NPTEL Online Certification Courses
COLLABORATIVE ROBOTS (COBOTS): THEORY AND PRACTICE
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Week: 05
Lecture: 22

Kinetostatic Measures for Robot Design

Overview of this lecture

- Kinetostatic Measures

- General Derivation of Velocity/Force Ellipsoid

- Manipulability Measures

Welcome to the third lecture of the module Robot Statics. This is about Kinetostatic Measures for Robot Design. In this lecture, I will be discussing Kinetostatic Measures, General Derivation of Force and Velocity Ellipsoids, and Manipulability Measures.
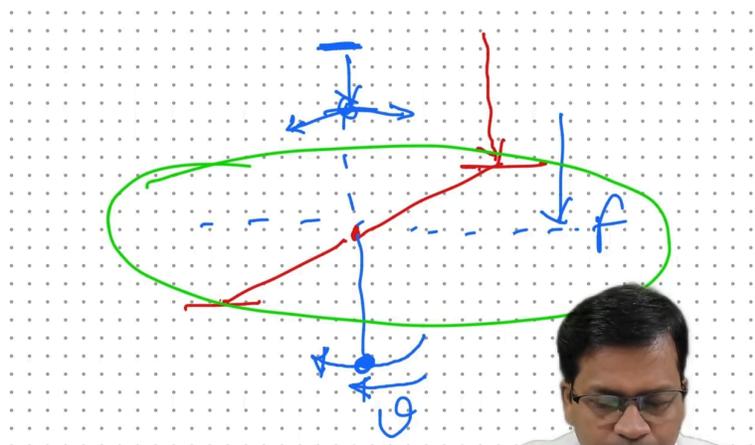
Introduction

Cycling

Maximum velocity, Fully stretched, Horizontal direction

Maximum Force, Fully stretched, Vertical

So, let me just give you a brief overview of how we work and how humans work. First, if you are cycling, what you have felt is that if this is a pedal, if this is a pedal, so at a particular position over here, you see whenever you start the cycle, you start keeping your

pedal something like this so that you can apply maximum force. You cannot put it like this. In this case, if you apply force, it can go either here or here, and at the same time, you won't be able to apply maximum force in this direction, okay? Whereas when you want to make it go fast, you keep the pedal like this so that you can make it go at the maximum velocity. So, at the tip over here and here, you want more velocity. Why? You don't want to get stuck at the top dead or bottom dead centres, and this is the place where you can apply maximum force and this is the place where you need more velocity and your legs are fully extended. Your legs are fully extended. Legs are fully extended. Almost bent over here, and this is the mid-range where you want to work, and this is the region where you can carry the maximum amount of force. So that is cycling. So maximum velocity is in the fully stretched condition, horizontal direction, and maximum force is in the vertical direction. So, you have two cases: Maximum Velocity and Maximum Forces.
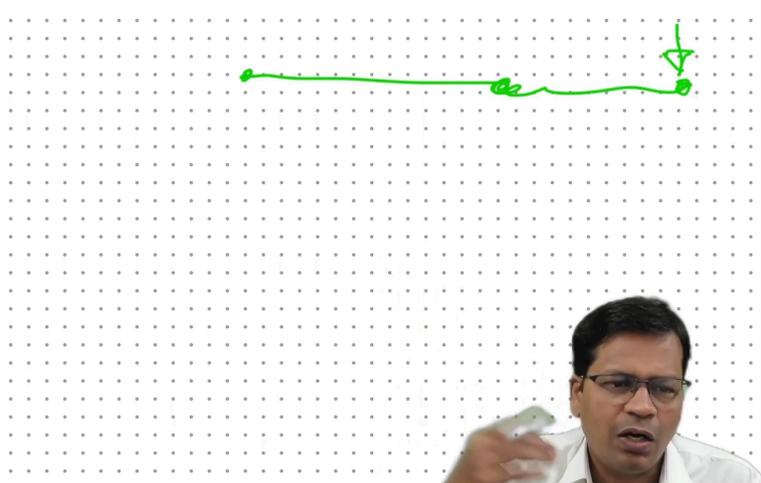

