Computational Fluid Dynamics and Heat Transfer
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Lecture – 06
Important Aspects of Flow Modeling-2

Good afternoon. We will start our lecture today on some more Important Aspects of Flow Modeling. We took up this issue in our last lecture. We discussed certain aspects. We will cover some more aspects today.

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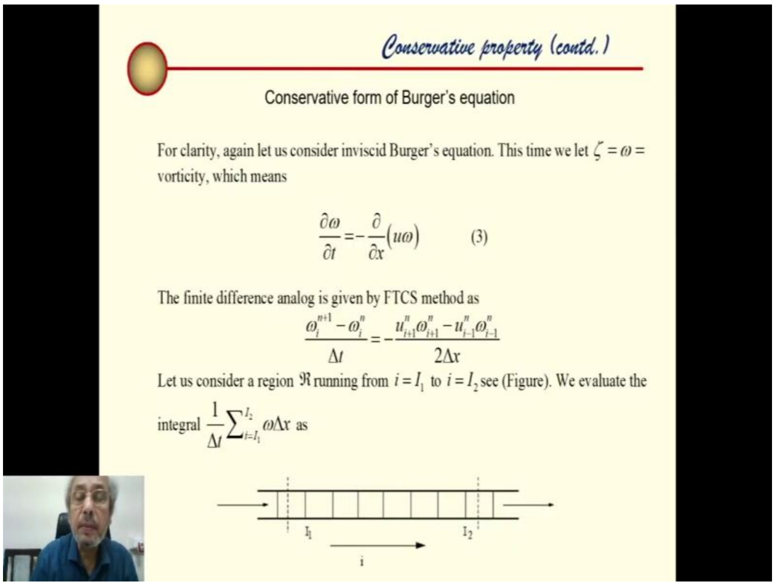
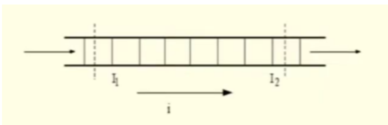


$$\frac{\partial \omega}{\partial t} = \frac{\partial (u\omega)}{\partial x} \qquad (3)$$

$$\frac{\omega_i^{n+1} - \omega_i^n}{\Delta t} = \frac{u_{i+1}^n \omega_{i+1}^n - u_{i-1}^n \omega_{i-1}^n}{2\Delta x}$$

Let us consider a region $\Re$ running from $i=I_1$ to $i=I_2$ see (Figure). We evaluate the integral $\frac{1}{\Delta t}\sum_{t=t_1}^{t_2} \omega \Delta x$ as



But before we start our discussion, we will just recapitulate something from the last class. In the last class we took up Burger's equation inviscid Burger's equation and then we tried FTCS method on it, but while taking up the equation we wrote a form as given in equation 3, we called it a conservative form and then we tried to discretize equation 3 in conservative form.

So, omega i n plus 1 minus omega i n by delta t is the left-hand side and right-hand side as we can see we have written in central difference. And then we considered one dimensional space as given in the figure and we are running from i equal to I 1 to I equal to I 2. So, left hand side basically we can write that is integral omega delta x running from i equal to I 1 to I 2 divided by 1 by delta t and right-hand side is summation of this quantity over the integral space.

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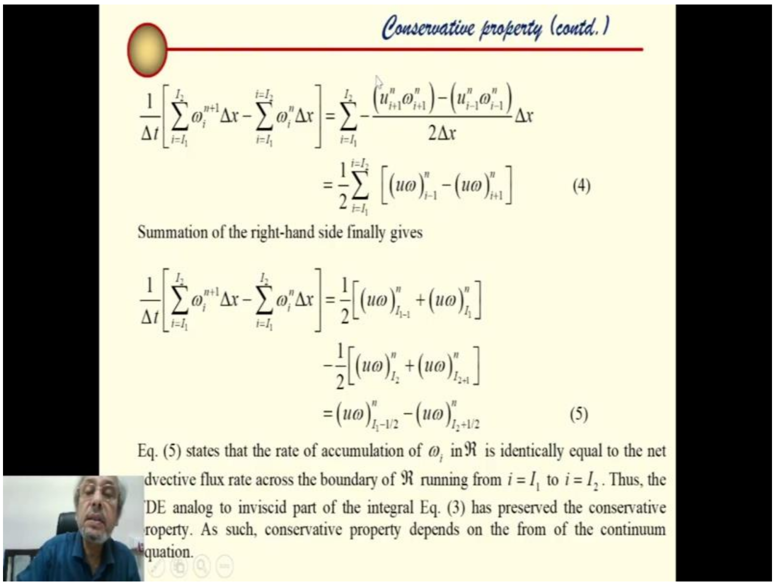

