

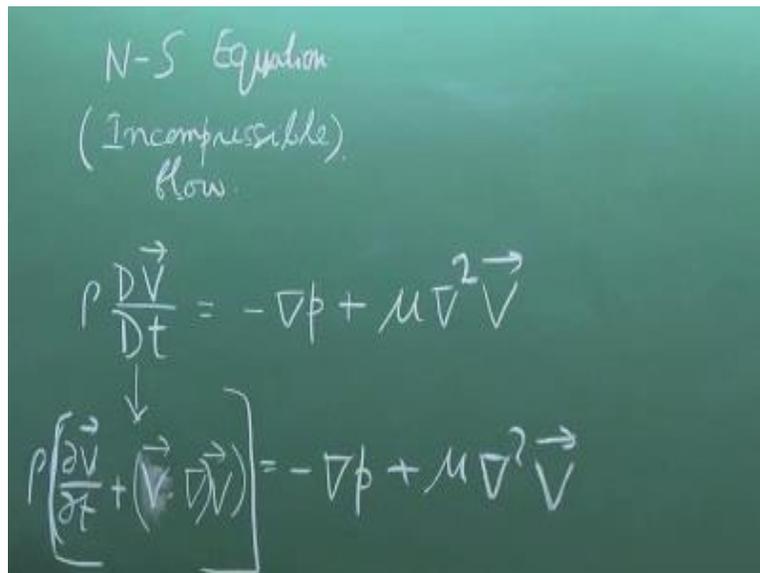
Conduction and Convection Heat Transfer
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Lecture - 23
Review of Fluid Mechanics – IV

Good afternoon and welcome to all of you to the session of Conduction and Convection Heat Transfer. Today, I will discuss the exact solutions of Navier-Stokes equation. So, last few classes from the Suman Chakraborty, derived the Navier-Stokes equation which is basically the equation of motion for the flow of a viscous fluid and you have already recognized that the Navier-Stokes equation is a non-linear, second order partial differential equation.

If I just write the Navier-Stokes equation again in vector form.

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The image shows a chalkboard with the following text and equations written in white chalk:

N-S Equation
(Incompressible)
flow.

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \mu \nabla^2 \vec{V}$$

↓

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right] = -\nabla p + \mu \nabla^2 \vec{V}$$

Let me write the Navier-Stokes equation in a vector form if you recollect, let us consider this in an incompressible flow; Navier-Stokes, N-S Equation, Incompressible flow for a simple case, for our understanding, we just exclude the term for incompressible flow then the Navier-Stokes equation stands like this, in the vector form—I am writing in a vector form, minus grad p + Mu Laplacian of the velocity vector del square.

Now this is the initial term, these are all per unit volume, this is the acceleration is the pressure term, pressure force per unit volume, this is the viscous force. Now this can be splitted, this acceleration part in terms of its temporal acceleration plus and this remain as it is. “Professor - student conversation starts” We done this already I think Prof. Suman Chakroborty has told you “Professor - student conversation ends”.

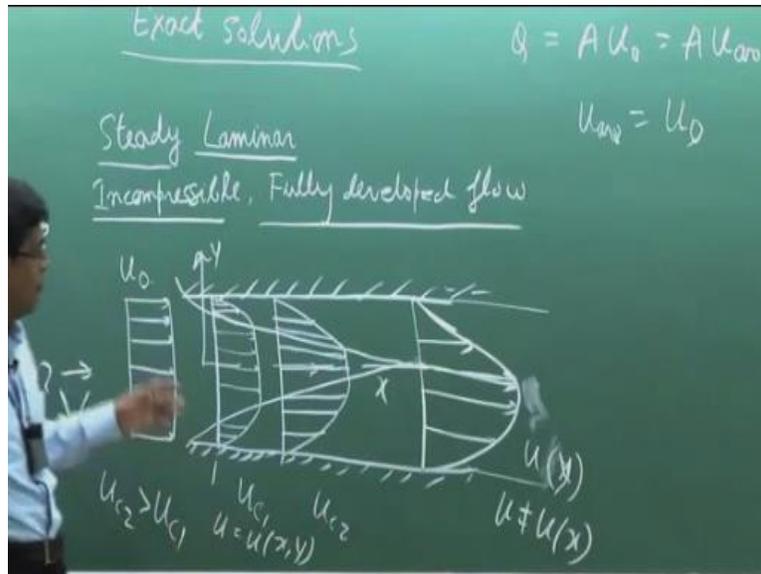
Now here, it is easy to recognize, first of all this is a—again I am repeating the thing, this is a partial differential equation because the velocity which is the dependent variable is a function of space coordinates and time. In this acceleration part of the initial term, the advection terms are the non-linear $\mathbf{V} \cdot \text{grad } \mathbf{V}$; $\mathbf{V} \cdot \text{grad}$ is an operator acts on \mathbf{V} – Vector \mathbf{V} . And this gives rights to the non-linear term.

This term is the linear term $\text{del}^2 \mathbf{V}$, Laplacian but this contains the second order term. Because of this non-linearity in the initial term even for steady flow this maybe 0 but convective term will not be 0. There is no exact analytical solution in close form for Navier-Stokes equation for all purposes of fluid flow. But there are certain classes of fluid flow where we have exact solutions.

We have exact solution when the initial term identically vanishes, all the term vanished and those situations pertain to a simple linear equation which is amenable to solution, analytical solutions, and those case are known as Exact Solution. And those cases where the initial term identically vanished become 0 and Navier-Stokes equation becomes a linear equation and we can have its analytical solution in close form known as Exact Solution.

And these cases in very short I will tell you pertain to this type of flow.

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Steady, Laminar, Incompressible, Fully Developed Flow. Now each and every term I think is known to you. Steady flow you already know from your fluid mechanics knowledge. Laminar flow is also known. Okay Laminar flow is also known – what is Laminar flow – if time permit I either myself or Prof. Suman Chakroborty will give you a brief idea about the Laminar and turbulent flow which you have already covered in your fluid mechanics class.

Similarly, the incompressible flow and the fully developed flow. This fully developed flow only I will describe today which will be again describe afterward when we will discuss the boundary layer theory. Probably all these things are known to you and I hope that fully developed flow also is known to you in your earlier fluid mechanics class, but still I try to tell you what is the fully developed flow.

This you have to understand better because these things are much easier in compare to understanding of fully developed flow. Fully developed flow is like this. Let us consider the flow through a channel comprising two fixed place. Let us consider a channel, this is the axis. And let us consider a uniform velocity, a flow with a uniform velocity of u infinity for example or u_0 approaches these channel made by two fixed place.

Then what will happen, the boundary layer will grow from the leading edge like this from the two surfaces and they will eventually merge at the axis, the boundary layer will grow. The

concept of boundary layer I told that within which the velocity gradient will exist that means, the viscous action of the fluid will be restricted within the boundary layer, because of the diffusion of momentum.

This is the consequence of the interaction between the fluid and the solid surface added by the-- or interfere by the fluid property viscosity. That already I told this concept while discussing the introduction to Convection. So, if it is showed then that boundary layer grows like this. One will expect a velocity distribution like this at the section very close to the inlet. Let this section is one, we will expect a velocity distribution like this.

That means, there will be a variation of velocity from both sides from 0, this is the no slip condition that means the surface is at, rest of the fluid velocity level will be 0. And beyond the boundary layer the fluid velocity will be constant there will be no variation of fluid velocity in the y direction. Let us this is the y and this is x. So therefore, at this section we see, very close to the plate there is a region that is known as boundary layer which we will be told afterward where viscous affect is this, the velocity gradient is there.

Beyond which where there is no viscous affections the velocity is like this one inlet velocity, uniform in the y direction. Now as the boundary layer growth in the direction of flow at another section downstream if you plot the velocity distribution you will see it is very simple to understand that, ultimately this is known as potential core and this is known as boundary layer region. Potential core or Potential region where the velocity is uniform and there is no shear stress.

Now this potential core or potential region is reduced, and ultimately it will be reduced to 0. But the interesting fact is that, the centre line velocity is increased for a steady incompressible flow. Why? That means if here we consider the centre line velocity section 1 is use u_{c1} and this centre line velocity use u_{c2} that means this centre line velocity—centre line means the potential core velocity—velocity in the potential region.