Introduction

Lifting a Weight

Maximum Force, Fully stretched, Horizontal direction

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In the second case, lifting a weight. You don't lift a weight, keeping your arm fully stretched. You don't prefer doing work most of the time, keeping your arm fully stretched. Why? You can carry a lesser amount of load. So, you want to remain somewhere in the middle, and while moving a jug, doing any household task, what do you do? You normally stay in the mid-range. You don't fully stretch. So this is what force is about when we try to lift any weight; you want to keep it mostly in the mid-range.

So, let's say you want to throw a cricket ball. What do you do? You normally stretch your arm to the maximum and throw the ball so that you can impart the maximum velocity to the ball. But at the same time, when you play cricket, what do you do? You hold the bat with both arms. You keep the bat in the mid-range and try to maneuver it to apply maximum force. These are a few instances where you notice that the configuration of your arm greatly affects the maximum force you can apply and the velocity you can achieve. With this as a background, let us begin.

Kinetostatic Measures

▶ **Dexterous robot**: Ability of a arm to reach all points within the workspace with all possible orientations of its end-effector, and can transmit motion and force comfortably.

▶ **Dexterity** w_d and **Manipulability** w_m are quantified by a value given by:
$$w_d = |\mathbf{J}|, \text{ and } w_m = \sqrt{|\mathbf{J}\mathbf{J}^\mathbf{T}|}.$$

▶ For a non redundant arm **J** is a square matrix:
$$w_d = w_m$$

▶ **Condition Number**: The ratio of the maximum to minimum singular values of a matrix (**J**), and can measure a singularity or other kinetostatic indices.
$\rightarrow$ index used to describe (1) the dexterity of a robot and
(2) the closeness of a pose to a singularity.

▶ **Manipulability**: Capacity to change the pose of the end-effector of a robot at joint configuration. It is a measure of how close a robot is to singularity!

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What is a Dexterous robot? A dexterous robot is the ability of the arm to reach all points within the workspace with all possible orientations of its end effector. You can reach somewhere, orient it anyway, and perform tasks here or anywhere else. It can transmit motion and force comfortably. At any location, you can transmit motion and apply forces in all directions. It is not that in a fully stressed condition, you cannot push it further, but you can only go in a vertical direction. If your arm is fully stressed, you cannot go further in this direction, but you can move like this.

So, dexterity is the measure of that. Dexterity w_d and manipulability w_m are quantifiable values for this. So, w_d is the determinant of the Jacobian, and wm is JJ transpose, which makes it a square matrix. The determinant of that and taking the square root of that.

$$w_d = |J|, \; and \; w_m = \sqrt{|JJ^T|}$$

So, what is that? Both are effectively the same. This one is meant for a non-square Jacobian in the case of redundant robots like KUKA iiwa and others. Whereas, for the 6 degrees of freedom robot, the Jacobian is already a square matrix, and you can directly use this.

So, for a non-redundant robot, J is already a square matrix, and w_d and w_m are both the same. So, this is the measure of dexterity. It measures whether the Jacobian is going singular or not. If it is going singular, as in the case of a fully stretched arm, so near to singular, it is no more dexterous.

So, the condition number is another similar measure. It is the ratio of maximum to minimum singular values of the matrix J and can measure singularity and other kinetostatic indices. So, the index is used to describe the dexterity of the robot. So, this gives you a kind of value to compare the dexterity of a robot, and it is the closeness of a pose to the singularity. So the same robot at different places, because you know the Jacobian is varying, so at this position, it will have a different dexterity measure than any other position where the Jacobian is different. Okay, so the same robot can have different dexterity at different locations.