$$\frac{1}{\Delta t}\left[\sum_{i=I_1}^{i=I_2}\omega_i^{n+1}\Delta x-\sum_{i=I_1}^{i=I_2}\omega_i^{n}\Delta x\right]=\sum_{i=I_1}^{i=I_2}-\frac{\left(u_{i+1}^{n}\omega_{i+1}^{n}\right)-\left(u_{i-1}^{n}\omega_{i-1}^{n}\right)}{2\Delta x}\Delta x$$

$$=\frac{1}{2}\sum_{i=I_1}^{i=I_2}\left[\left(u\omega\right)_{i-1}^{n}-\left(u\omega\right)_{i+1}^{n}\right] \qquad ....(4)$$

Summation of the right -hand side finally gives

$$\frac{1}{\Delta t}\left[\sum_{i=I_1}^{i=I_2}\omega_i^{n+1}\Delta x-\sum_{i=I_1}^{i=I_2}\omega_i^{n}\Delta x\right]=\frac{1}{2}\sum_{i=I_1}^{i=I_2}\left[\left(u\omega\right)_{I_{i-1}}^{n}-\left(u\omega\right)_{I_1}^{n}\right]$$

....

$$-\frac{1}{2}\sum_{i=I_1}^{i=I_2}\left[\left(u\omega\right)_{I_2}^{n}-\left(u\omega\right)_{I_{2+1}}^{n}\right]$$

$$=\left(u\omega\right)_{I_1-\frac{1}{2}}^{n}-\left(u\omega\right)_{I_{2+1}+\frac{1}{2}}^{n} \qquad (5)$$

And if we do that, we can see that we will be able to write simple algebra this half summation of i equal to I 1 to I 2 u omega i minus 1 at nth level minus 0 omega i plus 1 at nth level. Now, we have to run, substitute i equal to I 1 in equation 4 then i 1 plus 1 then i 1 plus 2. This way we will go closer to i 2.

We can say maybe i 2 minus 2 i 2 minus 1 then i 2. If we do these substitutions, we will see that the summation series that we will get from expression 4 in right hand side all intermediate terms different values of i's; that means, i equal to I plus 1 I plus 2 I plus 3 etcetera those terms will terms generated from this expression will get cancelled out.

Finally, we will get this term, intermediate fluxes are all cancelled out, left hand side remains same, right-hand side is half u omega i 1 minus 1 at nth level plus u omega i 1 at nth level. All intermediate fluxes if you write systematically, you will see they will cancel out. Minus again half u omega at I 2 nth level plus 0 omega at I 2 plus 1 at nth level; that means, then here u omega summation of whatever is value here plus whatever is value here and minus basically u omega again values at I 2 and I 2 plus 1.

Now, I 1 is this point and I 1 minus 1 is this. So, I 1 minus 1 plus I 1 average is flux at here. Similarly, I 2 I 2 plus 1 average is I 2 plus half flux at this dotted line. So, u omega flux at I 1 minus half minus u omega I 2 plus half at nth level. So, equation 5 states that the rate of accumulation of omega i in R is identically equal to the net advective flux rate across the boundary of R running from i equal to I 1 to I 2.

Thus, the inviscid Burger's equation analog of inviscid part of equation 3 we can say has preserved the conservative property. As such conservative property depends on the form of the continuum equation. Since this form of equation 3 this itself is conservative form. So, this preserves the conservative property.