Then u_{c2} is greater than u_{c1} that means the potential core is accelerated. Potential core the velocity is increasing. Why? Very simple. For a steady incompressible flow, the volume flow rate at each section is constant and if volume flow rate at a section can be written as area into average velocity. Now at the inlet if we write it will be A into u_0 that is uniform velocity. At any other section it will be A u avg. So therefore, you see u avg in this case is equal to u_0 .

That means u avg is u_0 . That means at each and every section u avg is constant because of the constant volume flow rate. Since area is constant the average velocity is constant and equals to the u_0 that is the inlet velocity. Since more is the boundary layer as the flow proceeds along its direction in the downstream that means more fluid is slowdown because of the viscous affect in the solid while interaction, the centre line or the central region velocity.

That is the potential core or potential region velocity has to increase to make the same average velocity. Because the potential core that uniform velocity is getting reduced. That means fluid is retarded more in the direction transverse to the channel. So, to have a constant velocity, this velocity has to increase and finally what will happen we will see that. After the boundary layer merges, we will have a velocity profile like this.

And here you will see difference is that, here the velocity profile, the shape is changing and at each section there is viscous zone and the potential zone. But when this merges there is no potential zone, entire zone is the viscous zone and the centre line velocity which is the maximum velocity that is reached, that is the potential for velocity and u here in this region u is a function of x and y . It is obvious u is a function of x and y . If you see at any x , u is a function of y .

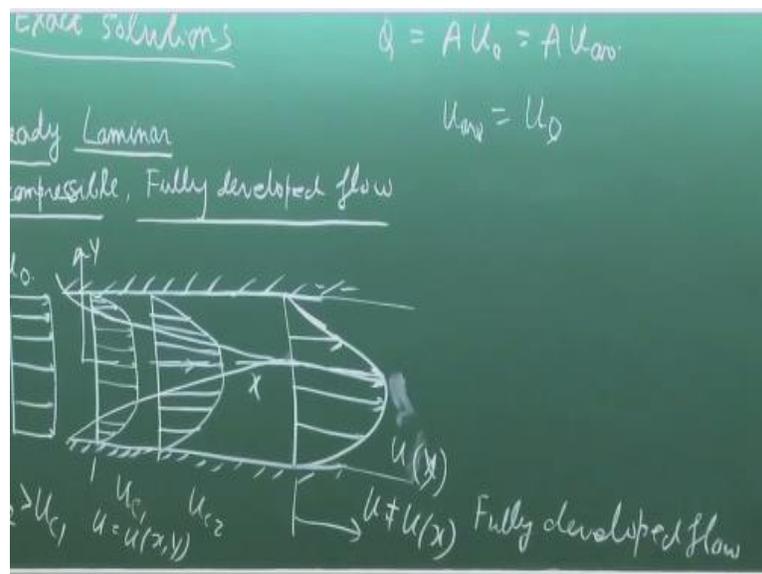
And at any y , u is a function of x . because it fixed in a y here, the flow is in the potential core but if you go along the flow direction this will be in the retarded boundary layer. So therefore, u is the both the function of the x and y . But when this – sorry u is the function of y only, u is not a function of x . When the two-boundary layer merges the velocity takes a fully developed shape, the entire region is the viscous region.

And the velocity becomes only a function of y and as you go along the downstream direction it resembles the same velocity all which is varying with the y , there is no change in the velocity same distribution with x . That means at any y , the velocity (u) (13:50) same with x . This comes from the similarity of the boundary layer. Similarity, velocity profile of the boundary layer. I tell you, that this part which I have told you today, it will be difficult to be appreciated only until and unless, you know the boundary layer correctly.

I do not know whether you have done it explicitly at your fluid mechanics class. If you have done it earlier, then you know because of the similarity in the velocity profile within the boundary layer, we have this type of distribution. So, within the boundary layer we can express the velocity profile as a function of a similarity parameter. Okay that will be discussed afterwards. Now, you know this thing that.

After this region in the velocity – when the two boundary layer merges we get a profile this.

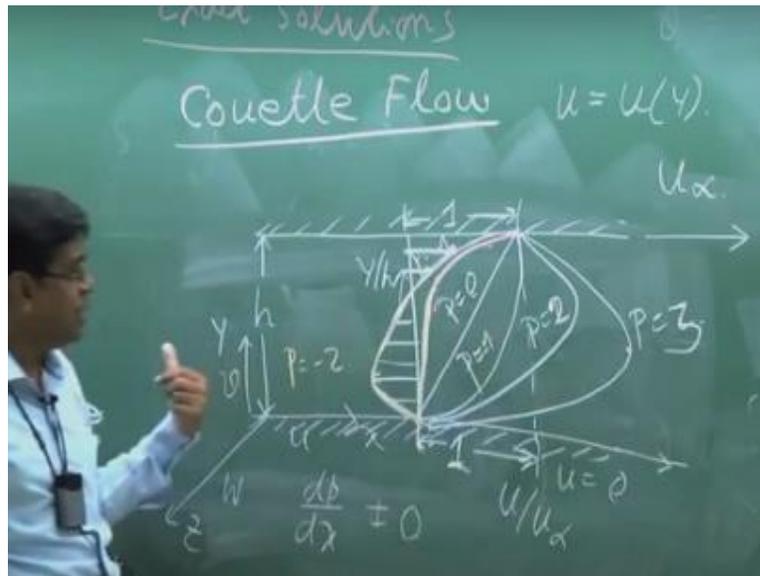
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Now, beyond this region flow is known as fully developed flow. And this region of flow where the boundary grows and yet – it is boundary layer grows and yet to merge this region is known as Developing region, Developing flow or Entrance region. This is a consequence of the boundary layer and his growth which will be taught afterwards. This is known as Entrance region. That means when flow if fully developed we have a similarity velocity profile.

That means the velocity profile is a function of only the transverse coordinate, here y is not the function of x that means the space coordinate along which the flow takes place. This is the concept of fully developed flow.

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Now with this concept we will make an exact solutions of a particular type of flow known as Couette flow. What is a Couette flow? Couette flow is a Steady, Incompressible, Laminar and Fully Developed Flow this is common for all exact solution cases. There are different cases but in all cases flow depicts these feature. But Couette flow, each and every flow has different flow geometry. Couette flow is the flow between two plates.

Couette flow is flow between two plates. Let us consider these two plates. Let us have two plates. And there is a steady, incompressible, laminar and fully developed flow. And let us have the distance between the plates as h . And one plate – let us consider the upper plate is moving with a velocity in u infinity with relative to the lower plate which means let us consider the lower plate is fixed that $u=0$, whereas upper plate has a velocity u infinity.

That means here, the flow is initiated or driven by the shear that means the motion of the plate drags the entire fluid in the channel to flow. But along with that we can impose a pressure gradient also, that means there is a high pressure this side, low pressure this side. That means it

will be also added by the pressure gradient that is known as pressure driven flow. That means if I consider this as x direction and this as y direction.

I can consider the flow to be also driven by the pressure gradient and this pressure gradient maybe negative or positive. Negative pressure gradient means the pressure upstream is more than the downstream that means it decreases the direction of flow, that is known as negative pressure gradient. Pressure decreases in the positive direction of x, the direction of flow. That is a favourable pressure gradient.

Because this pressure gradient favours the flow in this direction. Pressure maybe opposite, that means the adverse pressure gradient dp/dx is greater than 0. That means the pressure downstream is more than the upstream. In that case, the pressure or the pressure forced on a fluid element opposes the flow in this direction in the negative direction relative to the drag of the plate or shear of the plate, there is a tug of war between the two.

That will be manifested in the mathematical equations. So, let us consider in general a flow between two plates, one of which is moving with u infinity respect to other the constant velocity and drives the fluid by shear along with a imposed pressure gradient. That means it is both shear driven and pressure driven flow. Now come to the Navier-Stokes equation question which I wrote there.