So, the Manipulability measure is the capacity to change the pose of the end effector of the robot at any given joint configuration. So, let us say if you are fully stretched, you know you cannot do quite a lot of motions because it is already the tip is fully aligned with the robot and it has extended fully. So, it is a measure of how close, again, to the singularity. How close the robot is to the singularity, so that is what manipulability is.
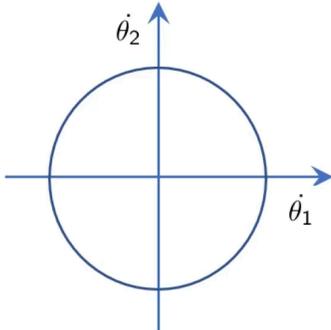
General Derivation of Velocity/Force Ellipsoid

End effector-velocities and joint-torques are given by:
$$\mathbf{v}_e = \mathbf{J}(\theta)\dot{\boldsymbol{\theta}} \text{ and } \boldsymbol{\tau} = \mathbf{J}(\theta)^T \mathbf{f}_e$$
where $\mathbf{v}_e, \mathbf{f}_e \in \mathbb{R}^m$; $\dot{\boldsymbol{\theta}}, \boldsymbol{\tau} \in \mathbb{R}^n$ and $\mathbf{J}(\theta) \in \mathbb{R}^{m \times n}$

$$(\mathbf{J}^{-1}\mathbf{v}_e)^T(\mathbf{J}^{-1}\mathbf{v}_e) = 1$$

$$\mathbf{v}_e^T(\mathbf{J}^{-1})^T\mathbf{J}^{-1}\mathbf{v}_e = 1$$

$$\mathbf{v}_e^T(\mathbf{J}\mathbf{J}^T)^{-1}\mathbf{v}_e = 1$$

$$\mathbf{v}_e^T\mathbf{A}^{-1}\mathbf{v}_e = 1 \text{ where } \mathbf{A} = \mathbf{J}\mathbf{J}^T \in \mathbb{R}^{m \times m}$$

Unit circle with variables $\dot{\boldsymbol{\theta}} = [\dot{\theta}_1 \ \ \dot{\theta}_2]^T$ is:
$$\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = 1 \text{ (hypersphere for } n > 3)$$

So, a General Derivation for Velocity and Force Ellipsoid, which direction, how much force you can do. So, that is what we will analyse here.

Let us say you have a 2-DoF robot like this. So, with joint angle, this is theta 1, this is theta 2. So, at this position, maybe it has some ellipsoid, which is something like this, in which you can apply force. So, an ellipsoid basically has an axis so that the amount of force which it can apply in different directions, okay? So, these are forces, so this is the ellipsoid. So, maybe over here, it may have a circular shape. If it is fully extended, it gets fully collapsed, so you can apply only the forces when it is fully collapsed, okay? So, this is the case. You see, you have a varying shape of this ellipsoid, ranging from a circle to fully collapsed, and maybe it looks like a circle over here. So, this is what we want to analyse for the velocity and for the forces that it can deliver in different directions.

So, starting with this. So, ve, that is, the velocity, is equal to the Jacobian times theta dot. By the definition of the Jacobian, you already know this. We have also derived tau is equal to J transpose of f_e. f_e is the end effector force. Jacobian transpose times that would give me the joint torques.

So, let us analyse this, where ve and fe have the dimension of m, that is, the task space dimension, whereas theta dot and tau are n-dimensional real numbers. So, that is in the joint space, which is directly related to the degree of freedom of the robot. The Jacobian

will have m cross m dimension. If you draw along theta 1 dot and theta 2 dot, the variables for that would be theta 1 dot and theta 2 dot. The unit circle equation for this would be theta dot transpose theta dot is equal to one.

So, in the case of theta one, theta two, theta three, a three-variable system, in that case, it would be a hypersphere for n greater than three, okay? So, for n greater than three, it would be a hypersphere, and for three degrees of freedom, it will be a sphere.

