NPTEL Online Certification Courses
COLLABORATIVE ROBOTS (COBOTS): THEORY AND PRACTICE
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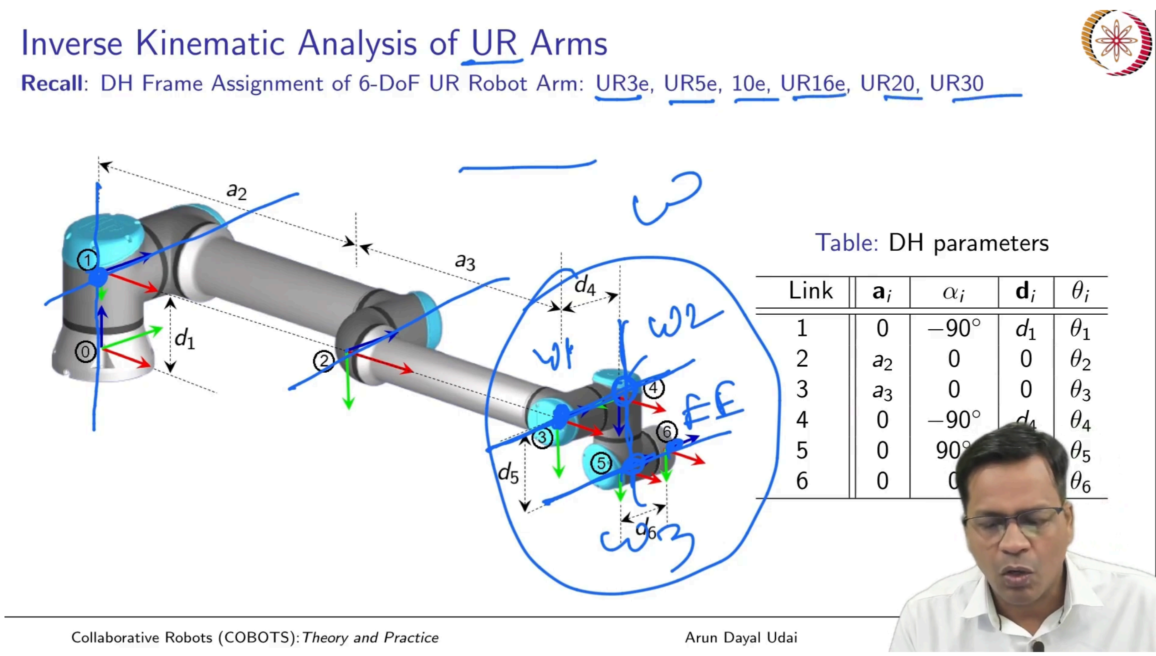
Week: 04
Lecture: 17

Inverse Kinematic Analysis of UR Arms

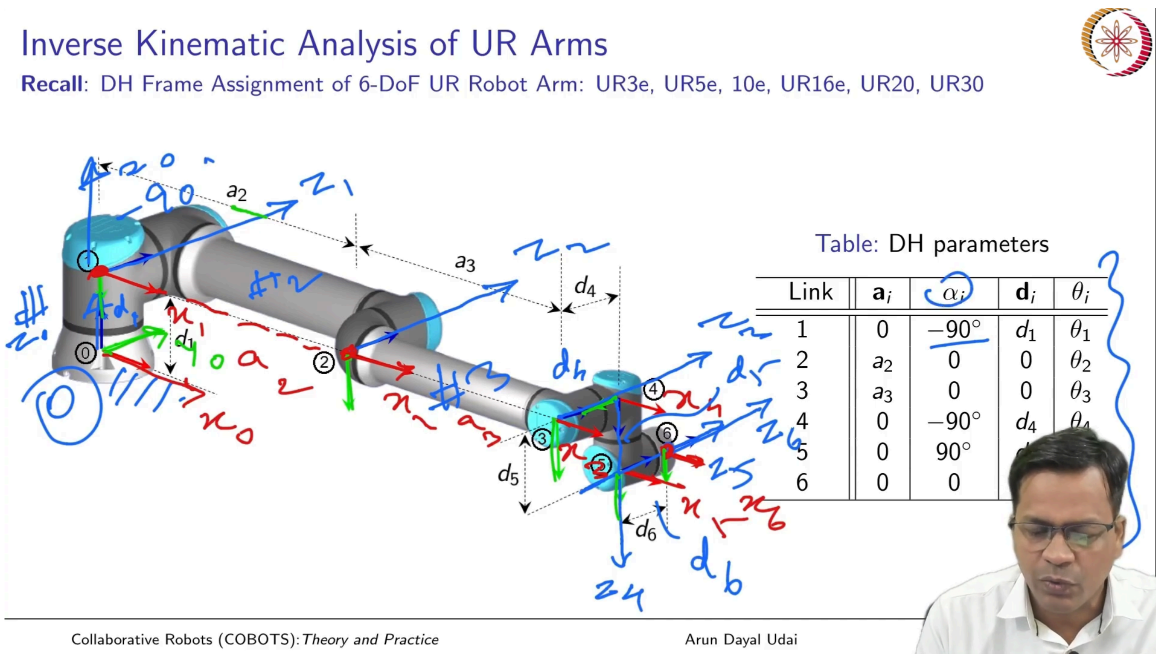
Welcome back to the course Collaborative Robots: Theory and Practice. In this lecture, we will discuss the Inverse Kinematic Analysis of Universal Robot Arms.



It looks like this, you see. It has a first axis that goes like this, and then you have a second axis, a third axis, a fourth axis, a fifth, and a sixth, and finally, this is your end effector.

So, working with this kind of robot, there are many which have a similar architecture. You see, it has no more than two consecutive axes which intersect at a point. So, it does not have a which has three axes all intersecting at a single point over here somewhere. So, it does not have a wrist which is a single wrist. So, instead, it has a wrist which may be divided into three different wrists. So, if this is your axis and this is your axis, this is one, and then you have another axis that goes like this. And finally, this is your end effector. So, this may be one axis also which is not intersecting, though. You can call it wrist 1, wrist 2, and wrist 3.