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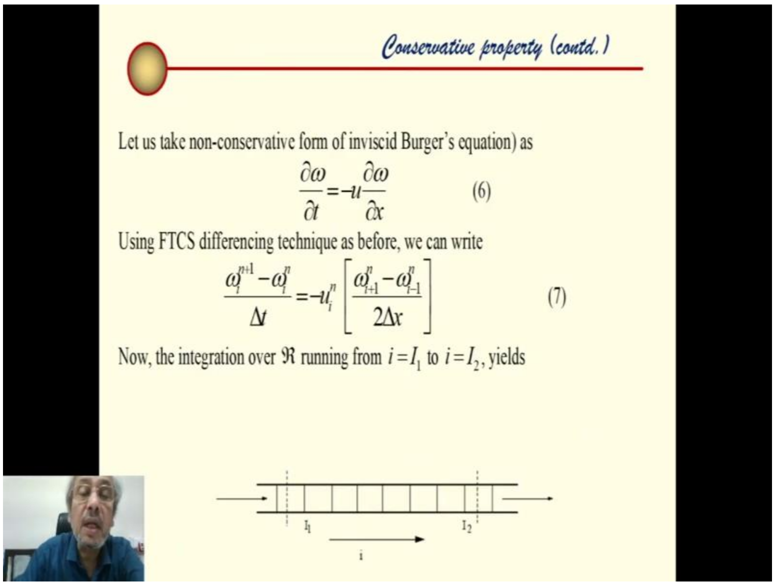


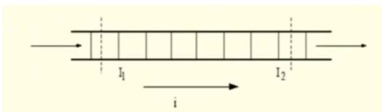
Consider a non-conservative form of inviscid Burger's equation as

$$\frac{\partial \omega}{\partial t} = u \frac{\partial (\omega)}{\partial x} \tag{6}$$

Using FTCS differencing technique as before, we can write

$$\frac{\omega_i^{n+1} - \omega_i^n}{\Delta t} = u_i^n \left[\frac{\omega_{i+1}^n - \omega_{i-1}^n}{2\Delta x} \right] \tag{7}$$

Now the integration over $\Re$ running from $i=I_1$ to $i=I_2$, yields



On the other hand, if we take non conservative form same Burger's equation instead of 3 if we write 6 where we do not write it in conservative form, we write it in non conservative form. Then again, we can apply FTCS. We will get this expression. Then again, we can integrate this expression between i equal to I 1 to I 2.

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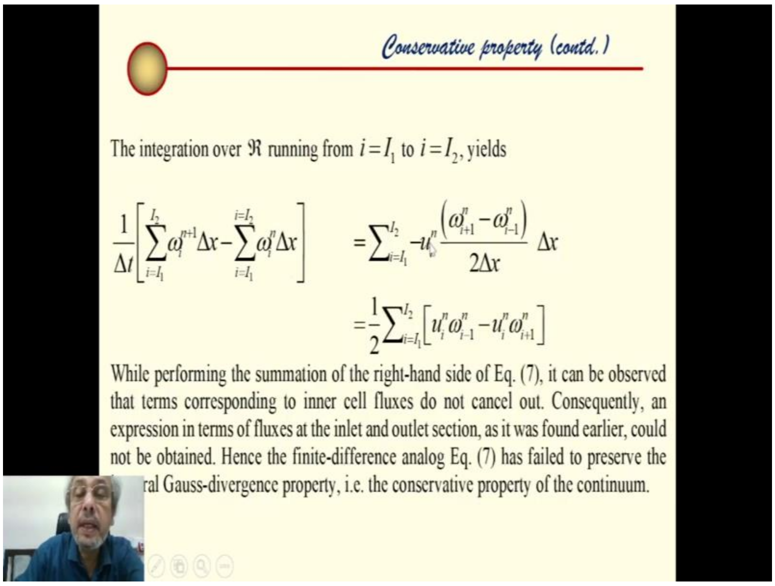

$$\frac{1}{\Delta t}\left[\sum_{i=I_1}^{i=I_2}\omega_i^{n+1}\Delta x-\sum_{i=I_1}^{i=I_2}\omega_i^{n}\Delta x\right]=\sum_{i=I_1}^{i=I_2}-u_i^n\,\frac{\left(\omega_{i+1}^n-\omega_{i-1}^n\right)}{2\Delta x}$$

$$=\frac{1}{2}\sum_{i=I_1}^{i=I_2}\frac{\left(u_i^n\omega_{i-1}^n-u_i^n\omega_{i+1}^n\right)}{2}$$