So, with this, now let us check this. So, if you substitute theta dot from here, in the case of the velocity ellipsoid, I want to find out. So, it should be J inverse of ve would give me theta dot. So, if I substitute that here, it becomes this.

$$(J^{-1}v_e)^T(J^{-1}v_e) = 1$$

If I reduce this further, it becomes this,

$$v_e^T(J^{-1})^T J^{-1}v_e = 1$$

And further reducing it will give me this.

$$v_e^T(JJ^T)^{-1}v_e = 1$$

So, ve transpose J J transpose inverse ve would become equal to one.

$$v_e^T A^{-1}v_e = 1 \text{ where } A = JJ^T$$

So this is equal to v_e transpose A inverse, where A is equal to J J transpose, that is equal to m cross m. So, this is now a square matrix.
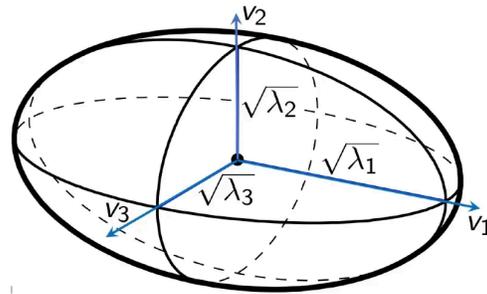
In case of a generic value of velocity $\mathbf{v}_e \equiv \mathbf{x}$

$\mathbf{x}^T \mathbf{A}^{-1} \mathbf{x} = \mathbf{1}$ where
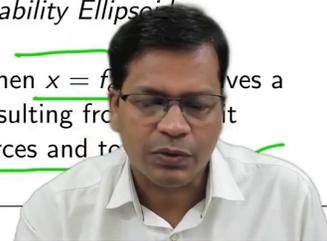$\mathbf{A} = \mathbf{J}\mathbf{J}^T \in \mathbb{R}^{m \times m}$ (symmetric, positive definite)

Eigenvalues of $\mathbf{A} = \lambda_1, \lambda_2 \cdots, \lambda_m$
Eigenvectors of $\mathbf{A} = v_1, v_2, \cdots, v_m$

The equation $\mathbf{x}^T \mathbf{A}^{-1} \mathbf{x} = \mathbf{1}$ for 3-dimensional space represents an ellipsoid.

If $\mathbf{A} = (\mathbf{J}\mathbf{J}^T)^{-1}$ then $x = f$ __ ves a
Force Ellipsoid resulting fr__ __t
sphere of joint forces and t__

Manipulability Ellips__

So, in the case of a generic value of velocity v_e x. So, this becomes this,

$$v_e \equiv x$$

That is any particular value if I put. So, it is x transpose A inverse x is equal to 1. This is symmetric and positive definite. So, A is to be a symmetric and positive definite matrix. So, the eigenvalues of A are lambda 1, lambda 2, lambda m. And eigenvectors will be v1, v2 up till vm. So, if the matrix is m by m, it is like this.

So this effectively equation X transpose A inverse X represents a three-dimensional space, represents an ellipsoid if it is a three-dimensional one. So, it is equal to V1, V2, and V3, which are basically the eigenvectors. Lambda 1, lambda 2, and lambda 3 are the lengths of these ellipsoids, these lengths, so lambda 1 square root is the axis of the ellipsoid. So, lambda 2 square root, lambda 3 square root. This is a mathematical representation. This is from mathematics; you can directly get it. So, this is the equation of an ellipsoid as such, where eigenvectors and eigenvalues represent the direction of the axes and the magnitude of those axes. So what you see here is if A is this (JJ transpose inverse), then x is ve. So, in case you started with the equation with velocity v is equal to J transpose of theta dot, so in case you start with the velocity Jacobian, so in that case, what you get is the velocities along each direction. So, the force ellipsoid results from the

unit sphere of joint forces and torques. So, if A is equal to JJ transpose inverse when X is equal to f_e, it gives a force ellipsoid resulting from the unit sphere of joint forces and torques.