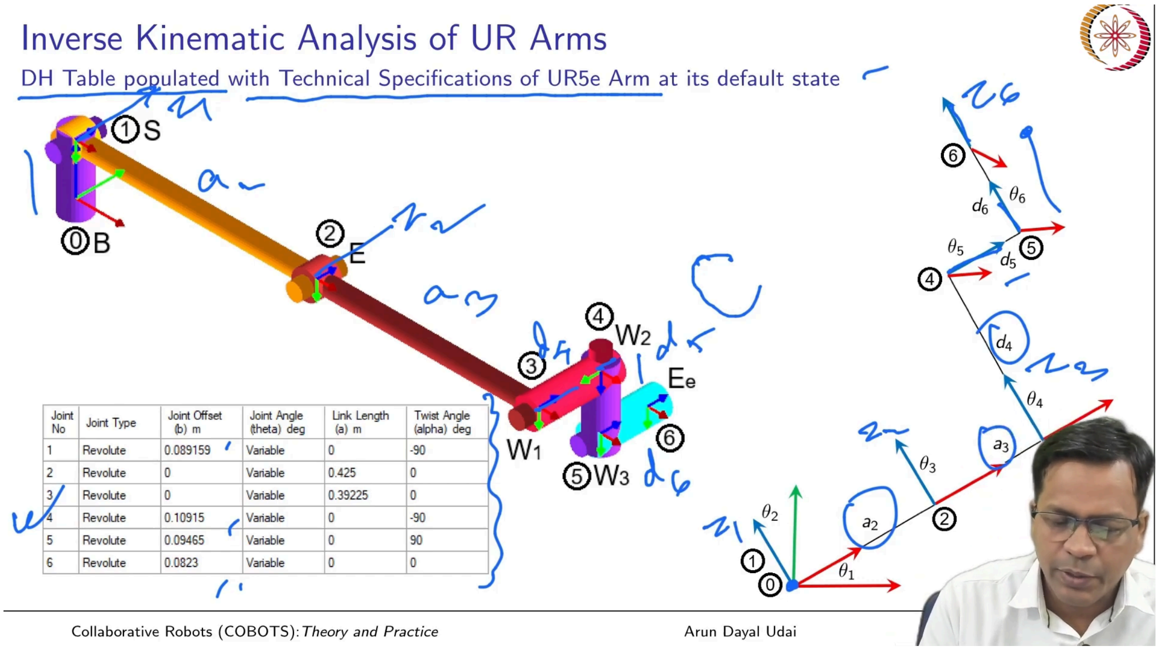
So, all together, in quite a lot of robots, at least in the case of industrial robots, you will find. They intersect all three axes at a point, and we call it a combined risk centre point. So, that is, axes four, five, and six intersect at a point, and that is the risk centre point. We have seen it earlier, okay? So, in this robot, that is not the case. So, it is like a snake kind of robot, and you see all the UR arms. UR here, I would mean Universal Robot arms, like UR3e, UR5e, 10e, UR16e, UR20, and now UR30. So, all the UR arms are of this configuration.



So, let us do Universal Robot arms here. So, we will quickly do the forward kinematics first. Forward kinematics we have already done earlier also. So, you see. You can directly put all the rotating axes as the Z axis. So, this is your ground. This is the frame that is attached to the ground, and then you have link 1, which ends up with frame 1, and this is your link 2. This is frame 1, and then you have frame 2; link 2 ends up with frame 2. link 3, frame 3. I am placing just the z-axis first. So, if this is z0, this is z1, z2, z3, z4, z5. And z6 is at the end, which is over here. So, z5 and z6 are collinear.

So, you have your end effector that lies here. So, now the common normals to all the z-axis are the x-axis, you know, for the ground frame, this is your x0. So, you have preferred putting the forward direction as the x-axis here. By default, the UR arm has the reverse direction as the x-axis, but that I am not considering here. So, I am preferring all the forward directions as the x-axis. So, this is x0, x1, x2, x3, x4, x5, and finally, this is x6, and the y-axis can be obtained using the right-handed thumb rule. x cross y should be equal to z, so z cross x is equal to y. So, you can get y0 like this, y1, y2, y3, y4, y5, and y6 like that.

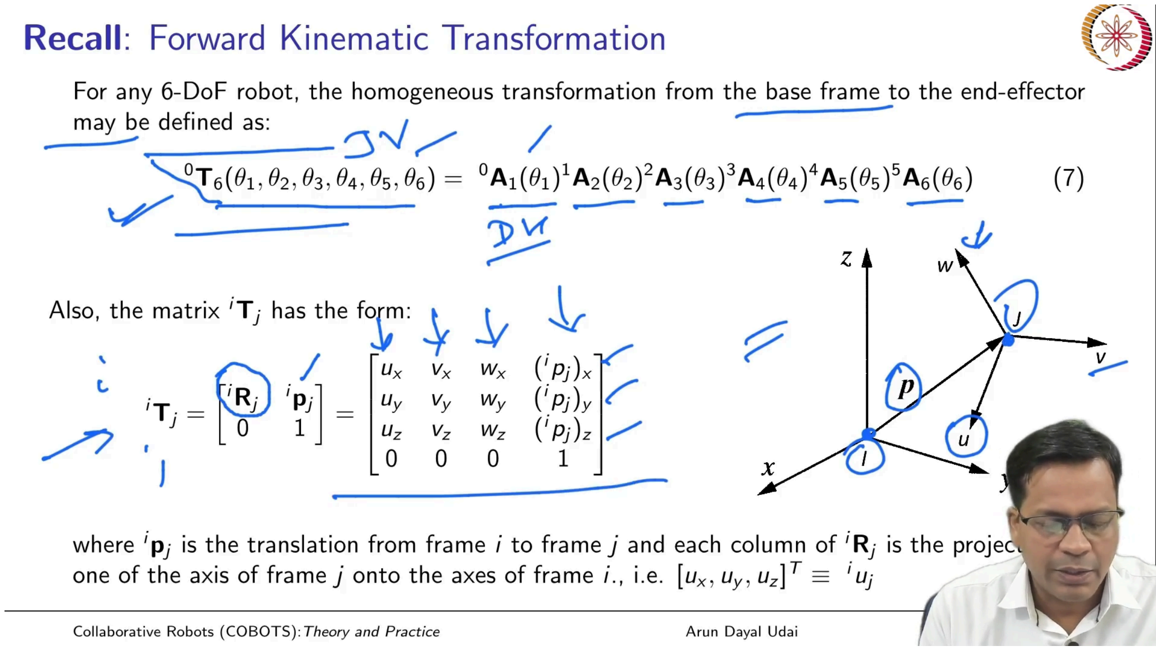
So, now you see, you have DH parameters that will go like this. So, the distance between two z-axes, the perpendicular distance, is the common normal. So, this should be your x-axis. So, from here to here, this is your a2. The distance measured along the z-axis is your d. So, that is joint offsets. So, this is your d1. The rest is your a2, a3, and then again, you will have all the ds here. So, it is d4, and this is d5, and this is d6. So, all the dimensions can be taken from the datasheet of this robot. So, this is how it looks like, and the DH parameter table can be made quickly like this. So, alpha is rotating from z0 to z1 along the x-axis. So, it is minus 90 degrees here, and similarly, other alphas can also be obtained. So, these are link twists. So, this is how you do forward kinematics. This is very, very important here because we will be using transformation matrices, taking their inverses a lot of the time. So, this will be very, very helpful.