Now if you write Navier-Stokes equation in Cartesian coordinate and at the same time you consider the z, the breadth or width of the channel in that direction is much larger than this channel height. So, with this in consideration you first start writing all the equation. Now first of all if the fluid is incompressible. “Professor - student conversation starts” This is a very important concept, I tell you. Many people even at the later stage lack this concept. Please listen to it. “Professor - student conversation ends”

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$$\frac{1}{\rho} \frac{D\rho}{Dt} = \frac{1}{\rho} \frac{D\rho}{Dt} = 0$$

$$\frac{1}{\rho} \frac{D\rho}{Dt} = \frac{1}{\rho} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{V} = 0$$

$$\boxed{\vec{\nabla} \cdot \vec{V} = 0}$$
 Continuity:
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = 0$$

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \vec{\nabla} \rho + \rho \vec{\nabla} \cdot \vec{V} = 0$$

For an incompressible flow, one can tell that basic requirement the definition of the incompressible flow is that, dv/dt that means the rate of change of volume with respect to time, the volumetric dilatation or the density. If you divided it by 1 by V then equality holds good-- with a negative sign only, because ρV is constant for any fluid element, consider it as a system, equals to 0 . I think in fluid mechanics you have learned this.

This two terms equal to divergences of the velocity vector, this has been proved in the kinematics is equal to one upon ρ that means the substantial derivative of density equals to the substantial derivative of volume and that one by v is equal to divergence from the velocity vector. Again, I tell you, this is the rate of change of volumetric dilatation per unit volume. It is the rate of changing of density per unit volume.

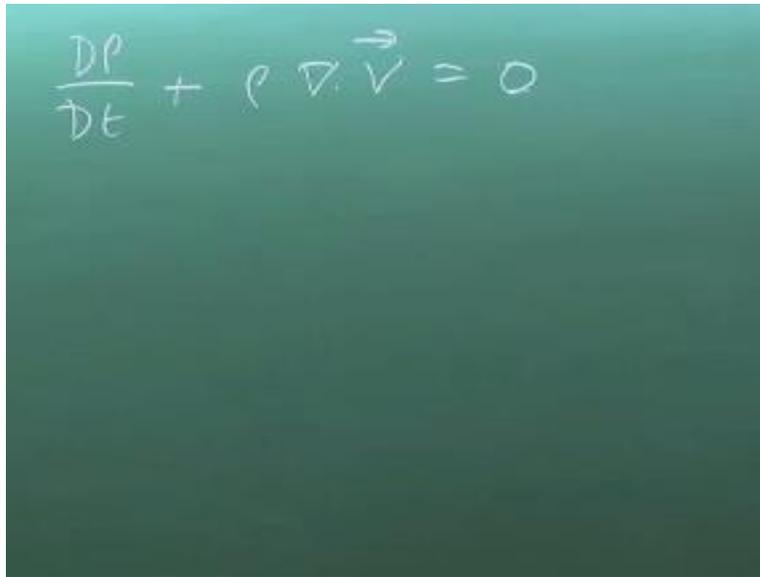
And this equals to divergence of V and this become 0 , i.e., for an incompressible flow divergence of velocity vector 0 comes from its basic definition of incompressible, it is nothing to do with continuity. Many people tell that it is consequence of continuity; No. Continuity equation is automatically satisfied. Sometimes many teachers teach wrongly. This is; this has been discussed that rate of volume volumetric dilatation is 0 by definition.

The rate of change of density is 0 and that becomes equal to divergence of velocity vector which comes from the kinematics. Now let us write the continuity equation. Continuity, let us consider

the Cartesian coordinate, continuity-- I know the continuity is $\frac{d\rho}{dt}$ plus, in a Cartesian coordinates system is $\nabla \cdot (\rho \mathbf{V})$ or rather I will write in vector form which will be valid for all coordinate system. Divergence of $\rho \mathbf{V}$, I think you know this thing is 0. So, if I – now expand this divergence of $\rho \mathbf{V}$ being a vector then this will be $\mathbf{V} \cdot \text{grad of } \rho$ plus ρ into divergence of velocity vector.

Now this two is the substantial derivative of density $\frac{d\rho}{dt}$ plus $\mathbf{V} \cdot \text{grad } \rho$. In Cartesian coordinate $\mathbf{V} \cdot \text{grad } \rho$ will be what? $u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z}$ where you u, v, w, r will be x, y and z component of velocity.

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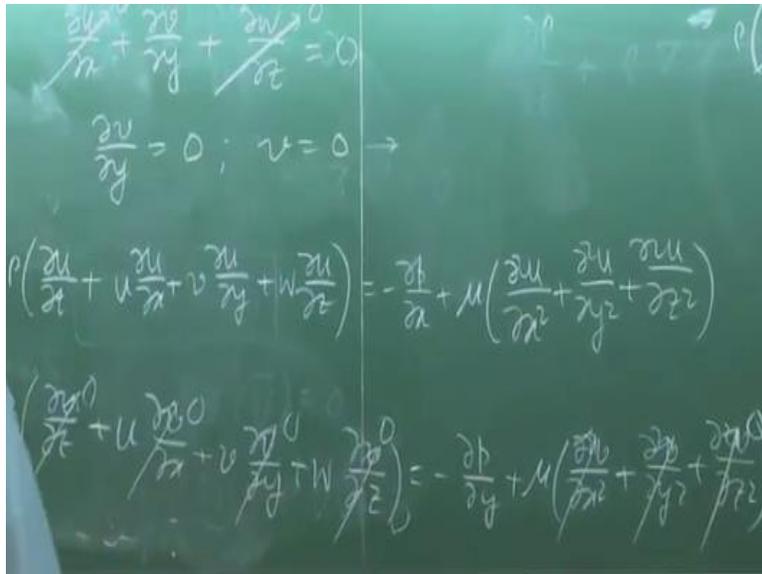


$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0$$

That means this become $\frac{D\rho}{Dt} + \rho$ of divergence of velocity vector that equals to 0. That means it automatically satisfies the continuity, because $\frac{D\rho}{Dt}$ is 0 and divergence of velocity vector is also 0, because this is implicitly connect with this $\frac{D\rho}{Dt}$ through this. So therefore, one should understand that the essence of the incompressible flow is the divergence of velocity vector 0 does not come from the continuity.

But it is – by the fundamental definition of incompressibility and continuity is automatically satisfied in that case, the continuity equation for an incompressible flow.

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Now if you accept this from an incompressible flow by definition, you get $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ is 0. Now here because of the geometry that z is so large, any change in the z will neglect. And moreover, this flow does not have any w component, so therefore this becomes 0. Now for a fully developed flow, $\frac{\partial u}{\partial x}$ is 0. So therefore, consequence of this is $\frac{\partial v}{\partial y}$ is 0.

And this $\frac{\partial v}{\partial y}$ is 0 since there is a plate at 0 velocity in the field, if you apply this condition you will be able to find that a y direction velocity is 0. For this plate also, any one of the plate. Two plates are non-penetration-- impermeable plate. So, if you apply this equation either to this plate or this plate, $\frac{\partial v}{\partial y}$ is 0, so you will get a velocity in the next point 0, so therefore if you go on – this is the understanding you will see that all the points will be 0.

That means in a flow field the two points because of the two plates which do not allow any penetration—impermeable plates. That means mathematically, since v is equal to 0 and both h 0, y 0, and y h, v is 0 throughout. So therefore, incompressible condition gives us that y component of velocity 0 that means this type of flow – a Couette flow is only having a u component of flow which is not a function of x , which is function of y .

Now let us write the Navier-Stokes equation. Navier-Stokes equation in each direction. Now let me write first the x direction that is the vector from $\frac{\partial u}{\partial t}$ plus – now this $\mathbf{V} \cdot \text{grad}$

operator with the vector V , if I put it u here then $V \cdot \text{grad } u$ will be $u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$, I write in the full form is equal to $-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$, alright.

Now I write the V direction equation. $\rho \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}$ is equal to minus $\frac{\partial p}{\partial y} + \mu$, I write the full form first, we consider everything is there, no term is 0, $\frac{\partial^2 v}{\partial z^2}$.

