Computational Fluid Dynamics and Heat Transfer
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Lecture – 09
Finite Volume Method-1

Good afternoon, everybody, today we will start a new topic on Finite Volume Method. As we did in our earlier topic, we studied certain important aspects of finite difference methods, in this topic we will learn the basic rules and ideas and mathematical background related to finite volume method. Next, we will do it for finite element method and then we will apply these methods to solve fluid flow and heat transfer problems.

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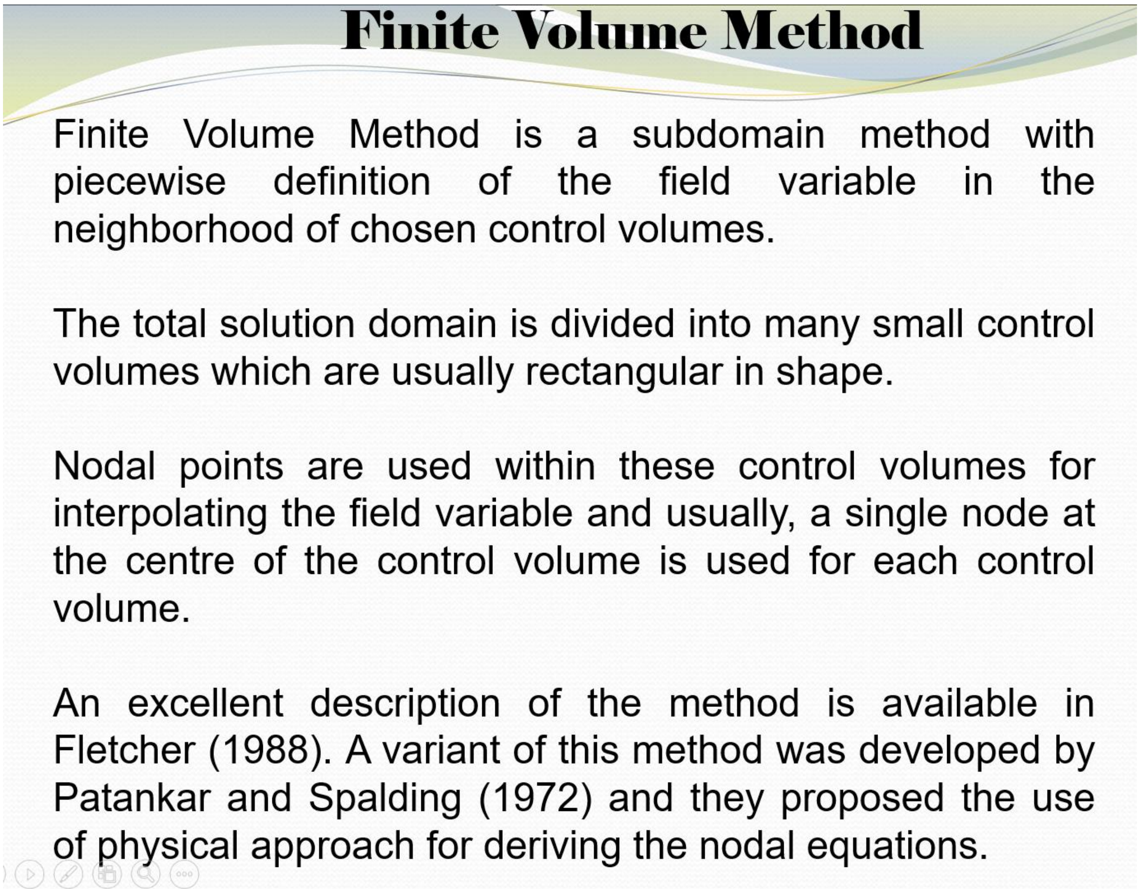

So, let us see what we aim at learning in this topic. Finite volume method is a subdomain method with piecewise definition of the field variable in the neighbourhood of chosen control volumes. The total solution domain is divided into small control volumes, which are usually rectangular in shape. This may not be exactly rectangular we will see later; it can have variation in shape.

Nodal points are used within this control volume; that means, this piecewise small control volumes; you can call it cells. For interpolating the field variables and usually a single node at the centre of the control volume that centre of the cell so, to say is used for each control volume. An excellent description of the method is available in the book of Fletcher also a variant of this method was developed by professor Patankar and professor Spalding and they proposed some physical ideas for the formulation of the problem and nodal equations.

Incidentally, you know the subdomain approach based on weighted residual method and this physical approach by professor Patankar and Spalding you will see that they are same.

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Finite Volume Method

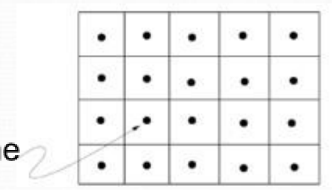
Consider 2-D, steady heat conduction in rectangular geometry (Figure 1). The 2-D heat conduction equation is

$$k\nabla^2 T + Q = 0 \qquad (1)$$

Where T(x,y) is the temperature field, k is the thermal conductivity and Q is the heat generation per unit volume.

Figure 1

Typical control volume



The two alternative ways of settings up the nodal equations are the weighted residual approach and the physical approach

Now, considered a two-dimensional steady state heat conduction in rectangular geometry like we have given in the finger here 2-D heat conduction equation as you know

$$k\nabla^2 T + Q = 0 \qquad (1)$$

where T is a temperature field, k is a thermal conductivity and Q is the heat generation per unit volume.