Joint No	Joint Type	Joint Offset (b) m	Joint Angle (theta) deg	Link Length (a) m	Twist Angle (alpha) deg
1	Revolute	0.089159	Variable	0	-90
2	Revolute	0	Variable	0.425	0
3	Revolute	0	Variable	0.39225	0
4	Revolute	0.10915	Variable	0	-90
5	Revolute	0.09465	Variable	0	90
6	Revolute	0.0823	Variable	0	0

So now, I have just drawn the stick model of this robot here. So, the DH parameter table is populated here with the technical specifications of the UR5e arm at its default state. So, I have put all the default states when all the joint angles are zero, okay? So, there is no initial angle. So, in the DH parameter table, anyway, that doesn't count here. So, it's this configuration. What you can see here is actually the default state of this robot, with all the joint angles equal to zero, okay? So, I have put all the values here, and this is how it is.

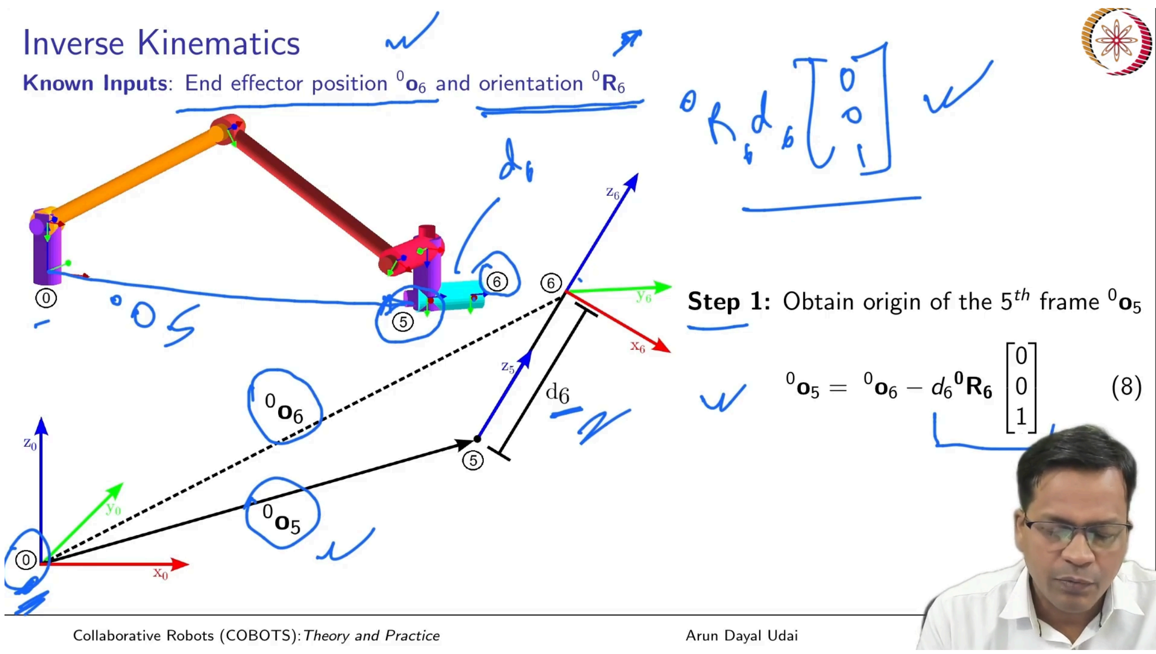
So, now, this is the stick model. So, it can further be simplified like this. If I look from the top, this robot may look like this when joint angle theta 1 has rotated by some angle. So, it goes like this. So, if I look at it from the top, it should look like this. So, from here, you see this is your joint 0 or 1; both will be one over the other. So, that is here, and then you have a2 that is visible here, a3 is here, and then you have d4. So, this is your d4. And then this is your d5. d5, I have assumed it has rotated by a certain angle, and then, looking from the top, this distance d5 is visible like this. so, this is your d5, and then you have finally d6. So, d6 would come like this, and at the end, you have this frame. So, all the z-axis, so at least you can see this is z1, this is z2, this is z3. So, all of them, that is, all the blue ones, Z1, Z2, and Z3, are all parallel to each other. And finally, you have Z4, Z5, and Z6. So, this is Z6 that comes at the end. So, this is how it will look like. Okay, so this figure I will be using it part by part in different parts of this analysis.



So, let us recall the forward kinematic transformation that is quite simple, you remember. So, T 0 to 6 ($^{0}T_{6}$) is the end effector configuration matrix with respect to frame 0, okay? That is given as that is a variable containing multiple parameters: theta 1, theta 2, theta 3, theta 4, theta 5, and theta 6; these are all joint variables. So, for any 6 degrees of freedom robot, the homogeneous transformation matrix from the base frame to the end effector frame is defined like this, whereas 0a1, 1a2, a3, a4, a5, and a6 ($^{0}A_{1,}{}^{1}A_{2},{}^{2}A_{3,}{}^{3}A_{4,}{}^{4}A_{5},{}^{5}A_{6},$) are our link transformation matrices. So, each one of them is a function of just one joint variable, and the remaining is the structure of the link: d theta a alpha. This comes from the DH parameter directly with a variable joint variable that is theta, theta 1, theta 2, and so on and so forth. The link transformation matrix or the total transformation from any point, let us say i to j, is given like this,

$$^{i}\mathbf{T}_j = \begin{bmatrix} ^{i}\mathbf{R}_j & ^{i}\mathbf{p}_j \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} u_x & v_x & w_x & (^{i}p_j)_x \\ u_y & v_y & w_y & (^{i}p_j)_y \\ u_z & v_z & w_z & (^{i}p_j)_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where the 3x3 matrix is the rotation transformation matrix, and this, the last column, is basically the position of that point. So, if this is the figure for that, it transforms from i to j by a position vector p. So, p j with respect to i along the x-axis, along y, and along z are the projections of the position vector p i to j along x, y, and z. So, this is the translation that has happened from i to j. So, components of that are reflected in the last column. What about u, v, w? u, v, w are the frame projections. You see u along x, u along y, and u along z. Similarly, v along x, v along y, and v along z. w along x, w along y, and w along z. So, they all are like this. So, this is how a standard transformation matrix will have all their components arranged. So, I'll move on. This is what you already know.

