Introduction to Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 36
Problems and Solutions

Let us try to work out maybe one or two simple problems to illustrate these equations that we have developed. We will deliberately try to consider examples, where we can use the stream wise and cross stream wise components, or maybe the polar coordinate systems like that.

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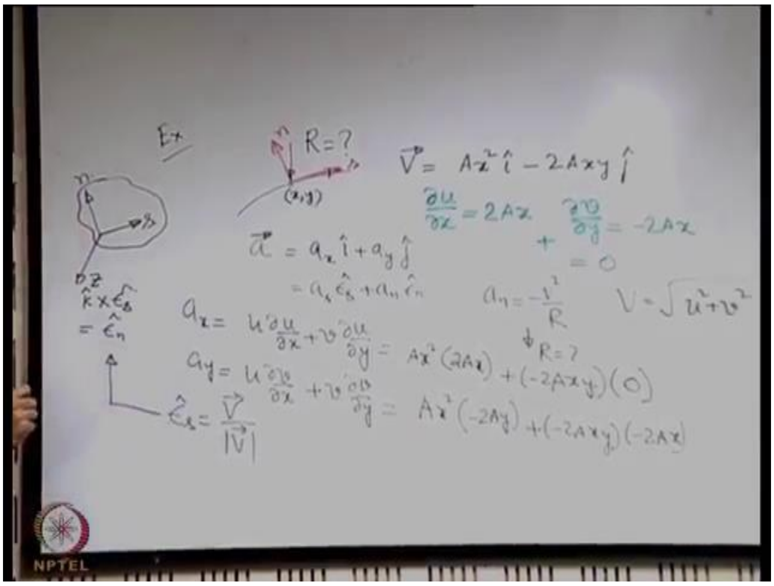

Let us take an example where, let us say that you have a streamline and the velocity vector is given not in the streamline coordinate system, but in the x y coordinate system. So, at a given point x comma y located on the stream line, the velocity vector is given like this, say it is A x square I minus 2 A x y j, where again A is a dimensional constant and x and y are the coordinates.

What we are interested to find out, so we are interested to find out what is the radius of curvature of the streamline at this point. Let us say we are interested to find out that, given this information. To do that, first let us see that what type of flow it is. So, that may not always be necessary for solving every problem, but it is not a bad idea to see that whether it is an incompressible flow or not. So, we need to check whether the divergence of the velocity vector is 0 or not. You can clearly see that the sum of these two is 0.

In fact, that is how I try to define the velocity field. So, that it becomes an incompressible flow. Now when you have this incompressibility in mind you can also try to write expressions in terms of stream functions and so on, but here our objective will be just to find out the radius of curvature. To do that, let us say that we are interested to find out the acceleration. See what can be our strategy, let us make a strategy and then we will find out relevant quantities. Say we find out the acceleration at the point a. So, this acceleration you can find out in terms of the x and y components. So, acceleration say it becomes a x i plus a y j. So, straightforward from the formula of acceleration it is possible to find out.

Now, this acceleration may be resolved in two components. This is also resolved in two components, but the components that we are very much looking for in the stream line coordinate system are s and n. So, this may be resolved, say along s, s is virtually like the tangent, but again I fundamentally told that it is not a tangent, it is it is basically located oriented along the streamline. So, s and n coordinates are there. So, this may also be resolved in such a way that you can write it as s into unit vector along s plus n into unit vector along n right. And once you get that you know that a n is equal to minus v square by r, where V is the square root of the, square of the components of the velocity. So, that should give us what is R. So, before calculating anything by a brute force it is not a bad idea to make a plan of, why we are going for such calculations.

So, now, we will not of course, go into the detailed calculation, because it is just a very trivial differentiation algebra, but at least we should try to figure out that how do we find out the unit vectors along s and n, because that may be the only botheration for solving this problem. So, when you write the acceleration components, say the let us quickly write that acceleration component, so what it acceleration along x? Here it will be. This is acceleration along x acceleration along y. So, you can write acceleration components along x and y. So, if the numerical values of the coordinates x and y are given you may substitute to get some values. Now, how do you get the epsilon s, how do you get the direction of the unit vector in the stream wise coordinates?

Student: (Refer Time: 06:51).

Yes.

Student: (Refer Time: 06:55).