So, this is an overall domain and you can see the domain has been split into small control volumes and this control volumes are basically arranged in such a way that will be we need to use their additive properties and each control volume is identified by a node at the central centre of the control volume.

Now, two alternative ways of setting up nodal equations as we mentioned earlier, we will start now. On the first one is weighted residual based approach; we can apply to any sub domain method.

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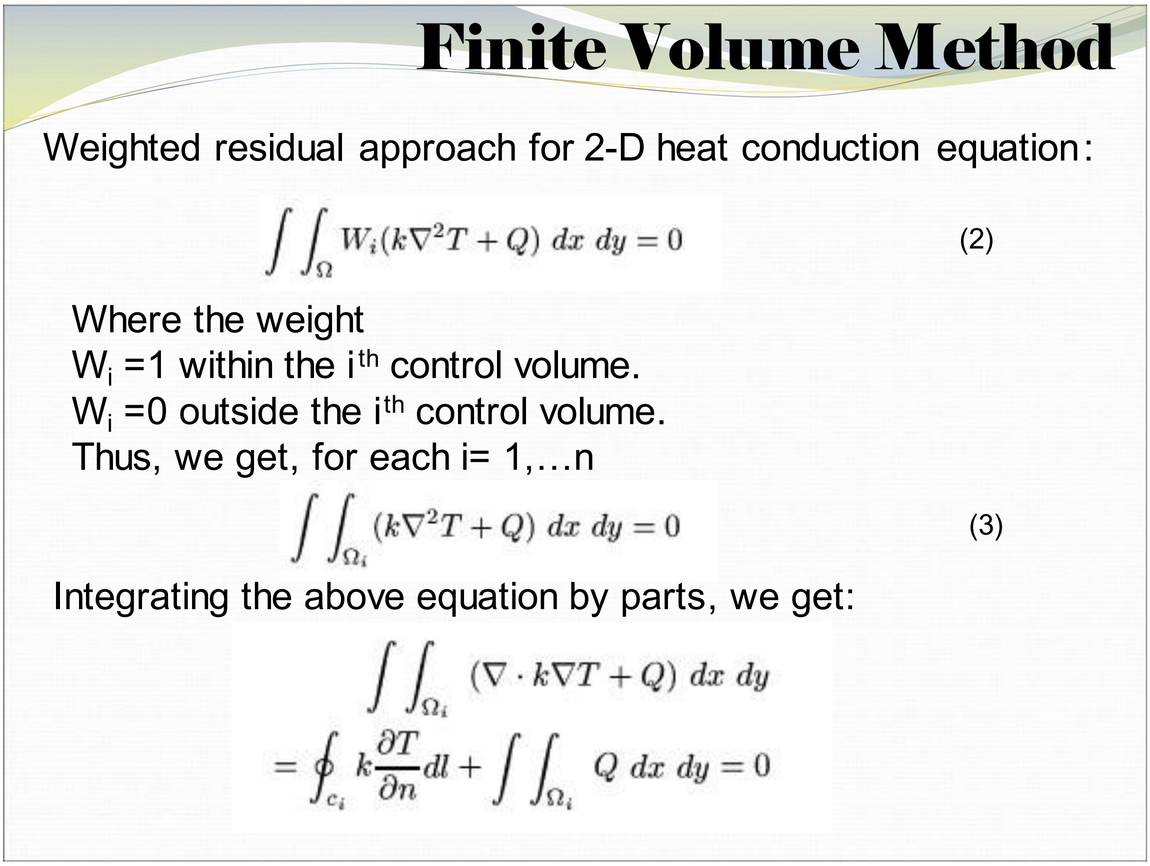

So, weighted residual approach for 2-D heat conduction equation we multiply the equation with a weight function.

You can see and integrate in that domain. So, this way it this can be integrated in all small control volumes and while integrating $W_i=1$ within the i^{th} control volume with whereas, $W_i = 0$ outside the i^{th} control volume when we are focusing at i. Thus, we get for each node i = 1 to n the equations.

And, then we use the additive property to assemble these equations. Now, we can write basically the volume integral, here although it is 2-D equation in a sense third dimension is having unit dimension third direction and we can use this volume integral which is equation 3. And this equation can be integrated using equation by parts yeah and then integration by parts.

And, then we can write this equation this term integral within the domain Ω_i is basically

$$\int\int_{\Omega_i} (\nabla \cdot k\nabla T + Q)\, dx\, dy$$
$$= \oint_{c_i} k\frac{\partial T}{\partial n} dl + \int\int_{\Omega_i} Q\, dx\, dy = 0$$

Q is the basically the heat generation rate.

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Finite Volume Method

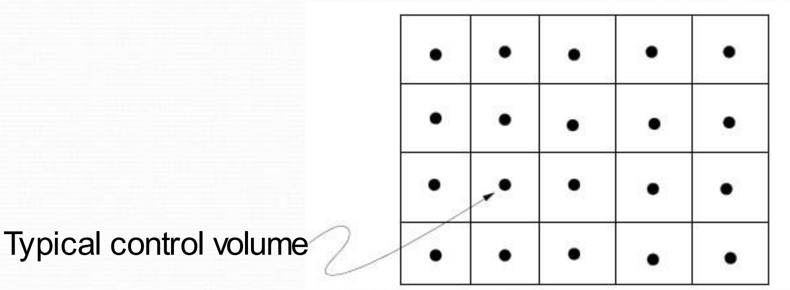

Figure 1: Grid Arrangement for the Finite Volume Method

Where the Gauss divergence theorem has been used to convert the volume integral to a surface integral.

$$\int\int_{\Omega_i} Q\, dx\, dy = \oint_{c_i} -k\frac{\partial T}{\partial n} dl \qquad (4)$$

So, where you can just refer to it the definition of the problem, then we have reached after applying this weighted residual method technique the last line and then we can say how did we reach here?