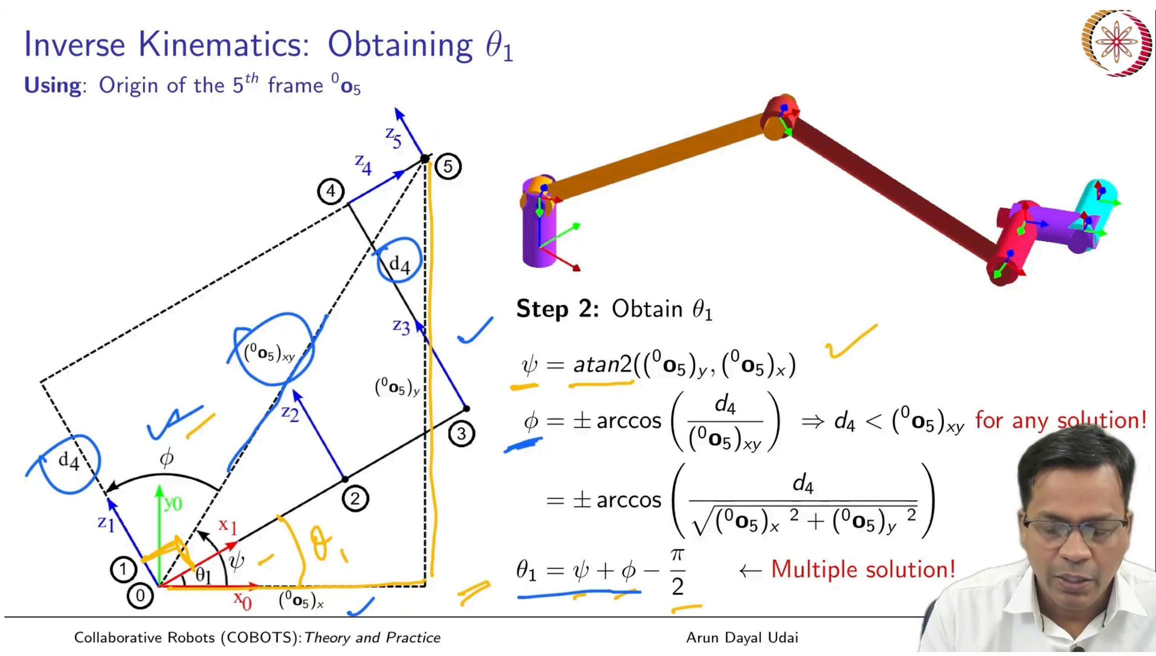


So, now the inverse kinematic problem of this robot. Actually, what is given here is the end effector position, which is ${}^0O_6$. So, from 0 to 6, that position is known, and the orientation of the end effector is also known. So, the end effector may be given in terms of Euler angles, maybe roll, pitch, and yaw. You can convert it to a rotation matrix, okay? The way we have done it earlier also, so in step one, what we will try to obtain is given frame six. So, I will try to figure out where is my frame five. In order to obtain that, this distance that is d6, so I will convert it to a vector. How can I do that? So, if this is the distance d6, so d6 is along the z-axis. So, d6 may be written as d6, 0, 0, 1, and this is in which frame? This is in frame 5. Frame 5 and frame 6 are both equally good here.

So, this is already there. So, if you know the end effector rotation matrix for frame 6 with respect to O, this gets converted to this vector. The vector which is here is now converted in frame 0. So, when you see this is in the local frame, d6, this is in the local frame. So, if I multiply this with ${}^0R_6$, it gets converted to the magnitude remains the same, but the vector is now expressed in frame 0. Now, the vector triangle is formed. What is given is $^0O_6$. This we have already obtained. So, using this vector triangle, I can quickly obtain the location of frame 5 with respect to frame 0. So, this vector is now obtained. So, this is $^0O_5$. This vector is now obtained using this equation.

$$^0\mathbf{o}_5 = {}^0\mathbf{o}_6 - d_6\,{}^0\mathbf{R}_6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This is using the vector triangle. So, we have obtained frame 5 now, ok. This is what we have obtained with respect to O.



Now, step 2. Step 2, what I have done, ok. What is given is $^0O_5$, which is the fifth frame that we have obtained in the previous slide. So, I already know the location of frame 5 that is sitting here, frame 5. So, this location is known. So, no,w if I look from the top, it looks like this. So, you see, you have Z1, this is Z2, and this is Z3. So, all three, Z1, Z2, and Z3, are visible here. And then comes your Z4, I assume it has rotated the way it is shown. So, it is Z4. And finally, you have Z5 that comes like this.

Looking from the top, it looks exactly like this. Now, we can obtain theta 1. How do you see? So, whatever the position of 5 is, we have already obtained 5 with respect to the O frame, the first frame, and the 0 frame, which is attached to the ground. So, this is known.

So, we already know the ${}^{0}O_5$ along x and along y. So, these two are known using that. So, I can quickly find O5 xy. So, along the xy, that means the square of this plus the square of this, and the square root should give me this. So, psi can be obtained directly from this. So, you take this, and you take this; you can calculate psi. In the second go, what will I obtain? I will try to obtain this one, phi (ϕ). This (ϕ) can be quickly calculated using this ($({}^{0}O_5)_y$).