So, V is oriented along s. So, you can write epsilon s as the V divided by the mod v that is the unit vector in the direction of s how do you get epsilon n.

Student: (Refer Time: 07:11).

Yes, they are perpendicular it is well known, but how you exploit that.

Student: (Refer Time: 07:22).

There are many possibilities in which their dot products may be 0. So, if I have a vector like this.

Student: (Refer Time: 07:32).

They are perpendicular and dot product is 0, but like it may give this instead of this right.

Student: (Refer Time: 07:39)

Well if you do not if you cannot do it with a dot product you have to think for a cross product. So, let us see why for what purpose we are going to do that. So, if you have, say think about the coordinate system. Let us say we have a coordinate system like s n and z. So, s n are the things which are happening in the x y plane basically, and z is perpendicular to that. So, this again forms like a system of orthogonal vectors, may be the coordinates is the curvilinear one, but still orthogonal one. So, when you have such orthogonal basis. So, just think s like x is n like y. Not that they are x y we know that they are different form x y, but just to draw an analogy with the usual unit vector components and vector components along these. So, can you relate the unit vector along n with the unit vector along s and unit vector along s, sorry unit vector along z and unit vector along s?

Student: (Refer Time: 08:50).

So, unit vector along z is k. So, k cross with unit vector along s should give the unit vector along n.

Student: (Refer Time: 09:10).

Right; that means, once you get unit vector along s you can easily find out, because unit vector along s is written in terms of i and j the unit vectors along x and y, and then it is possible to utilize the cross products k cross i n k cross j to find out unit vector along n in terms of the unit vectors along x and y, and once you find that out the remaining work is very trivial. So, you can write this in terms of the acceleration along s and acceleration along n. So, how do you write the acceleration along s and n? So, acceleration along n is what is our interest. So, what is acceleration along n, say you know what is acceleration along x and y. So, x y are the coordinate systems like say this is x and this is y.

And more importantly you know the resultant acceleration. So, this is the resultant acceleration. So, what is the acceleration along n this is nothing, but the result and acceleration dot with the unit vector along n; that will give it is component along the direction of n very straight forward vector resolution, and then you can equate it with minus V square by R and get the values of R. Let us look into another example.

Let us say that you have curve or a bend that we are discussing, say there is a bend in a pipe line, so that the stream lines which were originally parallel to each other, they made remain parallel to each other, but they are now curved instead of being straight.

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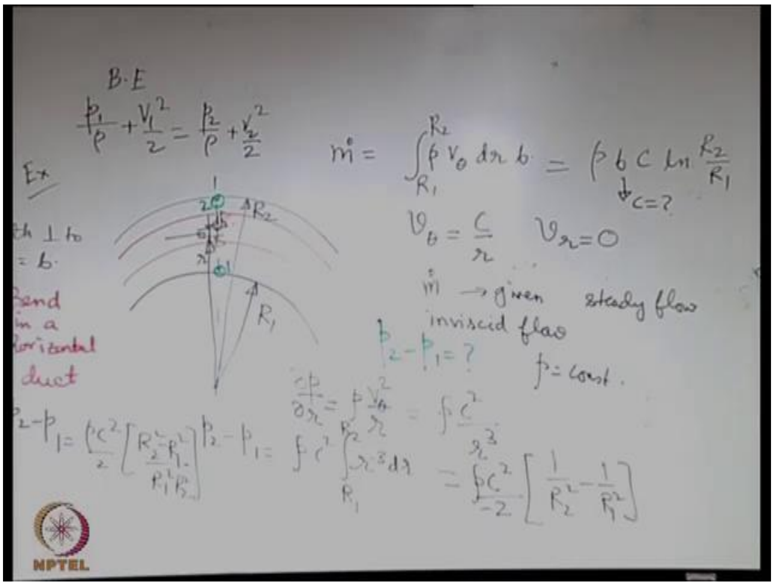

So, let us say that the stream lines are becoming like this. This is like a bend in a dot, bend in a horizontal dot. So, that we are not bothered about the change in height between various points, everything is taking place in a horizontal plane, and let us say that these are parts of circles. So, if these are parts of circles you have, let us say the inner radius of the bend as R 1 and outer radius of the bend as say R 2. Circular curves are special cases of general curve where the radius of curvature is a constant. So, when you have radius of curvature as a constant the good thing is, then if you use the polar coordinate system R theta that is equivalent of using the s n coordinating system, because then your R is always oriented along n, because the R is remaining constant and it is always the normal direction to the curve; that is the speciality of a circle which makes the manipulation much easier.