Because here we can see basically $\nabla . k\Delta T + Q$ and a we can apply Gauss divergence here theorem and then we can write convert it into basically volume integral into surface integral, but here it was surface per unit depth, so, volume per unit depth.

So, this you know by applying Gauss divergence we get basically line integral $-\frac{k\partial T}{\partial n}\,dl$ and other side is $Q\,dxdy$ (refer equation 4) over the domain integrated over the domain Ω_i.

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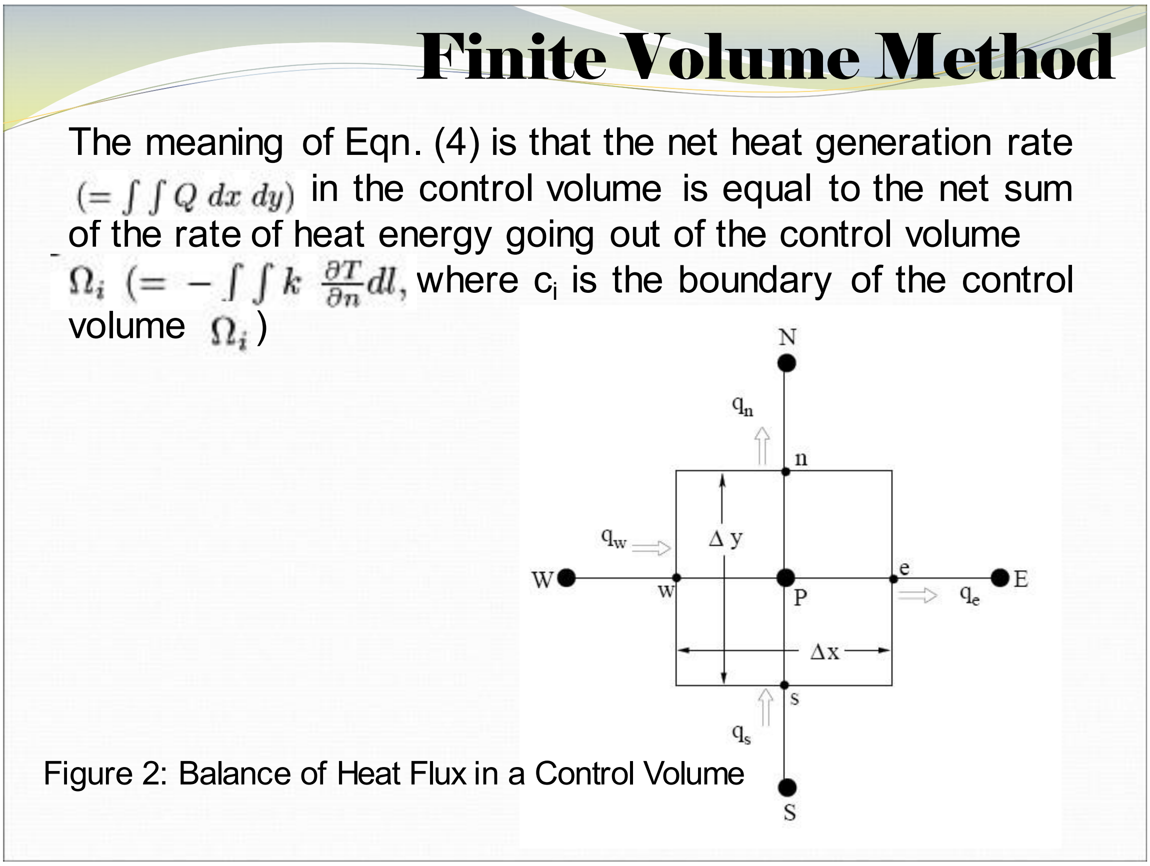

The meaning of equation 4; that means, the previous equation is that net heat generation rate which is integral over ($\iint q\,dx\,dy$) in the control volume is equal to the net sum of rate of heat energy going out of the control volume.

So, and this is you can see that $-\iint k \frac{\partial T}{\partial n}\, dl$ where c_i is the boundary of the control volume. Now, this can be translated in a physical sense by taking a single control volume where its centre is defined at P it has an eastern neighbour similar control volume that has a centre at E it has a western neighbour a similar control volume and the western side of it and it has a centre at W and a northern control volume which has a centre at N southern control volume which has a centre at S.

So, considering balance of heat flux and at the interface; that means, interface with the eastern cell and the cell of interest; that means, at the boundary, if we consider heat flux is q_e then similarly at the western side heat flux is q_w, the northern side heat fluxes this q_n and southern side heat flux is q_s.

And we can use the nomenclature if q_e is positive you know for the control volume is away from the confining surface and it is coming to the confining surface q_w. So, it will be the opposite sign.

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Finite Volume Method

Considering the balance of heat flux in Figure 2 for the control volume:

$$q_e \cdot \Delta y \cdot 1 - q_w \cdot \Delta y \cdot 1 + q_n \cdot \Delta x \cdot 1 - q_s \cdot \Delta x \cdot 1 = Q \cdot 1 \cdot \Delta x\ \Delta y$$

Net rate of energy diffusing out of the control volume through the boundary = Rate of heat generation within the control volume (CV) at steady state:

$$\sum \vec{q} \cdot d\vec{s} = Q \Delta x\ \Delta y \qquad (5)$$

So, that way

$$q_e \cdot \Delta y \cdot 1 \; - \; q_w \cdot \Delta y \cdot 1 \; + \; q_n \cdot \Delta x \cdot 1 \; - \; q_s \cdot \Delta x \cdot 1 = Q \cdot 1 \cdot \Delta x \; \Delta y$$

So, basically Δy is the dimension in y direction, q_e is going out and q_w again is coming in.