So, you see here, if you start with this, ve is equal to j theta dot, so in that case, this gives you a velocity ellipsoid, but if you start with this Jacobian, okay, so if you start from here, this also gives you a force Jacobian, in that case, this becomes a force Jacobian. So, this can either be a force on each side, and it becomes a force Jacobian, and in that case, it gives you a force ellipsoid, but if you start with this with the unit circle as I have shown here, I get what is a unit circle of one radius, and it is a velocity ellipsoid.

Manipulability and Force Ellipsoid

Manipulability Ellipsoid

▶ A circular shape would mean that the end-effector is equally manipulable in all directions.

▶ This defines the end-effector motion capabilities and captures how close the robot is to the singularity.

Force Ellipsoid

▶ Same principal axes

▶ Principal axes lengths are reciprocals of each other

▶ The force ellipsoid has the major axes orthogonal to the manipulability ellipsoid

So, Manipulability and Force Ellipsoid. So, this is known as a manipulability ellipsoid when it is with velocity, and in the case of force, I commence with it. So, it is known as a force ellipsoid that I get.
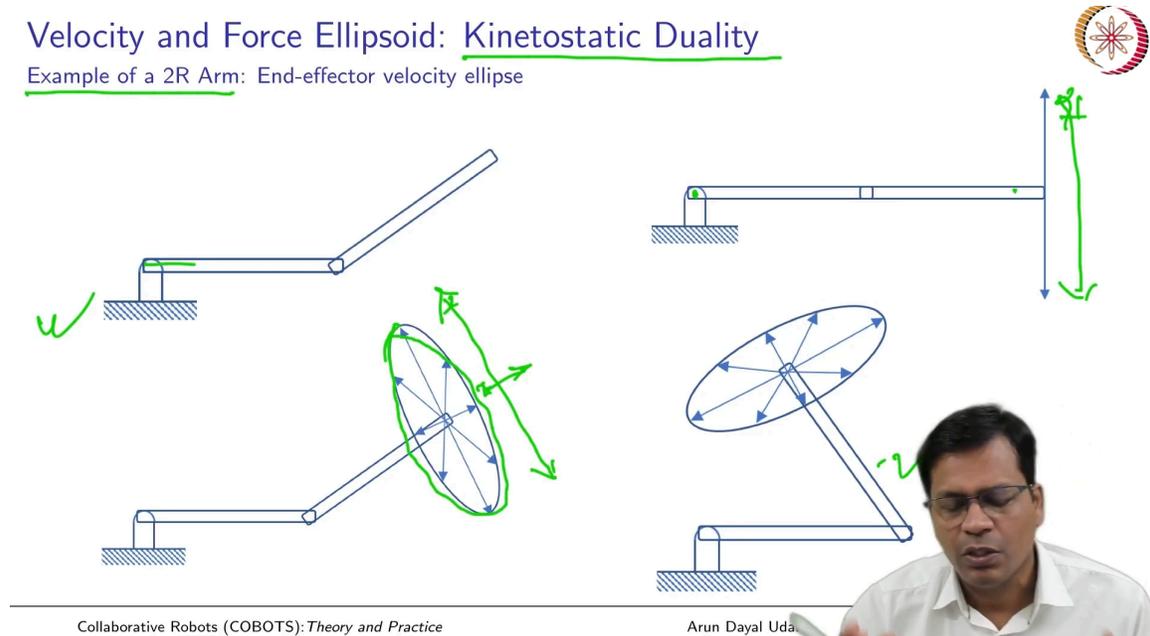
So, the manipulability ellipsoid, a circular shape, would mean that the end effector is equally manipulable in all directions. It defines the end effector motion capabilities and captures how close the robot is to the singularity.

Then what is the force ellipsoid? It is the same principal axis, so the principal axis of this and this would remain the same. The principal axis lengths are reciprocal to each other.