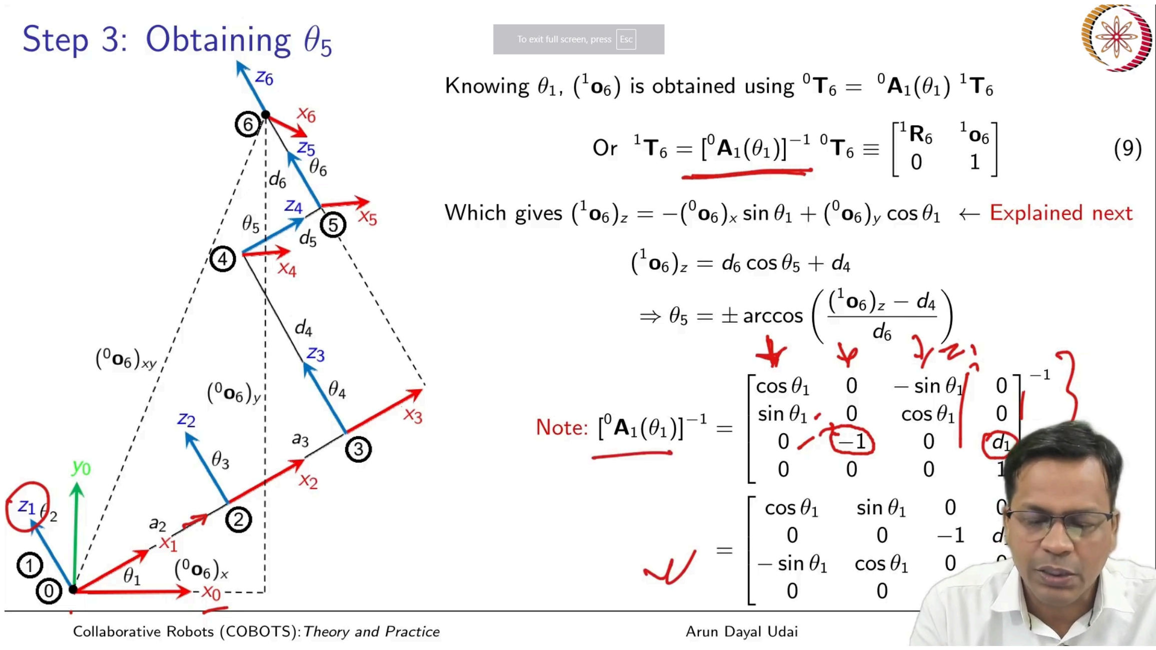
Now, we will consider another triangle, and in this triangle, you can calculate this. So, this is what if this is your d4; this also is your d4. This is the right-angle triangle. d4 is already known, and this we have obtained in the previous go over here. So, using these two, now I can calculate the angle phi. So, this I have put here. Knowing psi and phi, I can quickly calculate the angle theta 1 like this.

$$\Theta_1 = \psi + \phi - \frac{\Pi}{2}$$

So, theta 1 is equal to psi plus phi minus pi by 2 because this is pi by 2, and psi plus phi minus pi by 2 will give you theta 1. So, this is your theta 1 that can be obtained. So, I will just quickly recalculate.

So, psi is equal to using arctangent 2. I have used this, and I have used this; I calculated psi. Next, phi is equal to squaring these two, adding them, and taking a square root; I obtained this side, that is this one. This is equal to d4 from here. So, now I can calculate phi. Phi is obtained.

Using phi and psi, I have obtained theta 1. So, theta 1 is obtained. You see, it has multiple solutions, you see, because this is in plus or minus. If you put it here, this also has multiple solutions.

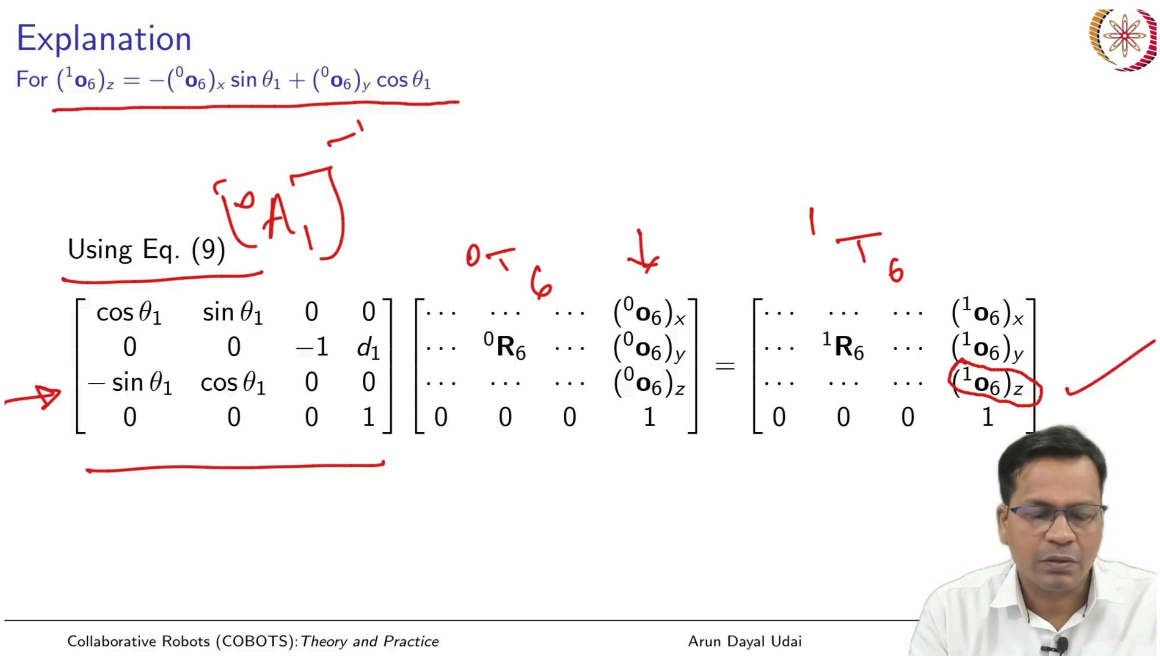


Now, moving to the next, I will try to obtain theta 5. How will I do it? Again, I will obtain. This is already given. You see, ${}^{0}T_{6}$, that is, the end effector frame position and orientation is known; that means ${}^{0}T_{6}$ is known. So, that is a product of now that I have already obtained theta 1 in the previous step. So, what will I do? I will use ${}^{0}A_{1}$, which is the transformation matrix from frame 0 to frame 1, that comes here into the transformation matrix from ${}^{1}T_{6}$. This is the link transformation ${}^{0}A_{1}$ into ${}^{1}T_{6}$, the transformation ${}^{1}T_{6}$. So, if I take the inverse, you can write it like this, and finally, you will get something like this. So, this is nothing but this is quickly obtained using theta 1; I can obtain the transformation from ${}^{1}T_{6}$. So, now, when I know the transformation from ${}^{1}T_{6}$, this is already there from ${}^{1}T_{6}$.

So, the component along z is used here, okay? How much is that, actually? So, see, this is the component along z. So, whatever is in the z1 direction, so in this frame, you are looking at frame six, okay? Frame six. So, the z6 component along z1 is along the same direction, okay? So, that will come out to be like this, okay. So, how much is that? So, this whole distance is actually $({}^{1}O_{6})z$. So, whatever is from 1 to 6, from 1 to 6, whatever is the vector, the position vector from 1 to 6, the z component of that will be along this direction, and that is equal to this. So, if this is the vector, let me just draw it. If this is the vector, the z component is this one. So, that is equated to two things here. I have just cleaned it up. So, this is your z component, which is here. So, that is equal to two things. First is d6 cosine theta 5. So, let me just draw it here once again. So, if this is yours, over here, you are measuring theta 5. So, whatever this one rotates, that is the angle made by the d6 branch. So, d6 makes an angle. So, d6 is something like this. So, if this is your theta 5, whatever is d6 over here, that will rotate by an angle theta 5, is it not? So, d6 cosine will actually come along the z6 direction.