Similarly, q_n is going out through the northern face through here and q s is coming through the southern face. So, we have written that and then Q the heat generation rate per unit volume into the total volume; that means, 1 into Δx into Δy. The net rate of energy diffusing out of the control volume through boundary, so, this mode of energy transfer we can call it diffusion because you know there is no conviction involved in it.

No, radiation involved in it this is energy that is being conducted or diffused. So, out of control volume through the boundary equal to rate of generation within the control volume at steady state. So, we can summarize this as

$$\sum \vec{q} \cdot d\vec{s} = Q \Delta x \; \Delta y \tag{5}$$

$\vec{q}.\overrightarrow{ds}$ is summation on all confining surfaces.

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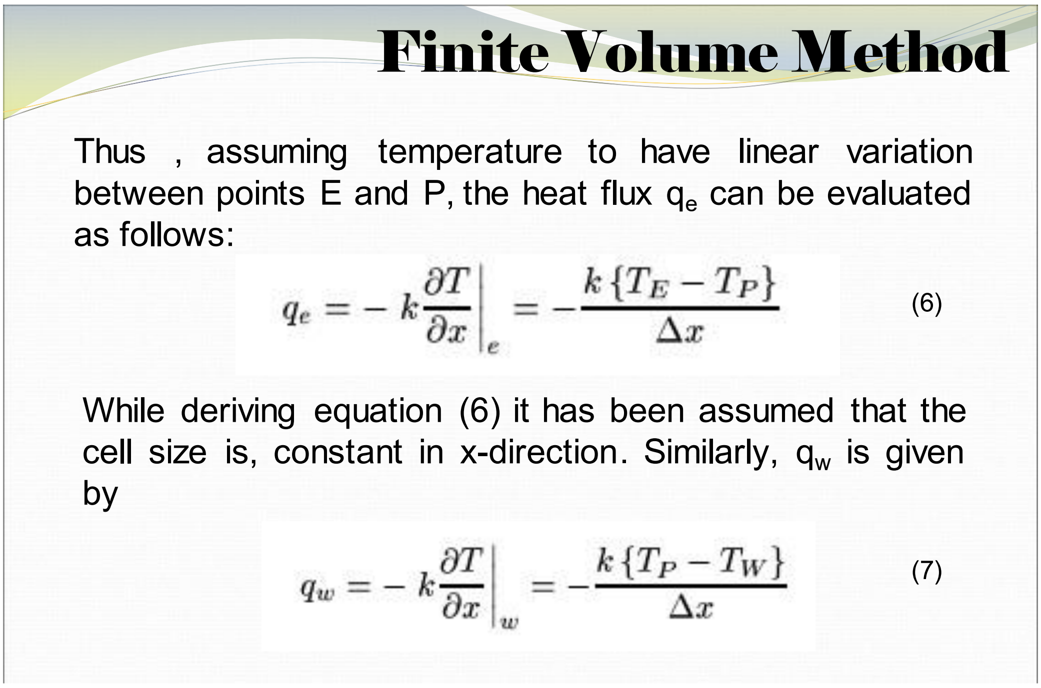

So, thus assuming temperature to have linear variation between point E and P; that means, again we go back to this figure. You know the whatever is the temperature at E and temperature at P difference between these two temperatures we will define the heat flux at q_e; similarly difference between temperature and P and W we will define the heat flux at the western surface. So, we are exactly doing that

$$q_e = -\,k\frac{\partial T}{\partial x}\bigg|_e = -\frac{k\left\{T_E - T_P\right\}}{\Delta x} \tag{6}$$

T_E that temperature the eastern point minus T_P temperature at the centre divided by Δx .

While deriving equation 6, it has been assumed that the cell size is constant in x direction similarly q_w is given by

$$q_w = -\,k\frac{\partial T}{\partial x}\bigg|_w = -\frac{k\left\{T_P - T_W\right\}}{\Delta x} \tag{7}$$