So, both of these, if it is like this, the force ellipsoid would be like this. So, that is orthogonal to each other also. The force ellipsoid has a major axis orthogonal to the manipulability ellipsoid. So, let me just vanish for a moment so that you can see it.


Now, look at the Velocity and Force Ellipsoid that is known as the Kinetostatic Duality. In the case of a 2R arm, the affected velocity ellipsoid would look like this. So, you see, in this case, Okay, this is the velocity ellipsoid. In the case of a 2R planar arm, it becomes an ellipse. So, you see, you can apply more velocity along this and less along this direction. So, when it is configured like this, it is like this, and in this case, when it is fully stressed, it is singular. In this case, you can only move along this direction. So, you see, your velocity ellipse is fully collapsed. You cannot move in this direction at all.

Example of a 2R Arm: End-effector force ellipse

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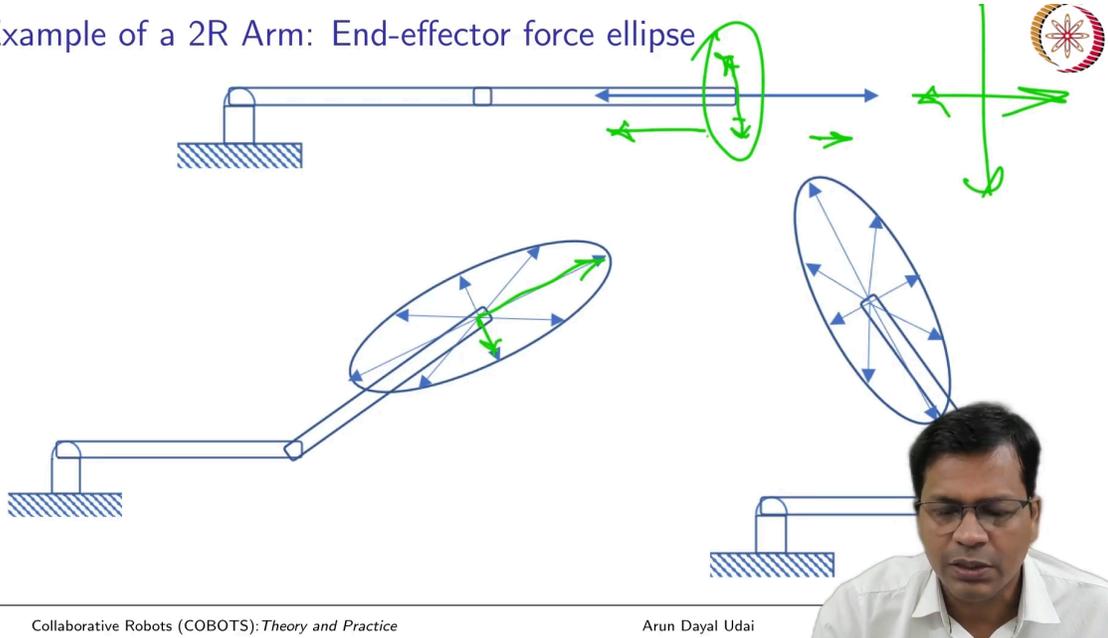
The same in the case of the force ellipsoid. You see, you can apply forces along this and this direction only. And this reduces to a minimum. You cannot apply forces more along these directions. It has fully collapsed once again. The velocity one was something like this. The force one is like this. So, you see, over here, you can apply more force and less force, and in this configuration, it becomes like this. So, you see, the force ellipse and the velocity ellipse are orthogonal to each other.

Manipulability Measures

- *Manipulability measure 1*: Ratio of the longest and shortest axes
 $$\mu = \frac{\sqrt{\lambda_{max}(A)}}{\sqrt{\lambda_{min}(A)}} \geq 1$$

- If $\mu = 1$: The robot is **isotorpic**, and it is equally manipulable in any direction.