What is our objective in this problem? Our objective in this problem let us say that the velocity field is given. Typically when the fluid flows in a bend it is velocity field may be approximated in this form. So, we will write the velocity components in the polar coordinate system v theta equal to c by r, and v r equal to 0. What is the name of this type of flow we have discussed earlier, free vertex. So, this not that it is really a free vertex see important, one important aspect in science and engineering is to approximate or model the reality with something which is very close to that. It does not mean that actually the flow through a bend is given by this velocity field, but it has a lot of closeness with this situation; that is the only reason why we have chosen this velocity field. Now you have seen. So, the two types vertices you have seen. So, this is like a free vertex, and you also have seen like a something like a force vertex. You have to keep in mind that like no matter whatever is the vertex, the free vertex or the force vertex you have a v theta component for a force vertex v theta is c into r and v r equal to 0.

So, in general what can be call as a vertex type of flow? A vertex flow is a flow which is only v theta component of the velocity and v r is 0. So, different descriptions of v theta may give different types of vertices. So, fundamentally vertex is something which will have only the cross radial component of velocity, not the radial component of velocity. And that cross radial component of the velocity is a function of the radial coordinate that is also a very important consideration. Now this velocity filed is given c is not given, but say somewhat we experimentally measures, what is the mass flow rate through this dot and say that is given m dot, m dot is the rate of mas flow per unit time say kg per second that is given and assume there is steady flow. So, this m dot does not change with time, and also assume that it is an inviscid flow. If it is an inviscid flow then many interesting things are possible. One of the interesting things is that if the flow is originally irrotational, it is likely to remain e rotational. So, you can clearly see we have discussed this earlier that the free vertex flow is an irrotational flow.

So, this flow is an irrotational flow. Now what is our objective? Our objective is to find out the difference in pressure between two points 1 and 2. One is in the just close to the or adjacent to the inner radius within the fluid, and two is just at the outer radius. So, we are interested to find out what is p 2 minus p 1.

There are certain ways in which we can do it. Let us look into the, what are the ways. So, one of the ways is, you have d p or like the expression that we just derived. Now the partial derivative of p with respect to n is equal to rho v square by, say local radius of curvature right. Just now we have derived this, it is not a very old derivate just now we have derived this. So, here n direction is like the r direction, and when you are writing here v, this v here has only v theta components it is a vertex flow. So, this is as good as v theta square by r. Yes.

Student: So, whatever the rho g (Refer Time: 17:32).

It is in a bend in the horizontal dot so; that means, there is no change in vertical elevation, see every key word given for the problem definition has some implication and it is important that you put enough attention to all the keywords, which are given to describe the problem. So, this is the form, and if you look into books you will see that this they start with this form this might give you a falls impression, that this equation is fundamentally derived for r theta polar coordinate system. No this is fundamentally derived for the s n coordinate system, but special case of a circular geometry will make n as good as r and special case of situation, where your v theta is only v component will give this as v theta square. So, that you have to keep in mind.

So, whenever you have. Again I am repeating whenever you have a formula type of thing just try to be aware of the origin of the formula, that will help you in solving a problem which is exactly not this, a bit change from this one. Now you can the reaming work is easy you can just substitute v theta square; that is c square by r square. So, this will be become rho c square by r q. So, you can integrate this with respect to r. So, you will get p 2 minus p 1 is equal to rho c square r to the power minus 3 d r from r is equal to r 1 to r equal to r 2. So, you will get this as rho c square divided by minus 2 1 by R 2 square minus 1 by R 1 square. So, the final expression p 2 minus p 1 is equal to rho c square by 2 into r 2 square minus r 1 square by R 1 square R 2 square that is the final expression.

Now question is, you do not know what is c? c is not given it is just given that v theta is invasive proportional to r. So, to find out c, you require the description of the mass flow rate. So, what is the mass flow rate? Let us say that we considered a small element at a radius r considered a small element of this is local r. So, at a local r we consider a small element of with d r there is a flow across this element. So, what is the flow across this element let us. So, we have to what is the shape of the section of the dot which is just like one of the it is a edge in a plane, but it has it is perpendicular direction.