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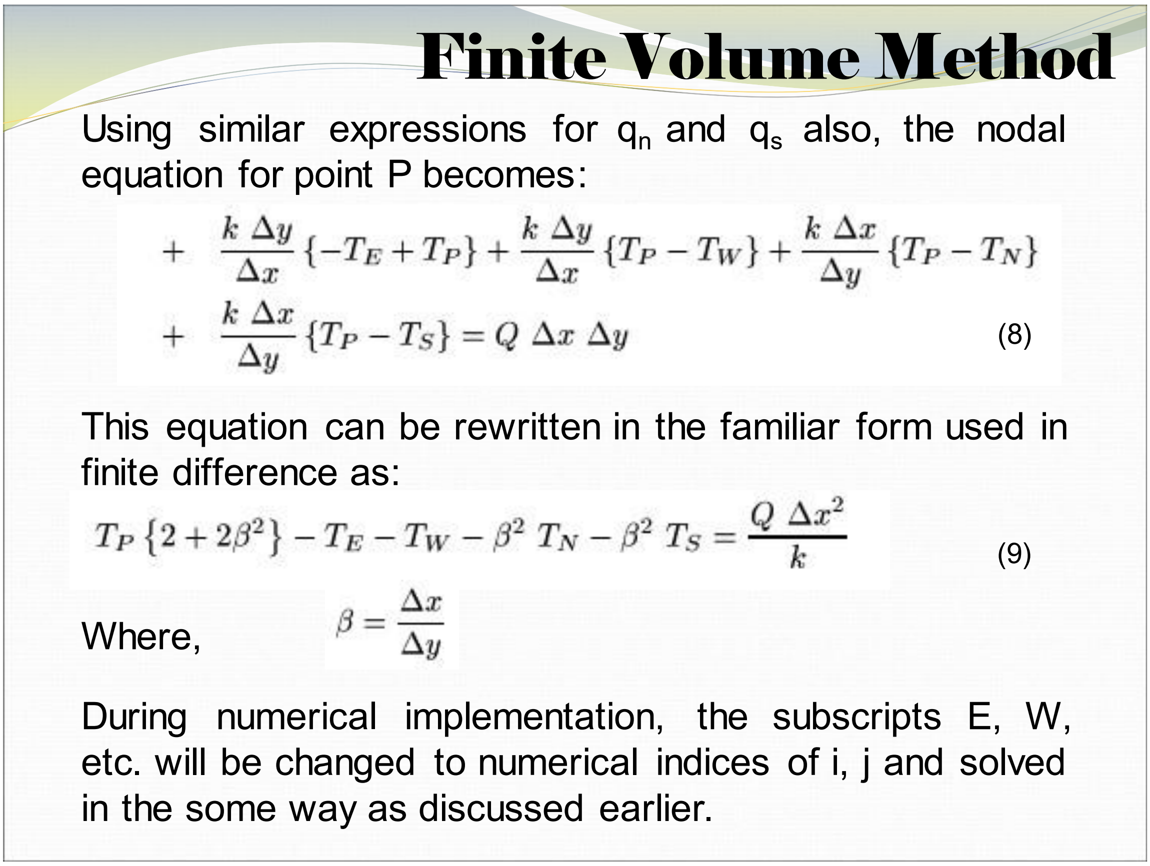

So, this way q_e, q_w, q_n, q_s. If we write them just you know what we wrote in equation (5). So, you are evaluating left hand side that left hand side now we can write

$$\begin{aligned} + \quad & \frac{k\ \Delta y}{\Delta x}\{-T_E + T_P\} + \frac{k\ \Delta y}{\Delta x}\{T_P - T_W\} + \frac{k\ \Delta x}{\Delta y}\{T_P - T_N\} \\ + \quad & \frac{k\ \Delta x}{\Delta y}\{T_P - T_S\} = Q\ \Delta x\ \Delta y \end{aligned} \tag{8}$$

just we are substituting the heat flux quantity multiplied by the area through which it is going out and coming in we will get equation (8).

Now, if we simply substitute $\frac{\Delta x}{\Delta y} = \beta$ the aspect ratio of the grade. So, we will get equation (9),

$$T_P\left\{2 + 2\beta^2\right\} - T_E - T_W - \beta^2\ T_N - \beta^2\ T_S = \frac{Q\ \Delta x^2}{k} \tag{9}$$

Now, just for the you know aspect ratio one grade Δx and Δy are same.

We can very easily see this equation will be 2 plus 2; that means, 4 this will be ($-T_E$ $+T_W$) this will be again minus $-T_N + T_S$ So, these are T_P means $T_{i,j}$, T_E means $T_{i+1,j}$, T_W is $T_{i-1,j}$, T_N is $T_{i,j+1}$ and T_S is $T_{i,j-1}$. So, this will get an algebraic equation (equation 9).

Exactly what we got following finite difference method if you recall. And from there either explicitly we find out what should be T_P and then we iterate it over every T_P; that means, every i,j or we do the implicit formulation, we express $T_{i,j}$ as unknown in terms of all is unknown neighbours and we run from cell 1 to the last cell and i, j varies we get the system of algebraic equations.

So, during the numerical implementation the we have written the subscripts E, W etc will be changed to numerical indices of i, j and solved the same way as we discussed earlier.

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Finite Volume Method

Neumann boundary condition

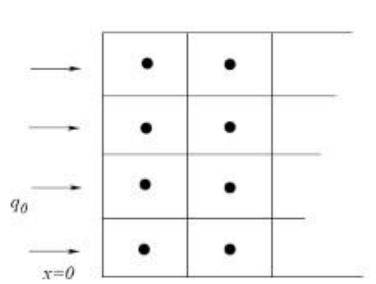

Figure 3: Prescribed Heat Flux at the Boundary

The control volumes adjacent to the x=0 boundary as shown Figure 3, the term q_w will be substituted by q_0 in equation (5). Thus,

$$
\begin{aligned}
& q_e\ \Delta y-(+q_0)\ \Delta y+q_n\ \Delta x-q_s\ \Delta x \\
= & -\frac{k\{T_E-T_P\}}{\Delta x}\ \Delta y-q_0\ \Delta y-\frac{k\{T_N-T_P\}}{\Delta y}\ \Delta x \\
+ & \frac{k\{T_P-T_S\}}{\Delta y}\ \Delta x=Q\ \Delta x\ \Delta y
\end{aligned}
\qquad (10)
$$

Now, we will discuss two special situations; that means, formulation part we have discussed, we have to discuss boundary conditions. Now, see here we have discussed about Neumann boundary; that means, see if on one phase if heat flux at x= 0 for example, if heat flux is defined that will be equivalent to Neumann boundary condition. So, the control volumes adjacent to x = 0 all these control volumes adjacent to x = 0 boundary as shown in figure 3.