And then we will get this expression on the right-hand side and we can see the right-hand side fluxes if we apply summation i equal to I 1 to I 2 on such fluxes u i omega at i minus 1 u i omega at i plus 1 at nth level. And if we keep on you know substituting i equal to I 1, I 1 plus 1, I 1 plus 2 up to maybe I 2 minus 2, I 2 minus 1 up to I 2 then we will see that intermediate fluxes will not get cancelled out.

And the final expression will not be outlet flux in the domain minus in inlet flux, so, which we obtained in the earlier case. Hence the finite difference analog here we are calling equation 7 is unable to preserve the integral Gauss divergence property that is conservative property of continuum. We discussed this conservative property and then said there are weak conservative form and strong conservative forms.

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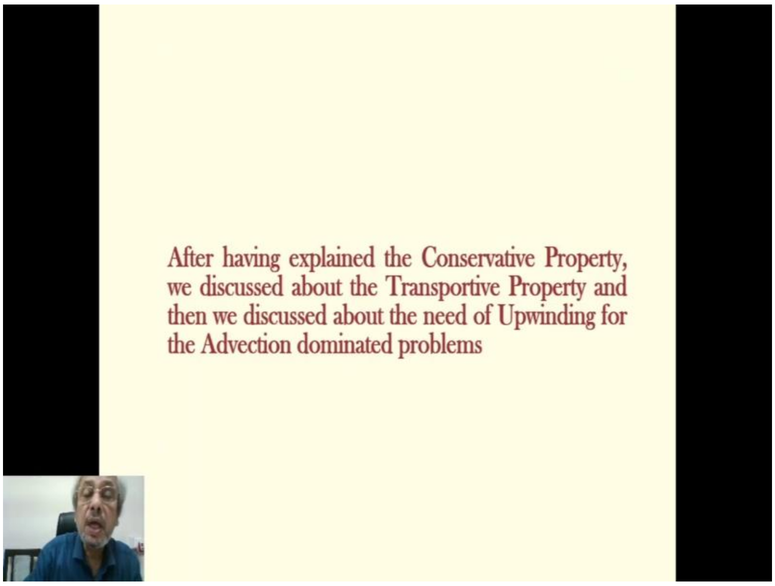

And then we discussed about transporting property and we also discussed about need of upwinding for the advection dominated flow problems.

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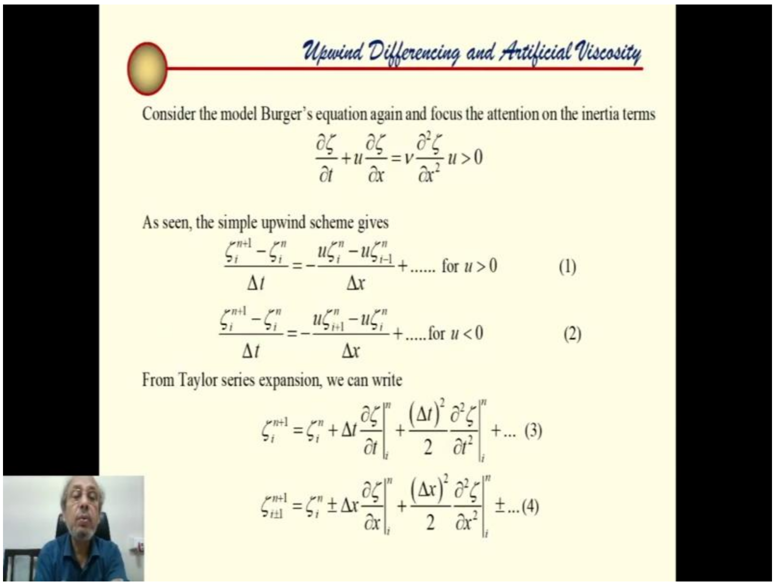