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$$\rho \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\frac{\partial v}{\partial z} = 0; \quad p \neq p(z)$$

$$\frac{\partial v}{\partial y} = 0; \quad p \neq p(y)$$

$$p = p(x)$$

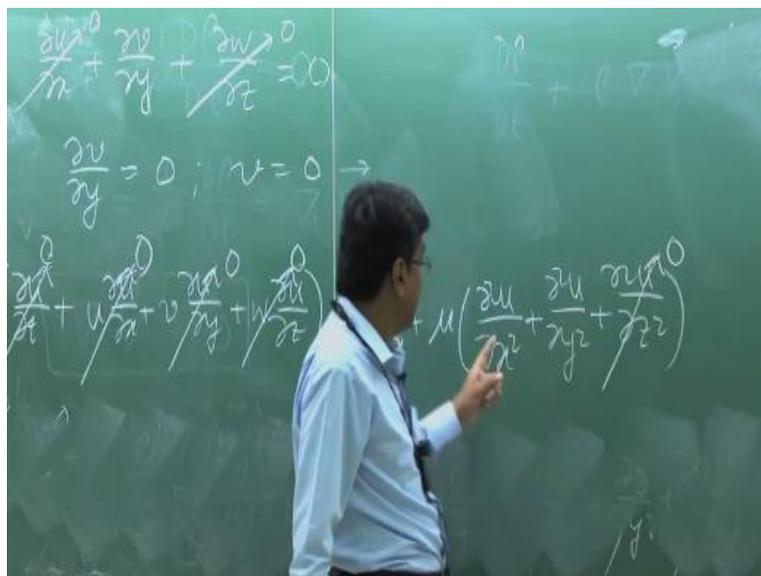
And if you write the equation for the third dimension that means the w component then I write $\frac{\partial w}{\partial t}$ plus similarly the convective term $u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}$. That means $V \cdot \text{grad } w$ is equal to $-\frac{\partial p}{\partial z} + \mu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$. Now one by one, first this thing there is no w , so therefore and steady state that this equation 0.

And the consequence of this equation is that $\frac{\partial p}{\partial z}$ is 0 which means p is not a function of z . What is this equation? Here also already I have proved that v is 0, so all derivatives of v is 0 there is no existence of v . So, in the similar way I have the conclusion from y motion, y direction motion, equation of motion or the Navier-Stokes equation $\frac{\partial p}{\partial y}$ is 0, p is not a function of y .

So therefore, you see, from the two equations y and z directions, I have this idea that p is a function of x only. If you have got any query you can ask me, do not ask your friends you can ask me. So therefore, before inspecting any term, I first change this. Because already I have got a result that p is a function of x only, so $-dp/dx$ in terms of ordinary differential, dp/dx , p is a function of x only.

Now let us see that that how all initial terms vanished here. Steady flow, the characteristic feature. Now u is their but $\frac{\partial u}{\partial x} = 0$, because the flow is fully developed, that is the beauty. And these two terms are 0 because v is non-existence and w is non-existence. So, you see the initial term are all vanishing. Even in the x component of Navier-Stokes equation. So therefore, we get – what we get from the x direction motion, x direction equation or motion, that is x direction Navier-Stokes equation, all terms in the left-hand side is 0.

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Here this is 0, why? Because z is so large compare to h, the variation with respect to z and that too second order variation, so scale is z square in the denominator, it is much smaller compare to other term we can consider it to be 0, $\frac{\partial^2 u}{\partial x^2}$ is 0 because $\frac{\partial u}{\partial x} = 0$, u is not a function of x. So again by (()) (32:26) the fully developed flow characteristic we have this thing.

Therefore, our final equation becomes $\frac{d^2 u}{dy^2}$ since u is a function of y , that already we have proved is equal $\frac{1}{\mu} \frac{dp}{dx}$. “Professor - student conversation starts” If you have anything to ask me, you ask me, do not talk in the class continuously. I do not understand that why this is happening here. Is there any problem? Hello, please tell, is there any problem? Hello. No why you people are talking I do not know, continuously. You ask me whatever you want to ask. “Professor - student conversation ends”

So, $\frac{1}{\mu} \frac{dp}{dx}$, so therefore we see that, this is the final equation which we get from Navier-Stokes equation from in three directions for a Couette flow. Couette flow thing has not yet come, that it means it is a flow which is steady, incompressible, laminar, fully developed and specified in terms of a Cartesian coordinate where x, y, z are the coordinate system and u, v, w is the respective velocity.

Because so far, this geometrical configuration of flow has not been imposed, so therefore you can consider that this equation is the generalized equation for all cases of Exact Solutions. That means steady, incompressible, laminar, fully developed flow. Now before integrating this to find out the value of u as a function of y , you appreciate one thing, u is a function of y , so $\frac{d^2 u}{dy^2}$ maybe either constant or a function of y depending upon the type of the function.

And if it is a quadratic function then it will be constant. But, if the power of y is more than two then it will be a function of y , so the option is that either function of y or constant, where p is a function of x that we have already proved. That means $\frac{dp}{dx}$ is either constant or function of x . If it is a linear function of x than it is constant, otherwise it will be a function of x . But, when the two are equal, so the only possibility is that they become constant.

So, with this logic at this state one can tell that p is a linear function in the direction of flow in case of such situations of a laminar, incompressible, steady, fully developed flow, p has to be a linear function in the direction of flow. That means two are equally constant.

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$$\frac{d^2u}{dy^2} = \frac{1}{\mu} \frac{dp}{dx} = \text{constant}$$

$$\frac{du}{dy} = \frac{1}{\mu} \frac{dp}{dx} y + C_1$$

$$u = \frac{1}{2\mu} \left(\frac{dp}{dx} \right) y^2 + C_1 y + C_2$$

Now we can integrate, we can integrate two times. First, we will integrate du/dy which is equal $1/\mu$ by $\mu dp/dx$ into y , oh sorry, $y + C_1$ and u is equal to $y^2 / (2\mu) dp/dx + C_1 y + C_2$. So up to this, I have not done anything with respect to Couette flow. So, this is the Cartesian coordinate. Velocity field for an exact solution situation, u is $1/(2\mu) dp/dx y^2 + C_1 y + C_2$, up to this everything is common.

Now for a Couette flow, depending upon the flow geometry I will find out C_1 and C_2 . How to find out? What are the boundary conditions?

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$$\begin{aligned} & \text{at } y=0, u=0 \\ & \text{at } y=h, u=U_{\infty} \\ & C_2 = 0 \\ & C_1 = \frac{U_{\infty}}{h} + \frac{1}{2\mu} \left(-\frac{dp}{dx} \right) h \end{aligned}$$

at $y=0$, u is 0, at y is h , what is the value of u ? What is the value of u ? u infinity. u is equal to u infinity. Now at $y=0$, $u=0$ means $C_2=0$. At $y=h$, if you put u infinity you get a value of C_1 . That is C_1 is equal to u infinity by h , this side will be minus but I will write this way, 1 by 2 μ this is the style $- dp/dx$ into y is h square. So, these are the values of C_1 and C_2 . And if you plot these values of C_1 and C_2 you get.

“Professor - student conversation starts” No, what happen I do not understand. Why are you talking? Is there—hello you, what happen to you? Come here. Why are you talking? I am really sorry, what happen can you tell me? C_1 is equal to u infinity by h plus 1 by 2 μ minus dp/dx into h , yes. C_1 —this one by 2 μ h square, C_1 is h . You tell that. Sir it will be h . Any one of you raise hand and tell sir, it will be h , that is the way of telling you. Hello, it is really.

What is the problem? Please tell me what is the problem? I am very sorry that this will be telecast, I will erase it, but what is your problem you tell me, why are you talking here? I am requesting those who are not interested you may leave this class. Is there any problem for you to understand? Hello, if anybody cannot understand you raise hand, I will be very happy to know, then I will it again, I will take him to my office and I will explain thing.

I will be very happy, please raise your hands, do not feel shy. Then why are you talking continuously I do not understand. Can you tell me the reason? Hello you. Yes, I do not know when I will finish. But really it is very disturbing, whenever I am teaching there is a sound going on that means you are all chatting here, except very few in the first 2 or 3 row. Am I true? Hello, you. Am I true? True or false?