So, that projection is now used. So, if d6 comes from here to here, So, this becomes your d6 cosine of theta 5. So, that is visible here, and d4 is the distance already along the z-axis. So, d4 plus d6 cosine theta 5 becomes the z projection of 1 to 6.

So, using that, I can quickly obtain theta 5, which is equal to this. So, this is further extended to this. So, now, how much is ${}^{0}A_{1}$ inverse that I have used here? That is very trivial, so you can just see from the top and calculate it manually also. Z projection, this is y, this is x. So, if you see your x1 along x0, it is x1 along x0, x1 along y0, x1 along z0. Similarly, y- y is directed downward, as you see in the original figure. So, it is y along z actually; it comes along z and downward. So, it goes like this. This is the projection of z1 along x0, z1 along y0, z1 along z0. So, that is here, and this is simply the distance from 0 to 1. So, it is d1 along the z-axis. So, this can also be obtained quickly. So, if you take the inverse, it comes to this. This I will be using next. So, I have used it here also.



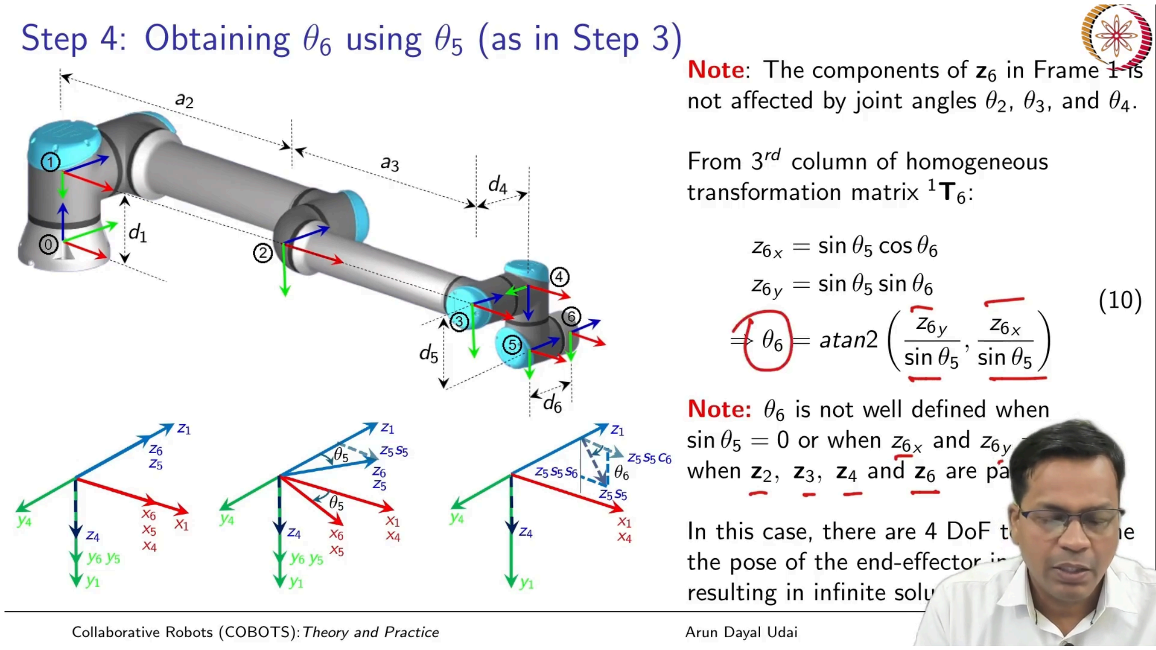
So, now using this equation here,

$$ {}^{1}\mathbf{T}_6 = [{}^{0}\mathbf{A}_1(\theta_1)]^{-1}\, {}^{0}\mathbf{T}_6 \equiv \begin{bmatrix} {}^{1}\mathbf{R}_6 & {}^{1}\mathbf{o}_6 \\ 0 & 1 \end{bmatrix} $$

I have just expanded it using equation 9. So, this is the inverse part of it ${}^{0}T_6$; this is ${}^{1}T_6$. So, this was the inverse of ${}^{0}A_1$. So, if I use it, I can quickly obtain this part.

So, this is what was used here. So, that you can quickly obtain. So, it is basically this part of this. So, you can quickly see it here. It comes out to be this into this. This into this gives you this. So, this is how I have obtained that. So, now you can recall all of this quite easily.

So, what is this now? So, the d6 projection along this direction is actually d6 cosine theta 5. Theta 5 is the angle moved by this link, okay? So, that cosine theta 5 d4 is already along this direction. So, the total distance along this direction is this, and 1 to 6 this vector projection along this direction is this. So, equating both of them, you can obtain theta 5. So, we have obtained theta 5 now. We have already obtained theta 1 in the previous step. Here, we have extracted theta 5 also.



So, now obtaining theta 6, okay? In Theta 6, you see three stages I have shown for three consecutive angles. The components of Z6, you see Z6, this is your Z6 in frame 1. So, this is your frame 1, which is not affected by joint angles theta 2, that is, this angle, this angle, and this angle or all the angles that are parallel to that. So, they do not affect the projection of z6. z6 on z1. Along this direction, there should be no change because it can only change the x and y components, but definitely, z along z is not changing because of any of these rotations as they are all parallel to each other, that is, theta 2, theta 3, and theta 4. The third column of the homogeneous transformation matrix ${}^1T_6$ should be this. Let us just derive it geometrically also.

So, this is the initial first condition when none of the joints have rotated 5 and 6. So, this is just a function of 5 and 6 because 2, 3, and 4 are not affecting it, and also, it is from one to six. It is from one to six anyway. So, theta one is also not affecting it. Two, three, and four are not present here. Theta one is not there, so it is only five and six that will come here. That is done now.