Consider the model Burger's equation again and focus the attention on the inertia terms

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} = \nu \frac{\partial^2 \zeta}{\partial x^2} \quad \text{for } u > 0$$

As seen, the simple upwind scheme gives

$$\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}=-\frac{u\zeta_i^n-u\zeta_{i-1}^n}{\Delta x}+\ldots\ldots\text{for } u>0 \tag{1}$$

$$\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}=-\frac{u\zeta_{i+1}^n-u\zeta_i^n}{\Delta x}+\ldots\ldots\text{for } u<0 \tag{2}$$

$$\zeta_i^{n+1}=\zeta_i^n+\Delta t\left.\frac{\partial\zeta}{\partial t}\right|_i^n+\frac{(\Delta t)^2}{2}\left.\frac{\partial^2\zeta}{\partial t^2}\right|_i^n+\ldots. \tag{3}$$

$$\zeta_{i\pm1}^{n+1}=\zeta_i^n\pm\Delta x\left.\frac{\partial\zeta}{\partial x}\right|_i^n+\frac{(\Delta x)^2}{2}\left.\frac{\partial^2\zeta}{\partial x^2}\right|_i^n\pm\ldots \tag{4}$$

Now, we will take up basically again Burger's equation and we have written del zeta del t plus u del zeta del x equal to nu del 2 zeta del x 2 for u greater than 0. Now, for upwind scheme we have seen for u greater than 0, we will write zeta at ith location will be zeta i i minus 1. So, zeta i at n plus 1 minus zeta i at n.

So, zeta i n plus 1minus Zeta i at nth level divided by delta t equal to we have taken the inviscid part minus u zeta i at n minus u zeta i minus 1 n by delta x for u greater than 0 and u less than 0. If u is negative then we decided to use i plus 1 minus i. So, basically if u is negative, we will use downstream point and if u is positive, we will use upstream point.

So, again we can write Taylor series expansion for temporal variable; that means, zeta i n plus 1 equal to zeta i n plus delta t del zeta del t plus delta t square by 2 del 2 zeta del t 2 etc. Similarly, in the spatial domain zeta i plus 1 is zeta i plus minus this is zeta i plus minus 1, so obviously, this sign is then plus minus delta x del zeta del x and plus delta x square by 2 del 2 zeta del x 2 again plus minus del x s cube by factorial 3 del cube zeta plus del x cube etc. So, this we will what we will do now? We will try to substitute in this expression.

For example, in this expression if we want to write zeta i zeta n plus 1 minus z i at zeta nth level divided by delta t. That means, this quantity minus zeta i n by delta t will get remaining terms on the right-hand side which is getting truncated. And we will try to get this truncated series for this term for this term and see what happens.

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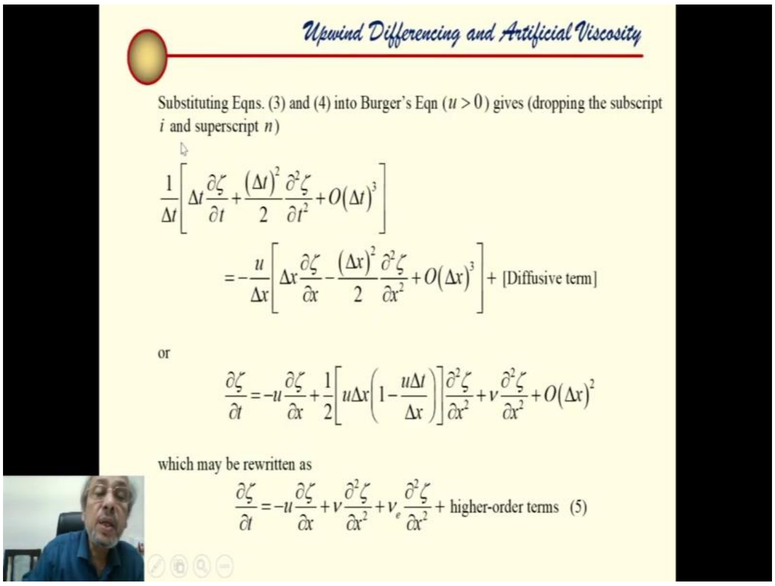