Nobody can tell it is false, because it is known you can ask him. Is it true? No what is the problem you tell me, yes sir it will be h , good. If there is any mistake, you tell me. “Professor - student conversation ends”

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$$\frac{u}{u_\infty} = \frac{y}{h} + \frac{h^2}{2\mu u_\infty} \left(-\frac{dp}{dx} \right)$$

$$P = \frac{h^2}{2\mu u_\infty} \left(-\frac{dp}{dx} \right)$$

$$\frac{u}{u_\infty} = \frac{y}{h} + P \frac{y}{h} \left(1 - \frac{y}{h} \right)$$

So C2 become 0 and C1 is this. Now if you put C1 and C2 in this equation u. If you put this C1 and C2 in the u with certain rearrangement, very simple, just you put in and you get, this equation, this type of expression what the velocity distribution, in terms of a non-dimensional quantity like this $y/h + h^2/2\mu u_\infty$, u_∞ is divided both the side that is why $-dp/dx$, it is rearrangement only, it is very simple y/h into $1-y/h$.

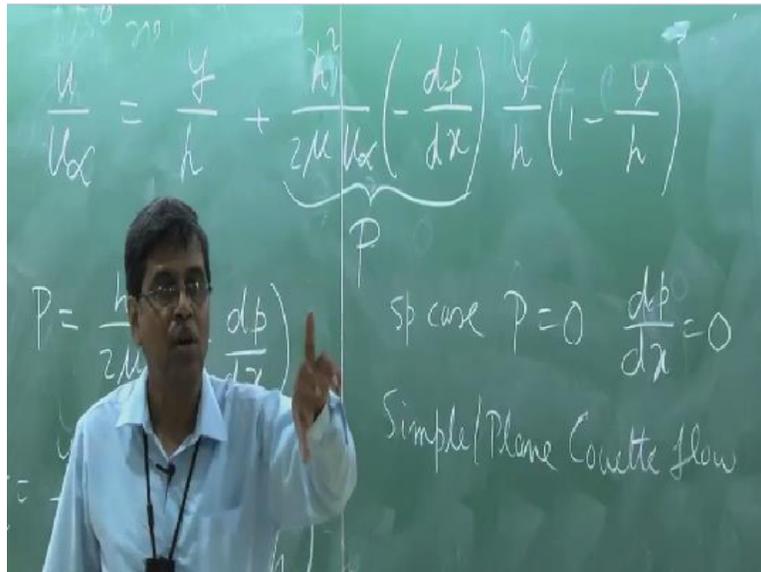
And this is the equation for the velocity distribution in the Couette flow. Now this term is represented by capital P which is known as non-dimensional pressure gradient. Because this term is non-dimensional y/h is non-dimensional, u by u_∞ is non-dimensional, this is non-dimensional variable so this has to be non-dimensional. So, group of this h^2 by $2\mu u_\infty$ into $-dp/dx$.

That means this group which contains the pressure gradient is denoted as capital P and is known as non-dimensional pressure gradient. So, if I write in terms of non-dimensional pressure gradient expression looks little simple that capital P y/h into $1-y/h$. Now you see this is a velocity distribution equation, where u is a function of y or y/h as a non-dimensional y with p as the parameter.

Now as a special case when there is no P, that means $P=0$, there is no pressure gradient, there is no dp/dx that means capital p is 0, that means the fluid in the channel is driven only by the shear

that means it is only a shear driven flow, then this term is 0 and we get a linear velocity distribution.

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Here if I draw this in the plane this is the u by u infinity and this is y/h then we will get a linear distribution, this is 1, that means at $y=h$ and this also 1, we get u is equal u infinity that has to be there, because the boundary condition has to be satisfied. That means, here the velocity is u infinity, no sleep condition, this will get a linear one and that is driven by $P=0$. And this situation is known as Simple or sometimes Plane Couette flow.

That means Couette flow without any pressure gradient. A flow is purely shear driven. In that case, we have a linear velocity distribution starting from 0 to the upper plate velocity or the velocity of the plate which is moving. Now we consider different values of P and see the behaviour of this function. This is very, very interesting. Let me do that. You consider different values of P and let us examine the behaviour of this velocity distribution.

One can definitely draw curve of different and values of P but now let me write again this equation.

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$$\frac{u}{u_\infty} = \frac{y}{h} + P \frac{y}{h} \left(1 - \frac{y}{h}\right)$$

$$\frac{du}{dy} = \left[\frac{1}{h} + \frac{P}{h} \left(1 - \frac{2y}{h}\right) \right] \cdot u_\infty$$

For max/min $\frac{du}{dy} = 0$ $\frac{d^2u}{dy^2} = -\frac{2P}{h^2}$

$$\frac{y}{h} = \frac{1}{2} + \frac{1}{2P}$$

u by u infinity is $y/h + p$ into y/h into $1-y/h$. Now to know this behaviour of this velocity distribution with different P , let us do one thing, let us first find out, du/dy . What is du/dy ? du/dy is simply $1/h + P/h$ into $1-2y/h$. This is the value of du/dy . Now let us find out whether this curve exhibits a maximal or minimal or not. A linear curve does not have any maxima or minima; this is linearly increasing.

Monotonically increasing function and the maximum value of the velocity is this one u infinity.

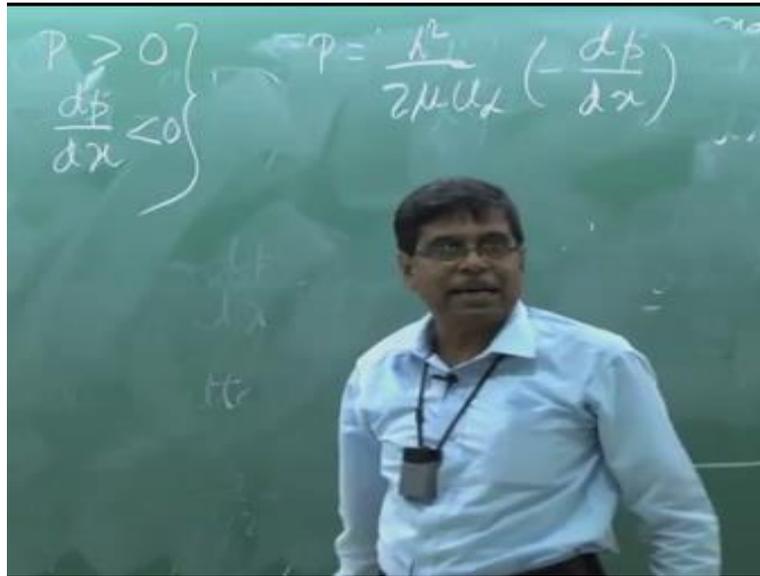
Now let us examine, whether this curve has a maxima or minimum, then we have to do this.

It has to be made 0 for maximum or minimum, max or min du/dy equals to 0, that means this equals to 0 which gives by a little rearrangement y/h is $1/2 + 1/2P$ simply you put is 0 and take y/h on one side and express it in terms of non-dimensional coordinate y/h $1/2 + 1/2P$.

And at the same time I see that, d^2u/dy^2 square is equal to what? d^2u/dy^2 square if you make, that means it is $-2P/h^2$ square to judge whether it is a maximum or minimum. u by u infinity is y/h , du/dy very good, I am in a mood to make it 0 that is why u infinity was missing. Fantastic, u infinity. So, if you make this mistake in using du/dy somewhere without u infinity some marks will be deduct, not full mark. So, this is u infinity. Very good.

So, $du/dy = 0$ means this is 0, u infinity cannot be 0. So therefore, I have this condition for maximum-minimum, for the value of y and d^2u/dy^2 square.

