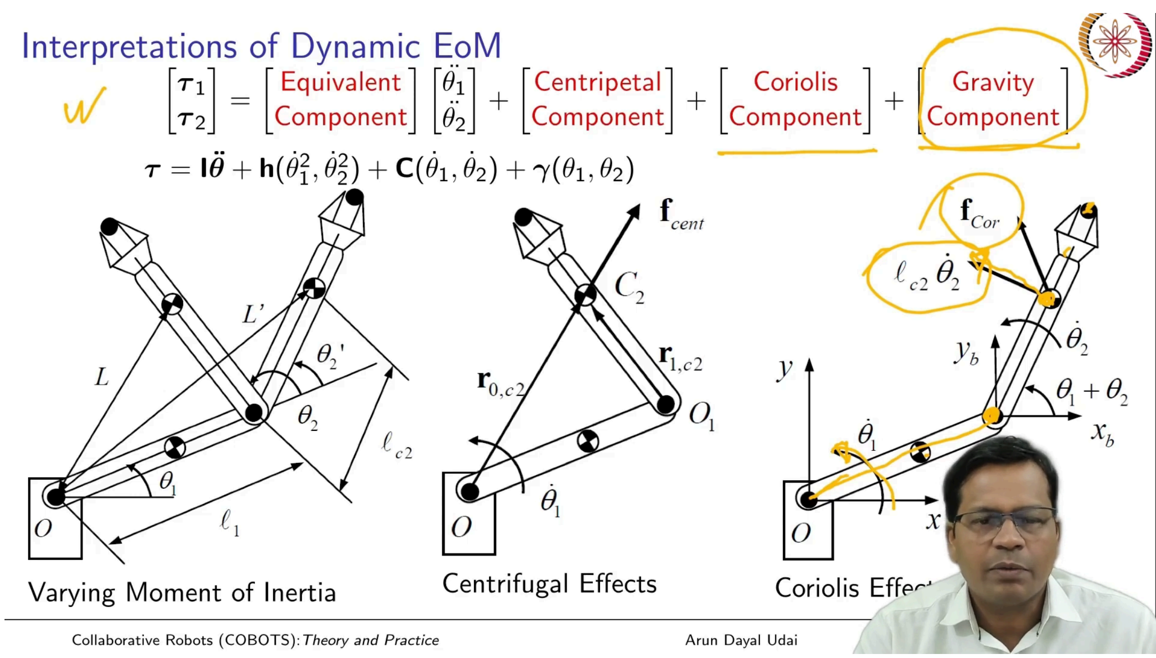
$$\tau = \mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{h}(\dot{\theta}_1^2, \dot{\theta}_2^2) + \mathbf{C}(\dot{\theta}_1, \dot{\theta}_2) + \gamma(\theta_1, \theta_2)$$

Now, packing both equations together, both are scalar expressions. Now, can I write it in vector-matrix form? So, it can be packed like this. You see, tau 1 and tau 2 can be written like this. All the terms, which have theta 1 double dot and theta 2 double dots, are expressed in a matrix form, matrix-vector form. So, this is your column vector for acceleration. And you are remaining with the terms, this one and this one for this. All the terms that have this will only remain. In this case, in the second one, you have this and this. So, those terms will appear here. So, this is a 2 cross 2 matrix, and now the second term it has theta 1 dot square and theta 2 dot square. Those can be packed together. All the terms which have got the square of joint angle rates come here again; you will get a 2 cross 2 matrix, which is here the third term has got the product of matrices that is theta 1 dot and theta 2 dot. Those are considered here, and again, you will get 2 cross 2 matrix and the terms which are left over last like these all the terms can be packed over here. They have g with us. Acceleration due to gravity terms will come like this. So, it contains four terms the first term will have joint accelerations. The second term will have a joint velocity square. The third term will have a product of joint velocities, and the term doesn't depend on joint velocities. It only depends on joint angles.

So, now, this is the general equation of motion for any robot. It can be expressed like this. So, first term you saw tau 1 and tau 2. That is arising out of the equivalent inertia component. So, this is equivalent to the moment of inertia of the system at any instant of time and inertia into acceleration. So, these are the accelerations. The second term is the centripetal component. As you see, it is I omega square. It has the dimension of that only, that has omega square terms which are here. The third term is the Coriolis component, so that has theta 1 dot into theta 2 dot. Those terms are here, that is, the product of the angular velocities term. The final one is the gravity component, which depends on g and does not depend on theta 1 dot or theta 1 double dot. Similarly, theta 2 dot or theta 2 double dot joint velocity and acceleration. They don't depend on that, so this is there even when the robot is static. So this is written as tau is equal to I theta double dot, h theta 1 dot square, theta 2 dot square. C is the function of theta 1 dot and theta 2 dot and gamma theta 1 theta 2, right?

$$\boldsymbol{\tau} = \mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{h}(\dot{\theta}_1^2, \dot{\theta}_2^2) + \mathbf{C}(\dot{\theta}_1, \dot{\theta}_2) + \boldsymbol{\gamma}(\theta_1, \theta_2)$$

So, this is the general equation of motion for any robot. So, in this case, all these matrices are 2 cross 2. In the case of any general robot of n degree of freedom, it will be n cross n.



So now let me further express it physically. So, these are some interpretations of the dynamic equation of motion, as you have seen. The first term is the equivalent component moment of inertia. So, let's say this robot at any instant of time is like this. It is like this for joint one. So, let us assume this is rigid. The joint is fixed over here. So, this is practically the shape of this link. So, the equivalent moment of inertia in this posture would cause torque at this joint, is it not? So, while considering the torque at joint one, the link is considered like this. But when considering the moment of inertia at joint two over here, the link is like this, and motion is because of the first joint motion and the second joint motion. Both of them will appear here, and that will cause torque over here. So, the shape here is very simple; it is straight, but in the case of the first joint, it is like this.

So, this is a Varying Moment of Inertia with time. The shape of the link changes. The configuration of the link changes with time. Variation is there in the configuration of the system. The whole of the moment of inertia keeps varying all the time.

So, now, the second term is the centripetal term. Again, due to the change in shape, the moment the centrifugal force because of the first CG is outward. It is very simple. Centrifugal force is outward; centripetal force is inverse, that is okay. But in the second link, due to the configuration of this joint this direction keeps on changing. This link configuration keeps on changing. Accordingly, this will change. So, this mass will create centrifugal force outward like this, and centripetal force will now create torque over here. So, these are two intrinsic aspects of this. You should understand this element-wise. That is the physical significance of each of these terms.

So, again, the Coriolis term is the second-last term you see in this case. This link is very simple, and it just has angular velocity here, and mass is here, whereas, in the second joint, This causes motion to the second link. In this case, it has both. It is both rotating and translating. This will definitely have a Coriolis component as well. It is not a pure rotation. So, the centre of mass is rotating as well as translating along the axises. So, the resultant velocity—the resultant motion—will create a Coriolis component.

The last component is the acceleration due to gravity; it is not because of motion. It is because the robot has to stay there, and gravitational forces are creating this component. So, yes, all these comprise the final equation of motion; all these are components of the equation of motion. All these form the equation of motion.

So, that is all for this lecture. In this lecture, we derived the equation of motion for a two-link manipulator, and we have interpreted that. So, in the next lecture, we will continue further with Robot Dynamics. We will learn what the Newton-Euler Approach is to derive the equation of motion. We will also extend it further to the Recursive Newton-Euler Approach.

That is all. Thanks a lot.