Substituting Eqns. (3) and (4) into Burger's Eqn ($u > 0$) gives (dropping the subscript i and superscript n)

$$\frac{1}{\Delta t}\left[\Delta t \frac{\partial \zeta}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 \zeta}{\partial t^2} + o(\Delta t)^3\right] = -\frac{u}{\Delta x}\left[\Delta t \frac{\partial \zeta}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 \zeta}{\partial x^2} + o(\Delta x)^3\right] + [\text{Diffusive term}]$$

or

$$\frac{\partial \zeta}{\partial t} = -u\frac{\partial \zeta}{\partial x} + \frac{1}{2}\left[u\Delta x\left(1 - \frac{u\Delta t}{\Delta x}\right)\right]\frac{\partial^2 \zeta}{\partial x^2} + \nu\frac{\partial^2 \zeta}{\partial x^2} + o(\Delta x)^2$$

which may be written as

$$\frac{\partial \zeta}{\partial t} = -u\frac{\partial \zeta}{\partial x} + \nu\frac{\partial^2 \zeta}{\partial x^2} + \nu_e\frac{\partial^2 \zeta}{\partial x^2} + \text{higher-order terms} \qquad (5)$$

So, if we substitute for here basically 1 by delta t within bracket zeta i at n plus 1 nth level minus zeta i at nth level, we will get all this entire truncated series 1 by delta t. Similarly, you know 1 by delta x then zeta i minus zeta i minus 1 or zeta i minus zeta i plus 1 whatever depending on the upwinding, if we want to write that we will be able to write minus u by delta x del del x of del zeta del x minus del x square by 2, we have taken for positive u.

So, basically minus u by delta x we are writing outside the bracket and then inside the bracket zeta i minus zeta i minus 1. From here truncated series we will substitute here plus diffusive terms.

And then if we try to reconstruct the equation, we will see that left hand side we will get del zeta del t, right hand side we will get minus u del zeta del x plus half u delta t 1 minus u delta t by del x del 2 zeta del x 2 and diffusive term was already there which is nu del 2 zeta del x 2 plus other higher order terms.

Now, apparently after constructing this term; that means, after multiplication this first term is producing u del zeta del x, but second term we have written as u you can see one delta x will cancel out delta x 1 minus u delta t by delta x and del 2 zeta del x 2. So, basically you know this term and the term which is coming out after approximating this term, the second derivative of time here, and the second derivative of space here, from these two whatever is the truncation this is otherwise truncated.

Now, if that is not completely ignored then from these two terms, we will get this. How? We will explain in the next slide. So, what we can say that del zeta del t equal to minus u del zeta del x. On the right-hand side first term has been taken up. The last term is nu del 2 zeta del x 2 is basically the viscous term which is already there. We have not disturbed it. We have written that here nu del 2 zeta del x 2, but we have got this additional term while reconstructing the equation and we are writing this as nu e del 2 zeta del x 2.

So that means, this is some effective viscosity or which is non physical arising out of our discretization technique. This entire term half u delta x into 1 minus u delta t by delta x, this is taken as nu e which is artificial viscosity it is behaving like since the second order term behaving like viscous term, but this viscosity is not a physical property. This is coming because if we rearrange the terms, the temporal derivative and the special derivative; that means, first order temporal derivative.

I mean if we write the highest term of the truncated series and first order special derivative, if we write highest term of the truncated series if we can organize them effectively, we will get this term. But while doing this see from here getting del 2 zeta del x 2 and u delta x square one delta x will cancel out by 2 is very simple. u delta x by 2 into del 2 zeta del x 2 is if we open this bracket, we will get that term from these bracketed terms that comes from here from the special derivative.