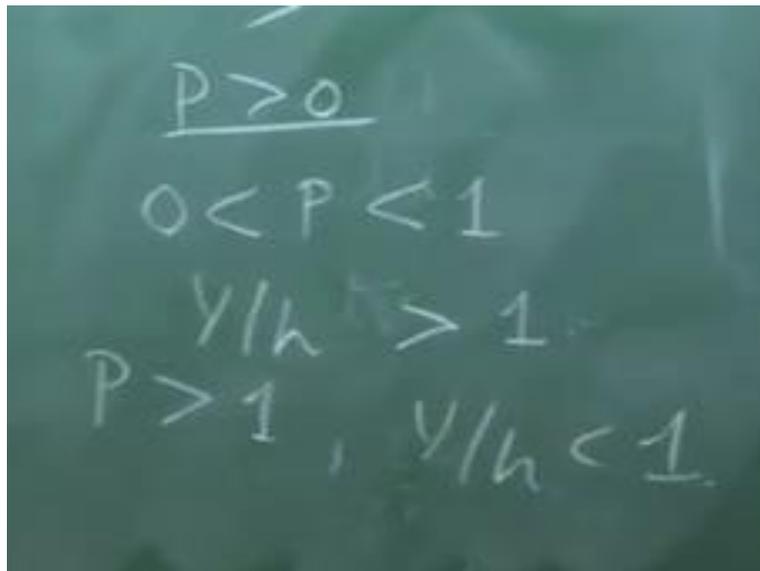
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Now consider, P greater than 0, what is P ? P equals to again I write, $P = h^2/2 \mu u$ infinity – dp/dx . P greater than 0 means dp/dx is negative, less than 0, this is the situation known as favourable pressure gradient which favours the flow, that means shear driven flow and the pressure driven flow are in line. In that case, if P is greater than 0, this is negative, that means we have a maximum not minimum.

Now again, there are interesting things. What are those locations where we get maximum for different values of P when P is greater than 0.

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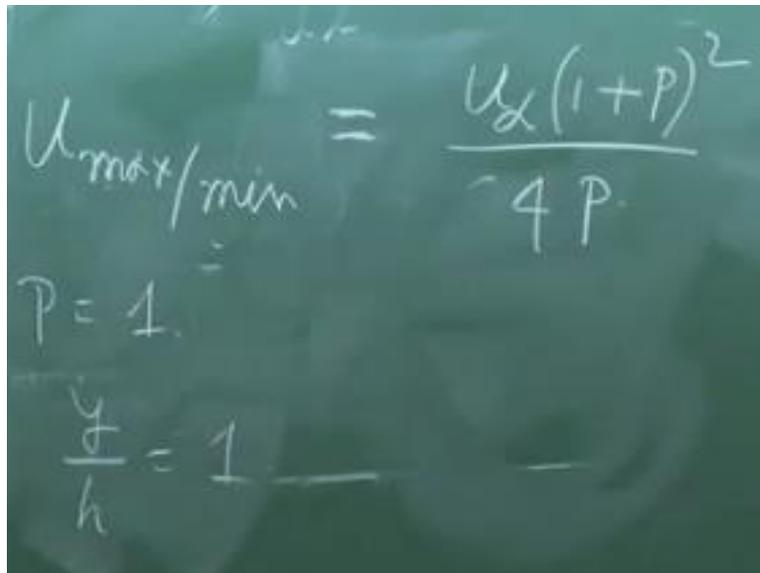


Now when P greater than 0 under this heading, I split it. Let us consider P greater than 0 but less than 1. This is because of the typical relation. Now when P is less than 1 between 0 to 1 greater than 0, less than 1; y/h is greater than 1 because this is greater than half. That means the curve mathematically has a maximum but that maximum is beyond the flow domain, I do not have any interest, so those values of P , my maximum velocity in the flow domain is u infinity.

The curve may not show a mathematical maximum a 0 slope there like in linear profile. Linear profile will never show any 0 velocity as a constant slope. So therefore, 0 to 1 value P the maxima will occur beyond P , means this y/h is greater than 1. But when P is greater than 1 then y/h is less than 1, that means I have a maximum within the flow domain that is y/h less than 1. And at the same time if we put this value of y/h $1 + 1/2P$.

You get the value of u max or u min that means if you substitute this y/h in this value in this equation expression of u you get the value u infinity into $1+P$ square divided by $4P$. So, when P greater than one the u max then it is a question of maximum is greater than u infinity, this is compatible because the flow has a maximum velocity within the flow domain somewhere below other plate at the moving plate which has to be add the new infinity then it will not be a maximum.

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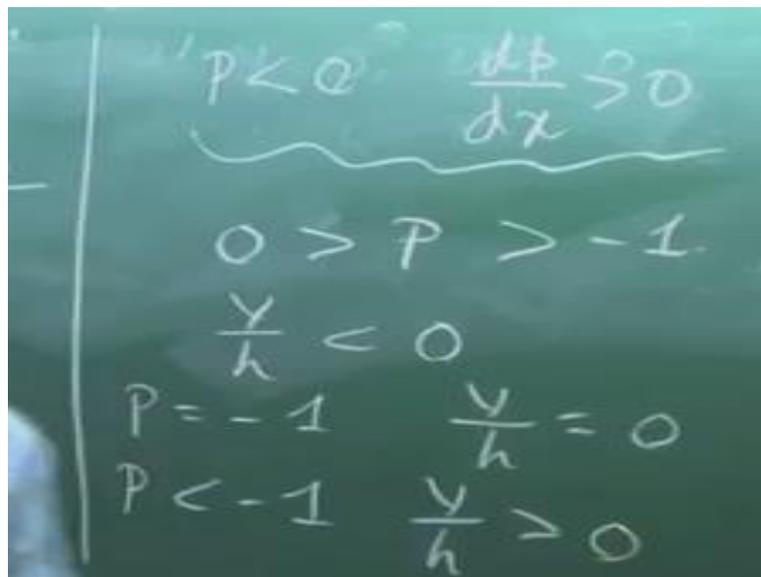


The image shows a chalkboard with handwritten mathematical equations. The main equation is $u_{max/min} = \frac{u_{\infty}(1+P)^2}{4P}$. Below it, there is a line $P = 1$. Underneath that, there is a fraction $\frac{y}{h} = 1$ with a horizontal line extending to the right.

Now let us examine a bottom line case $P = 1$ in that case, $du/dy = 0$, sorry $y/h = 1$ that means when $P = 1$, $y/h = 1$ that means the maximum occurs at the surface, upper plate. That means we can draw a curve like this, this is for $P = 1$ that means which shows a maximum at the surface because $du/dy = 0$ is $y/h = 1$ but at the surface boundary condition tells it is u infinity and if you put $P=1$ where u max is u infinity.

But when it is more than one that means favourable pressure gradient is more strong which make P more than one we get from this equation u max greater than u infinity and these things are like this. Here it is 1, but somewhere let us consider this is $P=1$ and if you go on increasing the P you will see your maximum point will come close to the stationary plate and a value of u infinity, sorry value of u max will be increased. Clear? Hello, okay?

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Now we consider the case when P is less than 0, means dp/dx is greater than 0 which is known as adverse pressure gradient, why adverse? Because dp/dx greater than 0, upstream pressure is lower than the downstream pressure that means the pressure force acts opposite to the direction of flow. This is commonly fluid mechanical terminology. Now when $P < 1$ I also spilt like this, less than 0. Let us consider $P < 0$ but greater than -1 that means $0 > P$ but $P > -1$.

Similar way the beauty is that it is just a mirror image when P is between 0 to -1 that means P is less than 0, but greater than -1, y/h is less than 0 because this dominates with the negative value.

So, in that case y/h is less than 0 that means this is not our concern this is not in the fluid domain, what happen in this case that between, 0 to 1 the maximum point occurs y/h greater than 1. It is just the same way y/h less than 0.

When, $P=-1$ special case than du/dy , $y/h=0$, 1 by $2 + 2P$, if you out $P= -1$ then it is $y/h=0$, that means it occurs at the surface, that means surface is the no slip condition 0 velocity that means if I draw the curve, the curve will slow with a 0 slow, the line will be vertical, this is $P=-1$. Now what happens when P is less than -1 ? That means $P<-1$, y/h is greater than 0. So, this is very interesting, the P is less than -1 y/h is greater than 0.

That means we come into the flow domain. Similar to the thing happen, similar to the same thing that happen for P greater than 0 cases. Then we find that y greater than 0, that means we come to the flow domain above this stationary plate and in this case the value of u min will by this with a negative sign, P is less than 0 that means in the opposite direction, that means flow reversal will take place, boundary layer separation.