Let us say that in the other direction it has a uniform width. So, let us say that width perpendicular to the plane of the figure is equal to say b. So, width perpendicular to the plane of the figure is b; that means that you know that area of this stream b into d r and what is the volume flow rate that is flowing through that see v theta is perpendicular to that. So, v theta into b into d r that into rho is the mass flow rate and integrated with respect to that from r equal to r 1 r equal to r 2 will give you to the total mass rate. So, you can write m dot as integral of rho v theta d r into b from r equal to R 1 to r equal to R 2. So, rho v theta you can write c by r. So, from here you can find out what is the value of c, and when you know what is the value of c you can substitute here to get if you know given mass flow rate why it is important, because by some measurement device you can measure the mass flow rate, and by that you can directly tell what is the pressure difference between these two.

Problem is solved, but as I have told quite a few times, our job may not be completely done. Let us try to look into the problem in a bit different way. Let us say that we are very crazy and we just simply interested to find out the pressure difference between the points 1 and 2 by the use of the Bernoulli's equation, why because we are always tempted to use such an equation, and anytime whenever a problem in fluid mechanics is given, I have seen that it is like a every student feel so happy that when the Bernoulli's equation may be used.

So, let us say that there is such a happy student, who is interested to use the Bernoulli's equation, and when that is being used. So, if you apply the Bernoulli's equation between points 1 and 2. So, a better student will say that, have you checked they are located along the same stream line, and we can clearly see they are not located along the same stream line, but the other students will say, that still let us apply that. So, if you apply that p 1 by rho plus v 1 square by 2 is equal to p 2 by rho plus p 2 square by 2 other assumptions are, they are like inviscid flow it is already given steady flow this is given, and we are assuming that rho is a constant; that is a additional assumption that we are making. In fact, that assumption we made earlier while taking out rho from the integral in the mass flow rate calculation rho is like a constant.

So, what is the logic of other students? Logic of the other students is somehow if I get my answer right, I will not clear that how I have solved the problem which is true for most of the students. So, then you can find out, what is p 1 minus p 2 and this is a very simple exercise.

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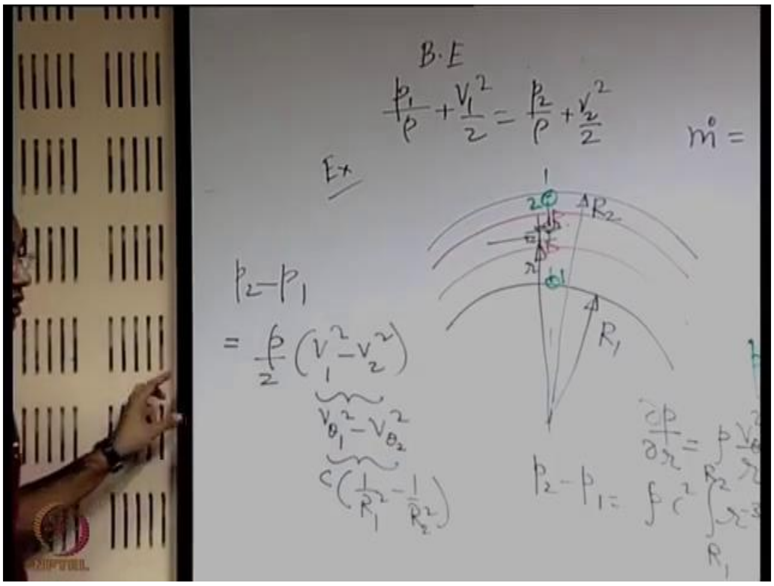

So, you can write p 2 minus p 1 is equal to rho into rho by 2 into v 2 square, sorry v 1 square minus v 2 square and v 1 and v 2 are like v theta 1 and v theta 2, because they have no other components, so this as good as v theta 1 square minus v theta 2 square. So, this is as good as c into 1 by r 1 square minus 1 by r 2 square. It is exactly the same what we got through this, and the reason is straightforward, because it is an irrotational flow. You can use the Bernoulli's equation between any two points, no matter whether they are located on the same stream line or not. So, this is not magical or a coincidence that it is like this, but you have to keep in mind that once you are applying the Bernoulli's equation between two points. See whenever you are solving problems what we see every year is just I am narrating experience people always writing the Bernoulli's equation between points 1 and 2.