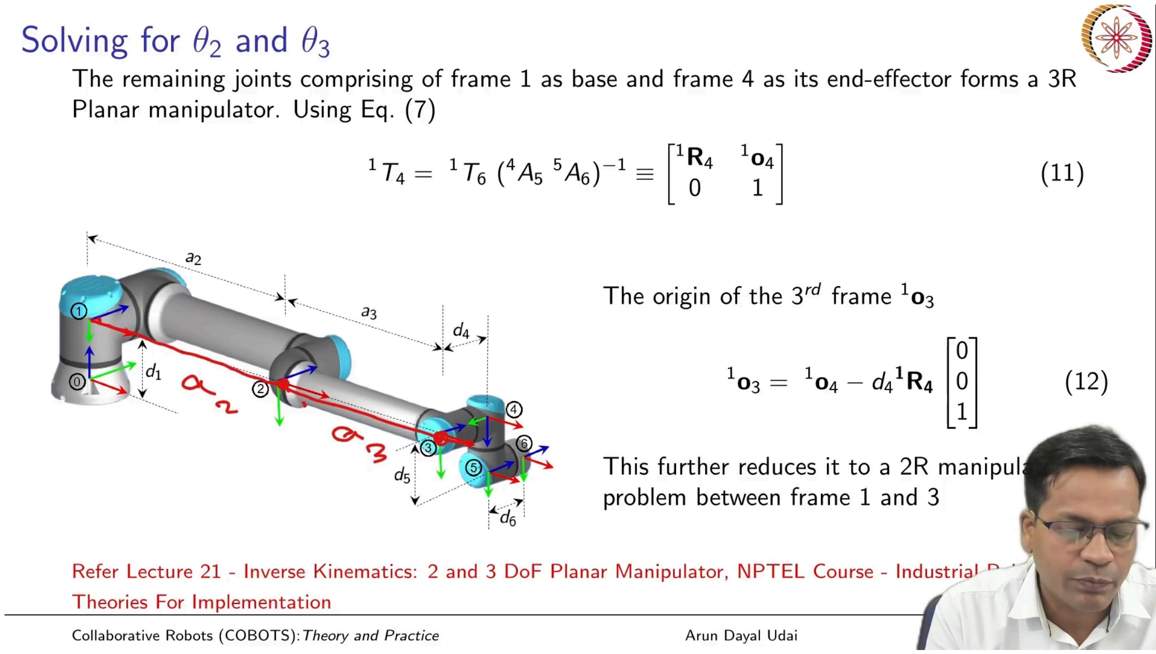
$$z_{6x} = \sin\theta_5 \cos\theta_6$$
$$z_{6y} = \sin\theta_5 \sin\theta_6$$

Let us understand the exact value of that. So, this is how all the frames are oriented initially when there is no rotation. It is in the default state. So, Z1, Y6, Y5, Z4 are all just copied to one single location. Now, let us say theta 5 has moved. So, what will happen? X5 and X6 come out of X1. Z5 and Z6 also come out of Z1. By an angle theta 5, the remaining remains as it is. So, Z4 along Z4. Y5, Y6, and Y1 remain as it is. This is the first rotation, and Z6 actually comes out from here to here. So, this is your new Z6 position.

Now, this Z6 again has a second rotation, that is, by an angle theta 6. So, what will happen to this theta 6 basically? This will actually pull out Z6 further and bring Z6 to this position. So, this is your final Z6. Now, I will project it back and see how much the projection of Z6 is along x and along y in frame 1 itself. So, that is what. So, what I will get here is this. When it came out by theta 5. It came here, and in the second go, it came here. So, the projections I have written here also. So, the same comes at these two locations. So, it is actually a unit vector. So, you will see only the sin theta 5 cos theta 6 as z6x. Similarly, z6y will be sin theta 5 sin theta 6. So, theta 6 can be obtained by these two. So, that is written as arctan2(z6y by sin theta 5, z6x by sin theta 5).

$$\theta_6 = atan2\left(\frac{z_{6y}}{\sin\theta_5}, \frac{z_{6x}}{\sin\theta_5}\right)$$

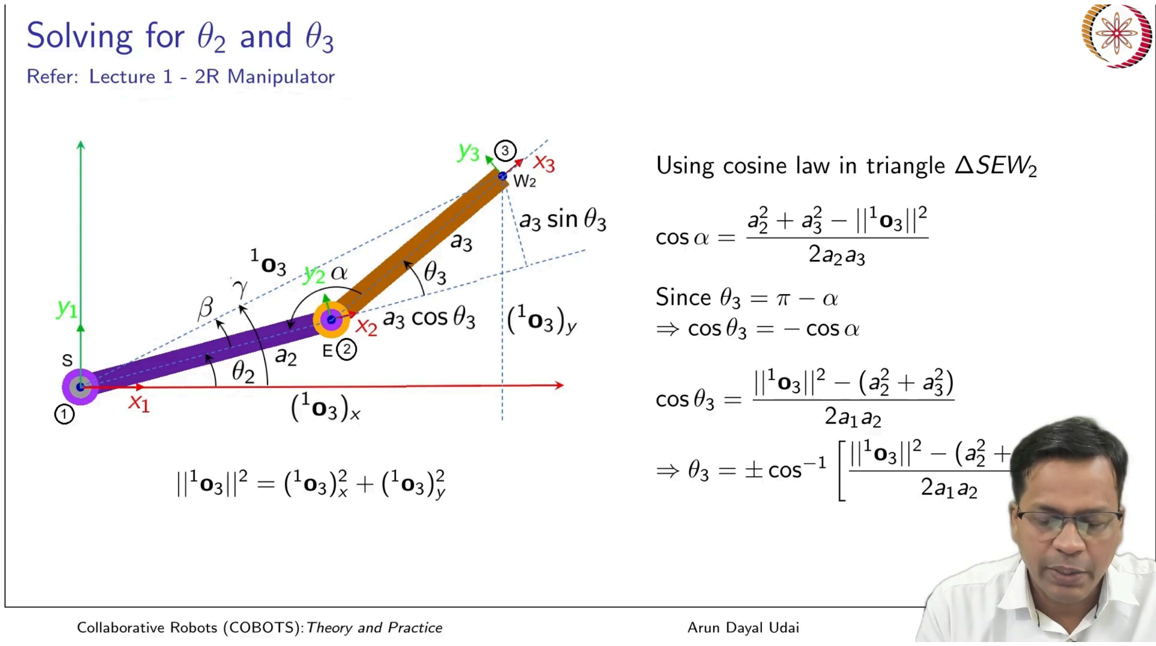
So, I could have struck it off, but yes, that helps me to understand when it is not solvable. So, theta 6 is not well-defined when sin theta 5 is equal to 0 or when both of them become equal to 0 that is, when z2, z3, z4, z6 are all parallel. In this case, there are 4 degrees of freedom, all the 4 to determine the pose of the end effector in the plane that results in infinite solutions. Let me just vanish for a moment. So, this is how I have obtained theta 6. So, what all angles have I obtained? I have obtained now theta 1, theta 5, and theta 6.