- A higher value would mean that the robot is more close to singularity.

- *Manipulability measure 2*: **Condition number** $= \mu^2 = \dfrac{\lambda_{max}(A)}{\lambda_{min}(A)} \geq 1$

- *Manipulability measure 3*: is proportional to the volume of the ellipsoid $\sqrt{\lambda_1 \lambda_2 \cdots \lambda_m} = \sqrt{|\mathbf{A}|} = \sqrt{|\mathbf{JJ}^T|}$

- Maximum volume occurs when axes are equal and it is a sphere. The robot can move with equal velocity in all directions.

- If the volume of manipulability ellipsoid becomes large the volume of force ellipsoid becomes small and vice-versa.
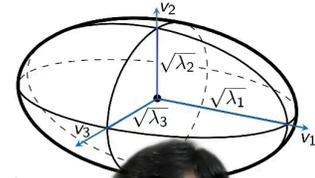
So, there are many other measures for manipulability. Measure 1 is the ratio of the longest to the shortest axis. So, it can directly be derived from the velocity ellipsoid.

$$\mu = \frac{\sqrt{\lambda_{max}(A)}}{\sqrt{\lambda_{min}(A)}} \geq 1$$

So, the maximum and the minimum axis sizes. So, lambda max, lambda min with square root. So, these are the dimensions of those axes, which should be greater than 1 normally.

Next, if mu is equal to 1, it is isotropic. It is equally manipulable in all directions. In that case, lambda max and lambda min both become equal, and it becomes a circle or a sphere.

A higher value would mean the robot is more close to the singularity. So, lambda max goes very high compared to lambda min. So, in that case, it is near to singularity. So, manipulability measure 2 is a condition number, mu square. It is nothing but the square of this. So, all these researches were reported differently at different times. So, they look similar, but they were originated and defined by different people of interest. So, mu square is equal to this, which is known as a condition number. This, again, is a measure of how close it is to singularity. Manipulability measure 3 is proportional to the volume of the ellipsoid. So, the volume of the ellipsoid mathematically can be given as the square

root of lambda 1, lambda 2, lambda 3, and so on. All the lambdas, it is the square root of a determinant or this. So, this is the volume of the ellipsoid. So, the higher the volume, the better it is. So, the best one is the sphere. If it is fully collapsed, then it has the least volume and is least manipulable.

Maximum volume occurs when the axes are equal and it is a sphere. The robot can move with equal velocity in all directions. The same is true for the force ellipsoid also. If the ellipse is becoming a sphere, that means it is able to apply forces equally in all directions.

If the volume of the ellipsoid becomes large, the volume of the force ellipsoid becomes small and vice versa. So, it is like this.

Notes on Types of Ellipsoids

The Jacobian $\mathbf{J} = \begin{bmatrix} \mathbf{J}_\omega \\ \mathbf{J}_v \end{bmatrix} \in \mathbb{R}^{6 \times n}$

$\mathbf{J}_\omega \in \mathbb{R}^{3 \times n} \to$ angular velocity/moment ellipsoids

$\mathbf{J}_v \in \mathbb{R}^{3 \times n} \to$ linear velocity/force ellipsoids

So, the Jacobian normally has two components: Jω and Jv. In the case of a 6DoF robot, each of them is 3×3, 3×n, 3×n. So, the overall Jacobian is normally 6×n, and it can be written as Jω is equal to 3×n, which is the angular velocity, and the moment ellipsoid can be found out using this. Jv will give linear velocity and force ellipsoids. So, using Jω, you can get angular velocity and the moment ellipsoid. With Jv, you can get linear velocity and force ellipsoids.

So, that is all for this lecture. In the next lecture, we will discuss all the models that we have learned this week. So, we will compile them all together, and we will try to create a

model, a mathematical model. We will try to put them in MATLAB Simscape and try to simulate them all.

That's all for this lecture. Thank you.