You have to be very careful that whether you can write Bernoulli's equation between those points 1 and 2 if irrotational flow, you do not care really where they are located, but if not you have to make sure that located along the same streamline. So, if they are not located in the same streamline, even if your final answer comes correct, it is fundamentally wrong, and it is not acceptable. Here you get read of that complication simply, because it is an e rotational flow. So, you can apply Bernoulli's equation between any two points in the flow field, no matter whether they located along the same streamline or not provided other assumptions of the Bernoulli's equations they are valid. Now, we will work out some additional examples to illustrate the unsteady Bernoulli's equation. So, till now we have discussed about the steady versions of these equations, what we have made a remark that it is possible that you have an unsteady version.

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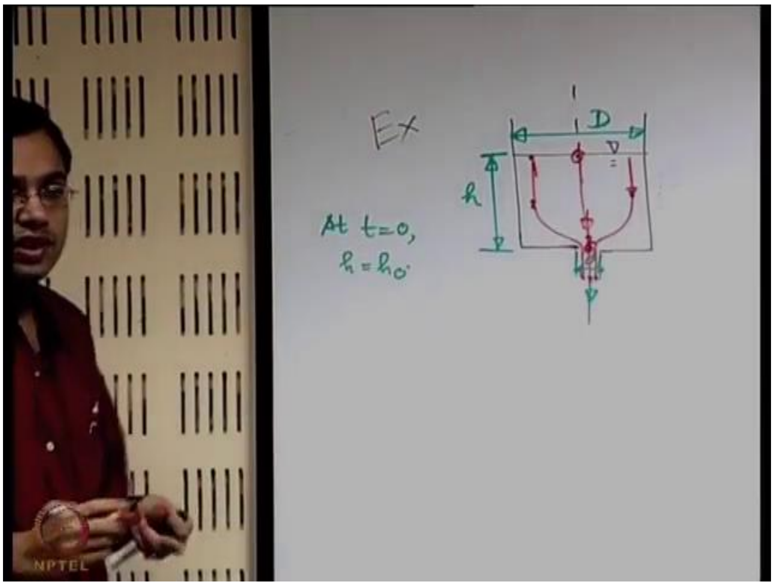

So, to look into such an example we will again consider a case which you have encountered many times in your earlier experiences with solving fundamental very simple flow problem, say you have a tank which has water to some depth, and there is a small opening at the bottom of the tank through which water is being drained out. So, you are interested to find out, how the height of the tank or the depth of the tank is changing with time, because of the water that is being drained out. Let us say that the original height. So, let us call this as h, and let us say that at time equal to 0, you have h equal to h 0 which is the original depth of water in the tank, and then you have opened say some exit line or escape root for the water. So, that water is coming out.

Let us say that this is this is a cylindrical tank. So, let us say that you have this as the diameter D, and the diameter of the escape as small d. So, our objective is to find out the time which is may be required to empty the tank as a special interest, but here the general interest will be, to write the equation of motion for this case. So, let us say that we are interested to write. Now here you can clearly see it is an unsteady flow. Since it is an unsteady flow, when you are using a Bernoulli type of equation you have to be careful, it is if you are not sure whether the steady or the unsteady version has to be used retain the unsteady term, and see that whether it is appropriate.

So, let us say that you have let us consider some streamlines. So, how will the streamlines look like originally they will be parallel? So, all the streamlines, but when they come close to the constriction, they will try to come close to each other, because the streamlines then have to be confined through this small hole. So, they are coming closer to each other. So, you can see that there curvatures of the streamlines they are created, close to the exit point. Similar thing is possible if you have a like a hole in the side not in the bottom, I mean fundamentally there is no big problem no big difference.

Now, if you take say two points 1 and 2 located on the stream line. Say you have one stream line and you have two points. So, 1 and 2 and let us say that other assumptions of the Bernoulli's equation are valid.