I think in fluid mechanics class, it was told. If you look that picture, it will be like this. Somewhere the flow will be minimum that means in the opposite direction, this is the minimum flow that y/h that means this is a particular. Sorry, I draw it by this colour, so that, it is all ((
(00:58:48). This is $P= -1$ and this curve is for $P< -1$, let us consider $P= -1$, this is the condition.

Therefore, we see a flow reversal that means a negative direction flow near the stationary plate when the adverse pressure gradient increases such a way that P is less than -1 . So, this is well-known. This is physically explained. Now let me explain this physically that when P is 1 here maximum, than P is 2, this is P is 2, I am sorry, this is let P is 3, this is 2,3 the flow velocity somewhere here is more than the upper plate velocity.

That means the favourable pressure gradient helps the flow along with shear so it is shear driven at pressure driven in the same direction which allows the fluid to flow at a velocity larger than the plate velocity. Another interesting thing feature is that at $P=1$ that means there is a value of

dp/dx negative dp/dx favourable pressure gradient at which point—here the velocity gradient is 0, that means no shear stress is there.

That means to drag the – to move the plate with the velocity u infinity does not require any force. If you keep the plate it will automatically flow, this is because the pressure gradient causing the flow keeps that velocity u infinity at that surface. That means a free surface is moving at some velocity because of the pressure gradient imposed on the flow, if you keep a plate, plate will automatically flow, so therefore there is no shear stress is required.

And for strong pressure gradient, negative pressure gradient or favourable pressure gradient the pressure driven flow adds the flow along with the shear driven to attain a velocity more than u infinity somewhere. Reverse is the true when the pressure gradient is adverse that means dp/dx is positive that is P is negative than with this value of P when it is -1 here it is 0, that means the slope is 0, this is the onset of flow reversal.

Any value of P lower than this, this curve is for $P=-2$ then there is a flow reversal. Some part of the flow region close to the stationary plate will flow in the opposite direction while this part the flow will be positive direction.

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The image shows a chalkboard with the following handwritten equations:

$$Q = \int_0^h u B dy$$
$$u = u_\infty \left[\frac{y}{h} + P \frac{y}{h} \left(1 - \frac{y}{h} \right) \right]$$
$$Q = \left(\frac{1}{2} + \frac{P}{6} \right) u_\infty B h$$

Now, find out what is the volume flow rate. Volume flow rate is found out at any section you can find out the volume flow rate like this, $u \, dy$, 0 to h . Why? (Refer Slide Time: 00:16:14) This is because you can think in this way that you can take a small elemental area, this height is dy and we consider the width to be unity. Then it is $u \, dy$ integrated h because da will dy into the width or you can take some width $B \, dy$.

B is the width in the z direction which is very large $uBdy$, so if you put the substitute the expression of u , u is what? u is u infinity into $y/h + P \, y/h$ into $1-y/h$, if you put that then you get an expression, 1 by 2 just you can slow it by putting this from 0 to h 1 by $2 + P/6$, u infinity into B into H , H will be there. That is very important. This is the flow rate, just you substitute it and integrate it.

And as you now average velocity in fluid flow in internal flow, average velocity is defined based on the volumetric flow rate.

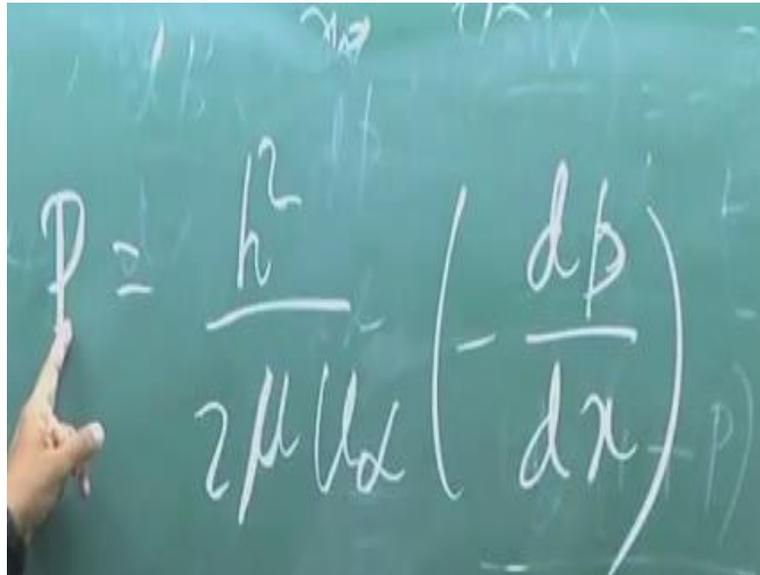
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The image shows handwritten mathematical derivations on a chalkboard. At the top, it states $u_{av} = \frac{Q}{A}$. Below this, it shows $u_{av} = \frac{Q}{Bh}$. Further down, it gives the expression $u_{av} = \left(\frac{1}{2} + \frac{P}{6} \right)$. At the bottom, it notes "When $P = -3$ $Q = 0$ " and $u_{av} = 0$.

That means the average velocity across any section is volumetric flow divided by the cross-sectional area. Here cross-sectional area is same which is equal to b into h and therefore $u_{av} h$ will be 1 by 2 that means average velocity is nothing but the scale of volume flow rate, scale is the cross-sectional area. Now here also another interesting thing is that when $P = -3$ u_{av} or is equal to $Q = 0$ both u_{av} is 0 and $Q = 0$.

Rather you should not write this way u_{av} is equal to 0 rather simply, $Q=0$ and u_{av} is also 0. When $P = -3$ what does it indicate? This is mathematics that indicates what?

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$$P = \frac{h^2}{2\mu u_{\infty}} \left(-\frac{dp}{dx} \right)$$

That indicates again negative value of P means adverse pressure gradient, P was defined as again h^2 you see many of times we have written this $u_{\infty} - dp/dx$. So, P greater than – sorry $P = -3$ corresponds to adverse pressure gradient with respect to $h^2 / \mu u_{\infty}$ that means for a given channel, channel of given height with the fixed, the moving plate, moving with a velocity u_{∞} is viscosity μ .

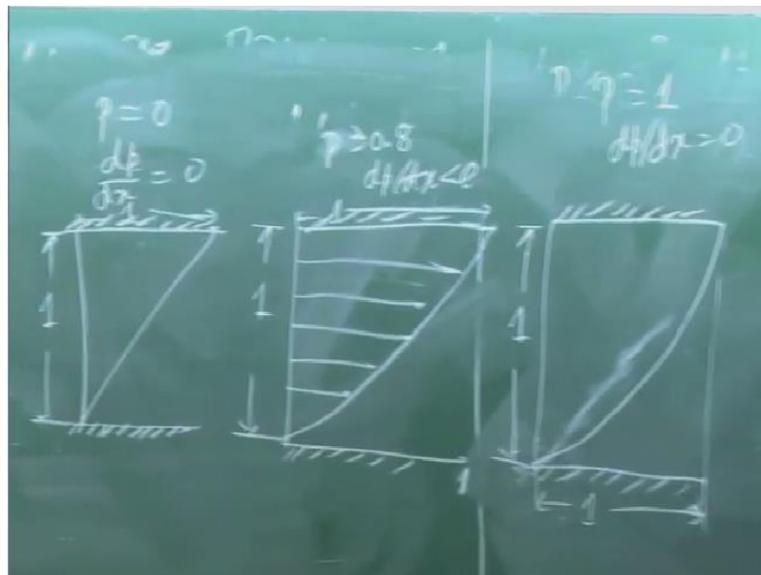
If the pressure gradient is such that this satisfies this equation that dp/dx is equal $h^2 / 2 \mu u_{\infty}$, that means if I write this dp/dx will be -3 that means the $3-3$ will be cancelled, $6 \mu u_{\infty}$ by h^2 that means this type of adverse pressure gradient will – I told that tug of war will erase the flow that means shear driven flow and pressure driven flow because the adverse pressure gradient that means pressure will drive the flow in this direction.