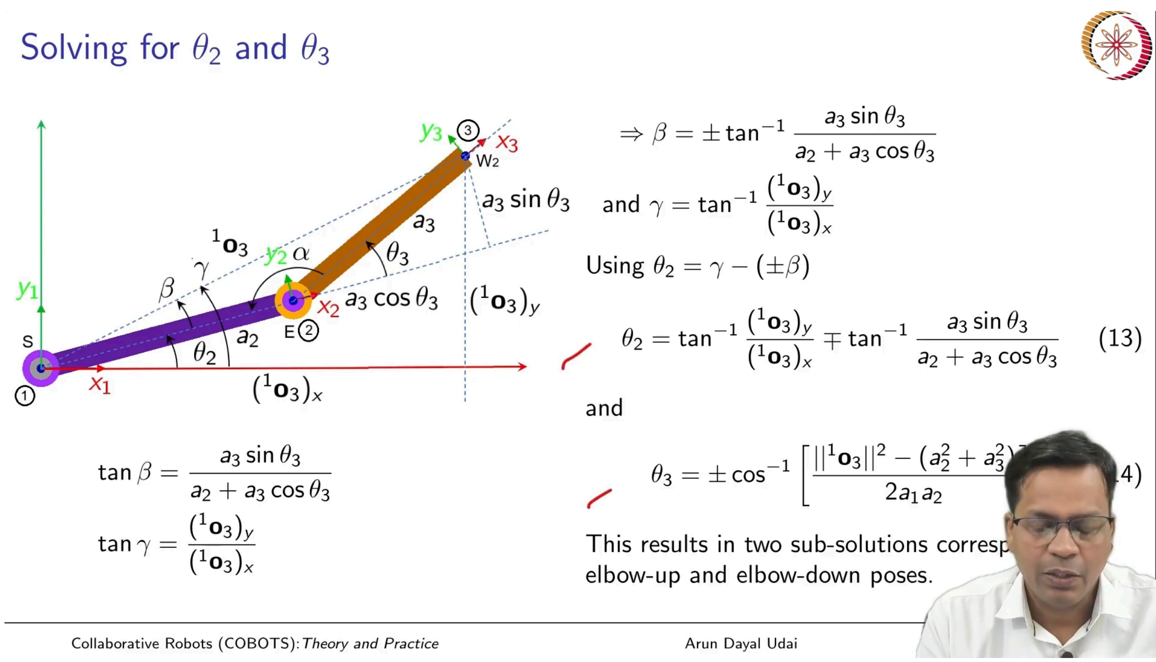
Now, solving for theta 2 and theta. Where are they? It is here and here. Now, using equation 7, what was that? ${}^1T_6$. So, ${}^1T_6$ was already there. In which if I post-multiply this with the inverse of this. So, actually, ${}^1T_6$ would go like ${}^1T_4$ into ${}^4A_5$ and ${}^5A_6$. So, this can be taken up here, and this is written ${}^1T_6$ was already given because you already have the end effector position and orientation. So, this was known; these two can be evaluated using theta5 and theta6. So, what I now know is ${}^1T_4$ from one frame to the fourth frame; that transformation is now known, the values I have.

The origin of the third frame, that is, ${}^1O_3$ from here to here, can be obtained similarly. You have to come back by the d4 vector. The d4 vector was in this frame, okay, the local frame. So, multiplying with ${}^1T_4$, that can be obtained from here. I can quickly create a vector triangle the way we did it, and I can obtain ${}^1O_3$. So, this is the vector that connects from frame 1 to frame 3.

Now, this further reduces it to a 2R manipulator. If you look carefully, so from 1 to 3, this is the first link, this is the second link, okay? That ends here with link length a2 and a3, okay? So, it is a two-link problem. It is a planar manipulator in this frame, in the first frame, okay? So, it can be quickly solved the way we did it for the 2R manipulator, okay?

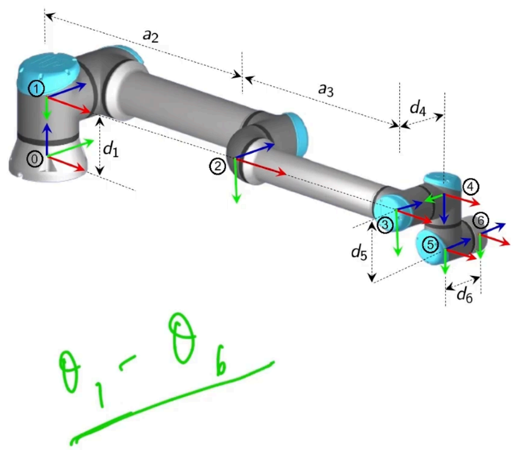


So, that is used here using the previous method only. You can refer to Lecture 1 for the 2R manipulator, and I have obtained theta 3,



and similarly, I have obtained theta 2. 2 and 3 are obtained using the 2R subset of this manipulator. Now, all the angles are known except angle theta 4. How to go for it?

Solving for θ_4



From equation Eq. (11)

$$^3T_4 = ({}^1A_2\ {}^2A_3)^{-1}\ {}^1T_4$$

From the first column of $^3T_4$,

$$\theta_4 = atan2\left[\ ({}^3x_4)_y,\ ({}^3x_4)_x\right]$$

Overall with all the combinations of multiple sub-solutions, the UR Arm has eight solutions for a given pose.

So, from Equation 11 that we have seen, we already had $^1T_4$. So, I will further take the inverse of $^1A_2$ and $^2A_3$ because we already know theta 1 and theta 2 here. So, using that, I can take the inverse and I can quickly find out $^3T_4$. Where is it? From 3 to 4. So, if I know $^3T_4$, what can I do? Theta 4 is simple, but it is this, okay. From 3 to 4, x projection along x, projection along y, this can be obtained from this, okay, from this, and that helps me to find out the joint angle theta 4. Got it? So, if this is your x. So x basically rotates by an angle theta 4 like this. This is theta 4. So, x rotates by an angle theta 4 like this. So, next x, it is the x4, comes out by angle theta 4. Projection of x4 on x3 and projection of x4 on y3 are utilised here. So, That can be obtained, so that is the first column of this matrix, basically. So I obtained theta 4 also.

So now, overall, with all the combinations of multiple sub-solutions, so all the solutions they had one or two solutions. So, if I combine them all together, UR arm has eight solutions for a given pose. Not all may be possible due to the kinematic limitations or the joint limits that are there. So, with that, I have obtained all the joint angles theta 1, 2, and theta 6 for the UR arm. URarm kind of; there are many other similar arms, not just UR arm. UR arms have all the variants on this architecture only, but there are many similar arms also. For all of them, you can use the same technique and find out the inverse kinematics, okay?

That's all for this lecture. So, in the next lecture, I'll discuss Differential Motion Analysis: Robot Jacobian, Velocity, and Acceleration Analysis.

That is all. Thank you very much.