Neumann boundary condition

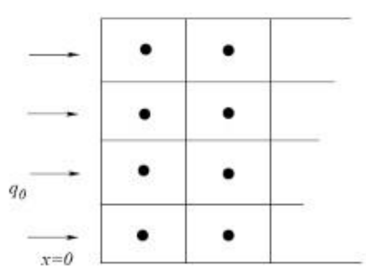

Figure 3: Prescribed Heat Flux at the Boundary

The term q_w will be substituted by q_0 because q_0 is specified, q_0 is basically thermal conductivity into the thermal gradient on this face which is $\frac{\partial k}{\partial x}$ if that is defined then; that means, q_0 is defined. So, then equation 5 means our earlier this equation, here we will just while evaluating we will write q_e into Δy eastern face, which is going out any cell we take through eastern face q_e into Δy through western face we can write q_0 into Δy which is coming in and then q_n into Δx through northern face minus q_s into Δx through the southern face and then q_e we explicitly right T_E minus T_P by Δx into thermal conductivity k the negative sign q_n also we write its expression q_s we write its expression (equation 10).

$$\begin{aligned} & q_e\ \Delta y - (+q_0)\ \Delta y + q_n\ \Delta x - q_s\ \Delta x \\ = \quad & -\frac{k\{T_E - T_P\}}{\Delta x}\ \Delta y - q_0\ \Delta y - \frac{k\{T_N - T_P\}}{\Delta y}\ \Delta x \\ + \quad & \frac{k\{T_P - T_S\}}{\Delta y}\ \Delta x = Q\ \Delta x\ \Delta y \end{aligned} \tag{10}$$

And basically, q_0 into Δy leave it like that because q_0 is now specified equal to Q it generation rate into Δx Δy equation 10, this will be the governing equation for all the cells you know that are having westerns from western side defined heat flux. So, in this symbolic equation if we can substitute the P as i, j and eastern neighbour western neighbour as i+1, j , i-1, j etc.

We will be able to set up again nodal equation on the nodes where the boundary condition is known.

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Finite Volume Method

Dirichlet condition:
T=T_L will be applied

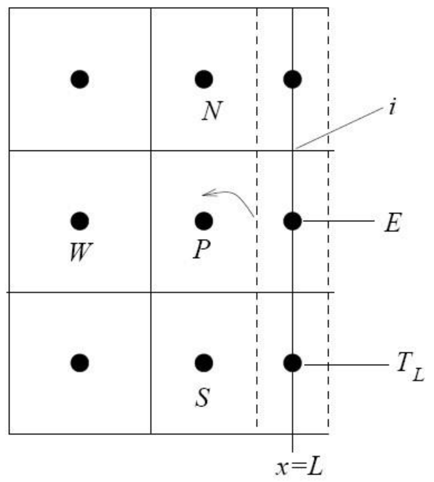

Figure 4: Boundary Condition, at x=L, T= T_L

Where $\Delta x_1 = \frac{3}{4}\ \Delta x.$

Note that T_L has to be substituted instead of T_E in the nodal equation, so that the boundary condition is directly applied.

Boundary condition is known, but Neumann boundary. Now, we will go for the Dirichlet boundary condition and we have given a typical grid distribution here you can see P is a point of interest, W is a western neighbour, S is a southern neighbour, N is a northern neighbour and the boundary condition at x = L is known.

Now, x = L ideally this point is the eastern neighbour the cell also eastern neighbour cell, but this point is falling on the edge or at the end of the boundary and we know the temperature here T_L, since it is Dirichlet condition. So, our formulation will be slightly more important. So, T_L has to be substituted instead of T_E in in the nodal equation and so, that the boundary condition can be directly applied.

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Finite Volume Method

The boundary cells will need no nodal equation as the $T=T_L$ will be applied. The nodal equation for the adjacent cell P will be written considering a shortened control volume:

$$\begin{aligned} & q_e \cdot \Delta y \; - \; q_w \cdot \Delta y + q_n \; \Delta x_1 \; - \; q_s \; \Delta x_1 \\ = \; & -k\frac{(T_L - T_P)}{(\Delta x/2)} \; \Delta y \; + k\frac{(T_P - T_W)}{\Delta x} \; \Delta y - k\frac{(T_N - T_P)}{\Delta y} \; \Delta x_1 \\ + \; & k\frac{(T_P - T_S)}{\Delta y} \; \Delta x_1 = Q \; \Delta x_1 \; \Delta y \end{aligned} \qquad (11)$$

So, if we write that if for eastern neighbour there is no problem, western neighbour no problem, northern neighbour no problem, southern neighbour no problem in writing this, but when we specifically substitute their value here you can see we have written - k (T_L-T_p).

So, eastern point it is T_L-T_P and the grid size is also changed divided by $(\Delta x/2)$, into Δy, then western neighbour remains similar expression (T_P-T_W)/Δx into k into Δy. So, northern neighbour remain same southern neighbour expression we have whatever we have done earlier and Q into but, here this for heat generation the volume is not $\Delta x\ \Delta y$, but this is basically we you can see we have written $\Delta x_1 = \frac{3}{4}\Delta x$. So, heat generation rate is $Q\Delta x_1\ \Delta y$ not $Q\Delta x\ \Delta y$. The Δx_1 means, basically three fourth Δx.