Here you see if you go on increasing the P more region will be the flow reversal and that value of $P = -3$ mathematically I see that u_{av} is 0 that means there is no net flow Q is zero across the section that means there will be a purely circulatory flow, some portion is negative it go up

positive and comes down. So, a locally re-circulating flow will take place and it will be 0. The average velocity will be 0, $P = -3$. Any question?

This figure, I think before I leave you today, I will draw again. So that you can very well understand the velocity distribution in a Couette flow. Please wait for 5 minutes. I will draw this figure very well because in this diagram sometime it is difficult.

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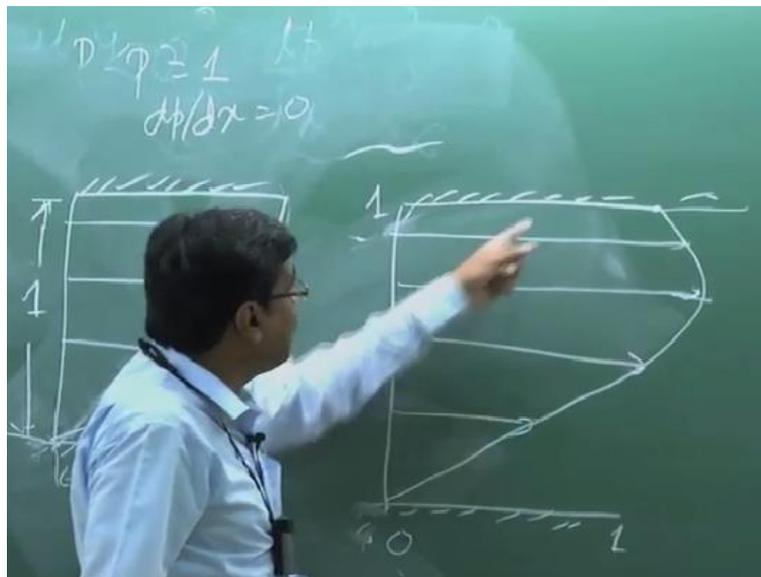
Let us consider this y/h and let us here u by u infinity. Let us have two plates, $P=0$ that means $dp/dx=0$. The velocity profile is like this. This is 1 and this is 1. This is very interesting. Let us have another situation where, $P = P < 1$, P is some value, let us consider P is 0.8 which corresponds to dp/dx less than 0 because P is positive but less than 1. In that case, the picture is like this, this is 1 same u by u infinity, this scale is 1.

The picture shows a non-linear one like this. It does not have a maximum within the flow domain. So therefore, practically maximum velocity is attained at $y/h=1$ which is nothing but the upper plate velocity that is u infinity that is 1, this is u by u infinity. So here the maximum will attain somewhere outside, we are not interested. But $P = 1$ when $P = 1$, when P is equal to – these are all favourable pressure gradient $P=1$ greater than 0.8.

And dp/dx is obviously 0 then the difference is that the maximum velocity, the difference is that that is not linear but the maximum velocity is attained at y/h . The value is u_{∞} . But it is mathematical maximum that means the value is u_{∞} but it gives the mathematical maximum. So therefore, a physical difference between the two is there. From the functional value, there is no physical difference, nobody bothers, it is okay.

You have u_{∞} maximum, but yet mathematically maximum, $du/dy=0$. So therefore here, there is a drag but here dragging 0 on the plate. No force is required to move the plate.

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And if you have more favourable pressure gradient so that P becomes equal to, let this is 1, this is 0 and this is 1, so this will be 1 but there will be a velocity gradient like this. That means, somewhere below the upper plate there will be a velocity greater than u_{∞} . And then this decreases and the interesting thing is that the velocity gradient changes its direction.

Because this is the decreasing part which means physically that when you increase the favourable pressure gradient then to keep the plate in u_{∞} you have to push it in the opposite direction. Try to understand the physics, that there is a pool of liquid, you have a plate, no pressure gradient you have to push the plate in a forward direction to make the flow in the same direction.

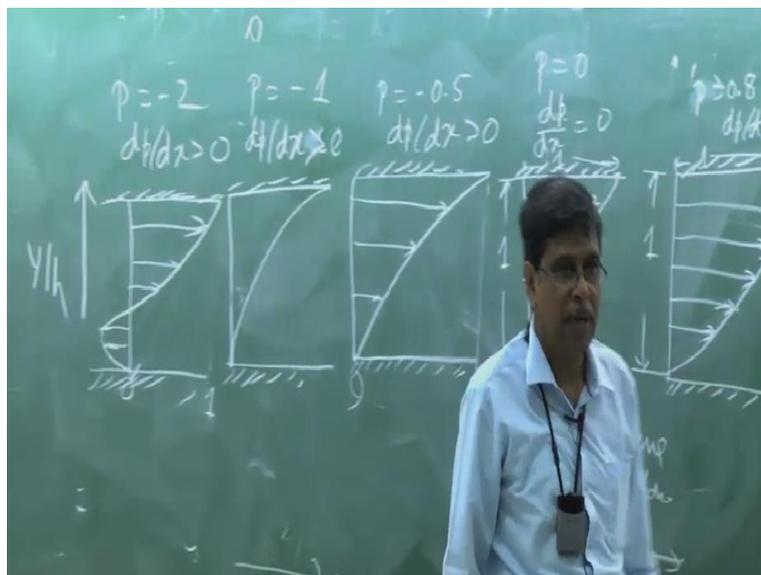
Now you impose the pressure gradient, see that we feel the pulse at some pressure gradient $P=1$, capital P corresponding to that dp/dx negative value, you do not have to whole the plate, it will automatically move that u infinity.

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But if the pressure gradient increases more than that then to keep a plate at that requisite velocity you have to $- dp/dx$ less than 0, we have to keep pull the plate in the opposite direction to maintain u infinity, you see from the figure, this is very clear that is why I am drawing it separately. Okay, that is the beauty of the Couette flow velocity variation with pressure gradient.

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Similarly, if we come to this direction we will see that, here at any pressure gradient – let us consider a negative value $P = -0.5$, dp/dx is greater than 0, that means it is adverse, in that case what happens – this is 1, this is 1, in that case there is a linear, but no minimum, mathematical minimum but minimum velocity is 0 velocity, no mathematical minimum, but minimum velocity the functional value is 0 here because that is flow domain is truncated.

But now if I have a condition exactly at -1 just the mirror image of that, that means you can place a mirror here, simple Couette flow, this is simple or sometimes Plane Couette flow. So here you will see the difference is like this, there mathematical minimum which says the onset of flow reversal. Here there is no danger flow reversal but here there is a danger, just after that the flow reversal will take place. That means this is $P=-1$, that means dp/dx domain is less than 0, sorry greater than 0.

Here $P = -2$ and dp/dx as usual greater than 0. Here you see that already flow has separated that means there is a flow reversal. That means if you show the figure like this, this is 0 and this is 1. Sorry, I am extremely sorry. this is 0 and this is 1. Clear? This is very important. But I spent more time in explaining the Couette flow. Next class, I will tell you the plane Poiseuille flow to Hagen-Poiseuille flow.

These are of the same kind; it will not take much time today. So, Exact Solutions of Navier-Stokes equation is possible if the flow is – this is today's, what is the today's learning? If the steady, incompressible, laminar and fully developed. And the solutions, final solutions depend upon the flow geometry. But in this case, we can reduce Navier-Stokes equation to an ordinary differential equation.

Linear ordinary differential equation by discarding the initial terms and we have only one single equation because the essence is that, flow velocity is not a function of the space coordinate along the flow, it is a function of the transverse coordinate, another third coordinate by the definition of the geometry becomes absent and pressure has a corollary of the y and z direction to transverse coordinate direction.

Navier-Stokes equation gives us is a linear function of the space coordinate along with the direction of the flow and accordingly it tells that u is a quadratic function – when you tell that d^2u/dy^2 is one by $\mu dp/dx$ is constant means P is a linear function of x and u is a quadratic function of y . For that, you do not have to solve the equation, but you solve the equation find the constant with the aid of the geometry of the flow that will vary from Couette flow to plane Poiseuille flow to Hagen-Poiseuille flow. Okay, these are the differences.