But, this term second term if we again open the bracket rationalize the quantities, second term is coming from the temporal derivative. How? We will see. Now, so, this term first of all we can call this as artificial viscosity and we can write this as half u delta x into 1 minus we have seen in the last lecture that we can call this term u delta t by delta x as a Courant number. So, this is half u delta x 1 minus Courant number. Now, in deriving equation 5 what we did?

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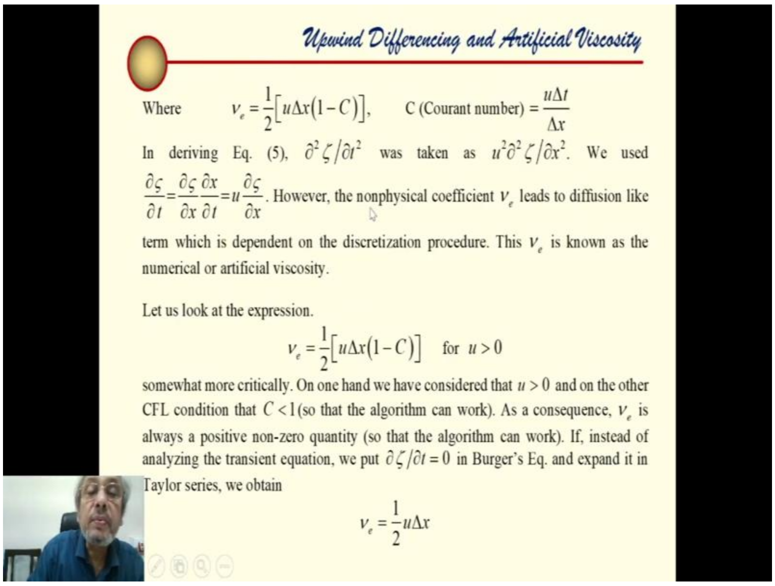

where $\qquad \nu_e = \frac{1}{2}\left[u\Delta x(1-C)\right]$, C(Courant number) $= \frac{u\Delta t}{\Delta x}$

$\frac{\partial^2\zeta}{\partial t^2} \quad u^2\partial^2\zeta/\partial x^2$. We used $\frac{\partial\zeta}{\partial t}=\frac{\partial\zeta}{\partial x}\frac{\partial x}{\partial t}=u\frac{\partial\zeta}{\partial x}$, however the non-physical coefficient ν_e leads to diffusion like term, which is dependent on the discretization procedure. This ν_e is known as numerical or artificial viscosity.

Let us look at the expression $\nu_e = \frac{1}{2}\left[u\Delta x(1-C)\right]$ for $u > 0$

$$\nu_e = \frac{1}{2}u\Delta x$$

This del 2 zeta del t 2 del 2 zeta del t 2 which was the having the highest contribution among the truncated series, we have written that del 2 zeta del x 2 as u square del 2 zeta del x 2. How? Because del zeta del t we can write as del zeta del x into del x del t and del x del t x is the spatial dimension del x del t gives a velocity. So, velocity into del zeta del x.

Again, if we take the second derivative of it we will be able to get yet you know another u and del 2 zeta del x square. So, it will be u square del 2 zeta del x square exactly that is what we got here ok. u square del 2 zeta del x square, but if you multiply this term outside the bracket with the terms inside the bracket will get multiplied of del 2 zeta del x 2 as u squared.

So, the non-physical coefficient of viscosity nu e leads to diffusion like term which is dependent on the discretization process. So, this nu e is known as the numerical viscosity or artificial viscosity.

So, when you are reconstructing getting back the equation effectively we are getting del zeta del t equal to minus u del zeta del x plus nu del 2 zeta del x 2. This is a physical equation, but as if we are adding when you are discretizing this way basically we are effectively adding new artificial viscosity into del 2 zeta del x 2. And the artificial viscosity is half u delta x 1 minus Courant number.

Now, somewhat we if you look at it more critically on we can see that we got this because this discretization we started as the with you know the possibility of u being positive. u being negative will give us the same conclusion similar conclusion not the same conclusion, but here u being positive we are getting u delta x 1 minus Courant number.