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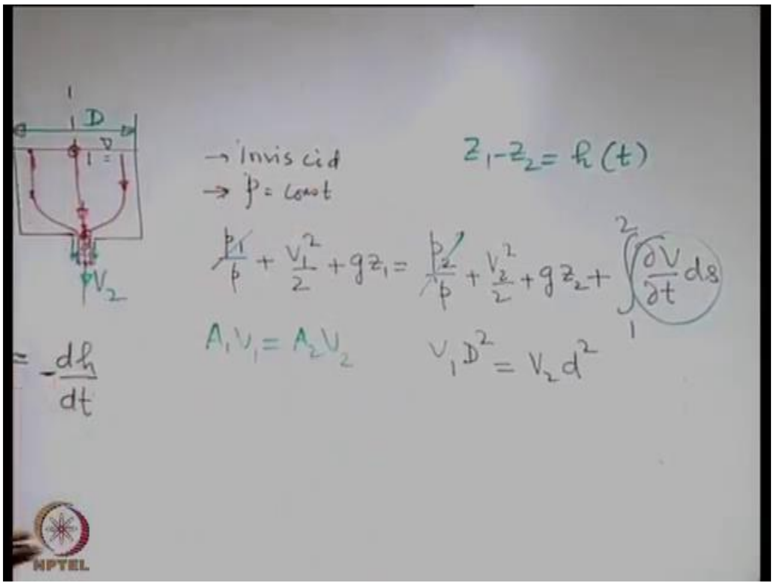

That is, it is inviscid flow rho equal to constant these two are the other assumption that we are considering. We are considering flow the equation along the streamline which is there on point 1 and 2, and let us write the unsteady Bernoulli's equation.

So, you can write p 1 by rho plus v 1 square by 2 plus g z 1 is equal to v 2 by rho plus p 2 square by 2 plus g z 2 plus. So, this extra term is there, because of the unsteady version of the equation. Now let us see when you are considering the points 1 and 2. So, 1 is exposed to atmosphere, and 2 is also exposed to atmosphere. So, at both locations the pressure is the atmospheric pressure. So, this two are the same, so they get cancelled out. You also can clearly see that the difference between the heights at 1 and 2 z 1 and z 2, this is h, which itself is the function of time, because as water is coming down h is decreasing. Now v 1 and v 2, you may relate v 1 and v 2 with the D and small d by writing like A 1 V 1 equal to A 2 V 2. Again so this is the form of the continuity equation for constant density, steady flows.

What are these V 1 and V 2? These are fundamentally the average velocities at the sections where 1 and 2 are located. But because we are considering inviscid flow, we are implicitly assuming that this velocity profiles are uniform; that means, V 1 and V 2 are as good as the velocities locally at the points 1 and 2, only valid if is a uniform velocity profile; otherwise it has to be replaced with an average velocity. So, A 1 is proportional to D square A 2 is professional to small d square. So, we can write this as V 1 D square D square is equal to V 2 into small d square. So, you can relate V 1 and V 2.

Now you can also write one important thing; that is what is V 1, V 1 is, it depends. First off all you can write that V 1 is dh dt, because what is the velocity at these section locally it is a velocity representative of the rate at which there is a change in level of this. So, h being a fixed height the rate with respect to which these h changes, is given by dh dt. Now we have to see that you must have an appropriate sign adjustment in this equation; think of A 1 V 1 equal to A 2 V 2, if you take V 2 as like this the velocity the velocity sends with which respect to which it comes out. So, V 2 is positive then.

But if you write V 1 and V 1 is dh dt is itself negative. So, it should be adjusted for this equation with the minus sign. So, minus sign or plus sign is based on your sign conventions. If somewhat you write this with the positive sign, actually there is fundamentally no wrong in that, because it is your sign convention. So, the sign convention is like whether you may say V 1 is positive in the downward direction or in the upward direction that is up to you but you have to keep in mind that V 1 and V 2 sign convention should be consistent. So, V 1 is like positive here and d h d t itself negative. So, sorry V 2 is positive downwards dh dt is itself negative. So, it has to be adjusted with the minus sign. So, you can relate V 1 and V 2.

The third thing is how do you approximate these term, how do you approximate these term is very important, because you explicitly do not have an information on how V varies with time. So, you have to make an approximation, see in engineering we always do approximations to get a feel of what is the magnitude of these term, and is it important with relation to the other terms if as compared to the other term other terms, this term is not important then we can drop it and we can utilize the study version of the equation, but if not we may not be in a position to drop it, and we should retain it even in an approximate form.

So, in the next lecture we will see that how to treat these special cases of this unsteady terms. So, we stop here for this lecture.

Thank you.