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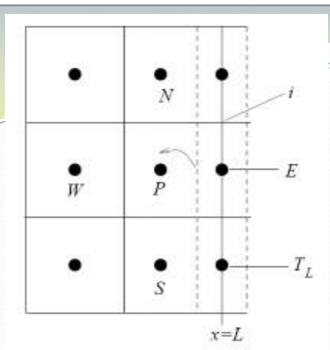
Finite Volume Method



When the boundary is specified, the control volume shapes near the boundary can be changed to facilitate the implementation of the boundary conditions. For instance, consider the condition $T = T_L$ on the $x = L$ boundary (see Figure). For the nodes on the boundary, an imaginary extension of the control volumes outside the actual domain can be considered in line with the finite difference methodology described earlier. The physical boundary is taken to be at the center of "boundary cell" of width $\Delta x/2$ (see Figure), while the widths if the adjacent cells are thus reduced to $\frac{3}{4}\Delta x$. Consider a typical control volume i near the $x = L$ boundary as shown in the Figure

So, let us explain this. Now, when the boundary is specified; that means, the eastern node all eastern nodes are now basically temperatures will be T_L.

When the boundary is specified the control volume shapes near the boundary can be change to facilitate the implementation of the boundary conditions for instance consider the condition T =T_L on the x=L. Say the domain starts somewhere at x=0, at x=L all T_L s are known. For the nodes on the boundary and imaginary extension of the control volume outside the actual domain can be considered in line with the finite difference methodology described earlier.

So, we are thinking as if this node is surrounded by a cell and this is a cell although this one part of the cell is falling outside the boundary and this is not really truly physical. So, that is what we have done the physical boundary taken to be the centre of the boundary cell width $\Delta x/2$.

So, this is the physical boundary you know of the neighbouring cell sorry and this is the basically this $\Delta x/4$, we have done these extensions and the physical boundary is taken to be centre of the boundary cell of width $\Delta x/2$ while the width of the adjacent cell reduced to $3\Delta x/4$.

So, this cell width is not now full this Δx because one forth has been taken away while defining the pseudo the cell which is not pseudo cell real cell, but this real cell is having a dimension this $\frac{\Delta x}{4}$ in either side.

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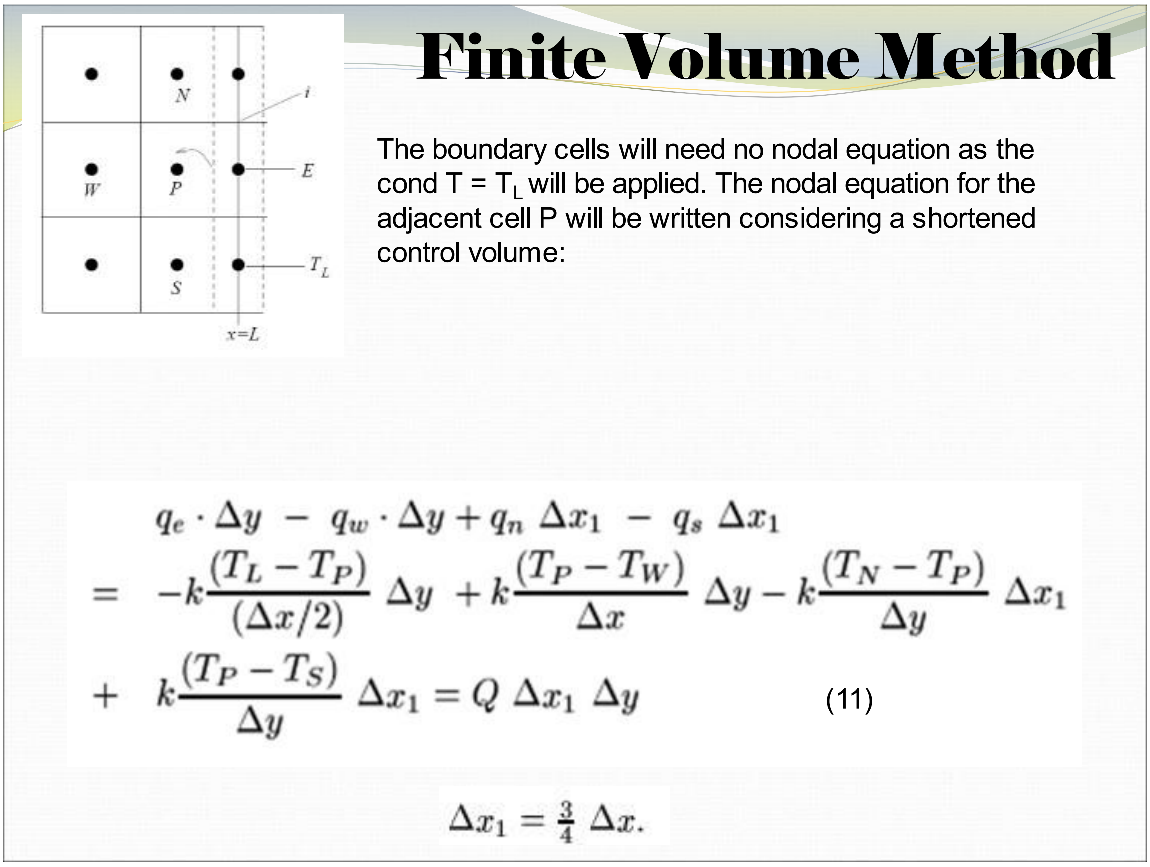

So, basically the boundary cells will need no nodal equation as the condition $T=T_L$ will be applied. The nodal equation for the adjacent cell P will be written considering then shorted control volume. So, again let us go for what we wrote earlier,

$$
\begin{aligned}
& q_e \cdot \Delta y - q_w \cdot \Delta y + q_n \Delta x_1 - q_s \Delta x_1 \\
= & -k\frac{(T_L - T_P)}{(\Delta x/2)} \Delta y + k\frac{(T_P - T_W)}{\Delta x} \Delta y - k\frac{(T_N - T_P)}{\Delta y} \Delta x_1 \\
+ & \; k\frac{(T_P - T_S)}{\Delta y} \Delta x_1 = Q \Delta x_1 \Delta y
\end{aligned} \tag{11}
$$

One can see the corresponding changes in cells at the x=L have Δx_1.

So, basically you can see that these nodes we are directly writing T_L, we are not generating any equation for them, but the equation for P with respect to all neighbours when we are expressing that we write here the flux T_E - T_P or T_L-T_P by $\Delta x/2$ here we write T_P-T_W by Δx.

Here, (T_N-T_P)/Δy and here (T_P – T_S)/Δy and the when we multiply the area here through which flux is coming here it is Δy into 1 here it is Δy into 1, but here it is 3 by 4 Δx into 1.

So, that is the difference and this way now we can set the equations. So, we think about it how the boundary condition when it is Dirichlet, how the boundary condition is defined and the since it is directly on the node no nodal equation is generated for you know these points, but the inner points you know for example, when this is P what should be the equation we have come to know.

So, philosophy is same heat flux in this direction only thing while deciding heat flux it will be decided k(T_E-T_P)/$\left(\frac{\Delta x}{2}\right)$ and here it will remain same k into (T_P - T_W)/Δx here it will be (T_N-T_P)/Δy, here it will be (T_P – T_S)/Δy and k is obviously, they are with all the heat fluxes.

And as I said area of concern here it is delta when it is heat flux between T_L and T_P then area through which it is crossing is basically Δy into 1 between P and W again Δy into 1, but when northern node and P node are involved this heat flux; that means, q_n here the area multiplying area will be three fourth Δx into 1.

And similarly, between the southern node and then node P, the heat flux will be multiplied by 3 by 4 of Δx into 1 and energy generation rate also will be considered within this control volume. This dotted control volume; we do not write any equation; this is represented by a point while temperature value is directly known on the value of the dependent variable is directly known.

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Finite Volume Method

In this way by adjusting the control volume spacing and the placement of nodes, nodal equations can be obtained at all nodes and these can be solved simultaneously by the matrix inversion technique, line-byline technique or point-by-point technique as discussed earlier. The point T_P will be substituted by $T_{i,j}$, T_E will be $T_{i+1,j}$, T_W will be $T_{i-1,j}$ so on and so forth.

Having done the above exercise, we may like to look at a more generalized (in curvilinear coordinates) description of the finite volume method.

So, to today's first lecture on control volume, we will we can say that what we have studied is the following that by adjusting the control volume spacing and the placement of nodes, nodal equations can be obtained at all nodes and this can be solved simultaneously by the matrix inversion technique.

So, we are talking about implicit formulation; that means, you know basically node of interest P we will express in terms of neighbours and while expressing if neighbours are you know represented if they are by you know they are node number, then the they will come straight away.

If the neighbours are represented by boundary condition, then we will adapt the boundary condition; that means, you know all this while formulating the problem neighbouring nodes to the boundary those are to be paid special interest a special attention. And then we will be able to get the algebraic equations, if we run *i* from 1 to imax (i maximum) and *j* is a node number from 1 to jmax (j maximum).

So, this can be solved simultaneously by matrix inversion technique, all line-by-line technique or point by point technique as discussed earlier the point T_P will be substituted by $T_{i,j}$, T_E will be substituted by $T_{i+1,j}$, T_W will be substituted by $T_{i-1,j}$. Similarly, T_N will be substituted by $T_{i,j+1}$, T_S will be submitted by $T_{i,j-1}$ and so, having done the above exercise we may like to look at a more generalized.

So, this is you know basically Cartesian coordinate we have used here. So, grids are straightforward what we try to focus at the formulation is based on integral technique everywhere we are integrating and finding out the nodal equation and this is different from the finite difference method. We are not life in a different method we are not following the or forming the different quotient and substituting different quotients in the differential equation.

Whether we are unlike that we are integrating it over the control volume like as I said that equation 11 or equation any equation you take or equation 9, these although since the geometry is simpler, it is in Cartesian coordinate and in Δx and Δy the grid spacing is also they will learn.

So, we will get exactly if we express T_P, T_E, T_W, T_N, T_S by $T_{i,j}$ $T_{i,j-1}$ $T_{i,j+1}$ or northern node and $T_{i+1,j}$ or eastern node and for $T_{i-1,j}$ for T_W then we will get an equation which is; which we also get in finite difference method, but here the methodology we will follow is different. We integrated the fluxes on the phases you know through which fluxes are passing through or going from one cell to another cell or going out of the domain as a boundary and boundary condition is being defined.

So, this is the difference, but you know even then expression what we got expression is the same, if we take you know Cartesian coordinate and we integrate it on the Cartesian control volumes smaller control volumes. But as I mentioned here the last line that in complex geometry grids will be curvilinear grids the and grid mesh will be complex.

And there then the same integration technique, but we have to improvise we have to see how to improve this integration technique; so, that the integration can be done on the curve surfaces in order to balance the fluxes. So, we will stop here today we will take up these aspects in the subsequent classes.

Thank you very much.