Now, u is positive quantity. Courant number has to be less than 1. Now, if Courant number for stability that we have seen. So, if Courant number is less than 1 this quantity is a positive quantity and this quantity is a positive quantity. So, we will see that this artificial viscosity is positive quantity and this will definitely make some contribution in the final numerical solution. Obviously, our effort will be to reduce artificial viscosity in such a way that it does not influence the solution.

Now, this expression we got because we took unsteady state term also that del zeta del t full Burger’s equation unsteady term. It is a steady equation or steady term is not there then we can clearly see there will be no Courant number and the effective viscosity will be this artificial viscosity will be basically half u delta x, C will be 0.

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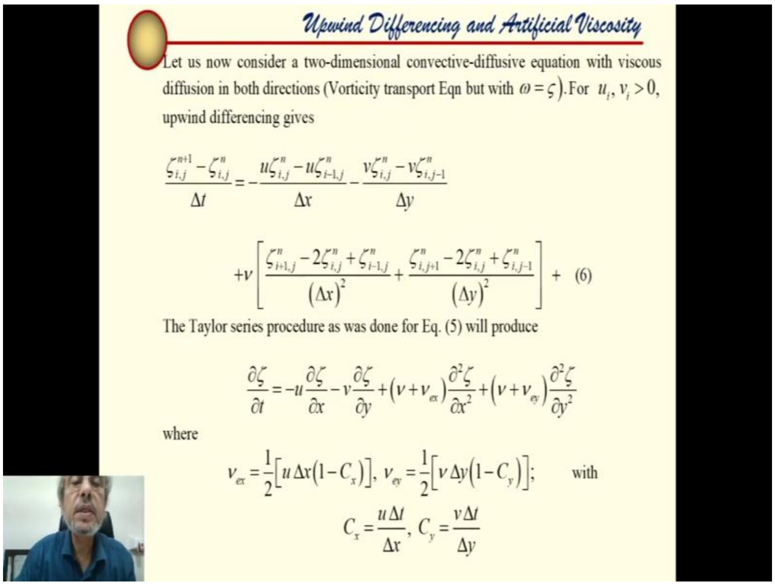

Let us consider a two-dimensional convective-diffusive equation with viscous diffusion in both directions (Vorticity transport Eqn but with $\omega = \zeta$). For $u_i, v_i > 0$, upwind differencing gives

$$\frac{\zeta_{i,j}^{n+1} - \zeta_{i,j}^{n}}{\Delta t} = -\frac{u\zeta_{i,j}^{n} - u\zeta_{i-1,j}^{n}}{\Delta x} - \frac{v\zeta_{i,j}^{n} - v\zeta_{i,j-1}^{n}}{\Delta y}$$

$$+\nu\left[\frac{\zeta_{i+1,j}^{n} - 2\zeta_{i,j}^{n} + \zeta_{i-1,j}^{n}}{(\Delta x)^2} + \frac{\zeta_{i,j+1}^{n} - 2\zeta_{i,j}^{n} + \zeta_{i,j-1}^{n}}{(\Delta y)^2}\right] + \qquad (6)$$

The Taylor series procedure done for equation (5) will produce

$$\frac{\partial \zeta}{\partial t} = -u\frac{\partial \zeta}{\partial x} - v\frac{\partial \zeta}{\partial y} + (\nu + \nu_{ex})\frac{\partial^2 \zeta}{\partial x^2} + (\nu + \nu_{ey})\frac{\partial^2 \zeta}{\partial y^2}$$

$$\nu_{ex} = \frac{1}{2}\left[u\Delta x(1 - C_x)\right],\ \nu_{ey} = \frac{1}{2}\left[v\Delta y(1 - C_y)\right];\ \text{with}$$

$$C_x = \frac{u\Delta t}{\Delta x},\ C_y = \frac{v\Delta t}{\Delta y}$$

Now, let us consider a two-dimensional convective diffusive equation with viscous dissipation in both direction that and we will use the vorticity transport equation. And substitute zeta by omega and our velocity u i and v i both are positive. So, if we do that the equation is del zeta del t plus u del zeta del x plus v del zeta del y equal to nu del 2 zeta del x 2 plus del 2 zeta del y 2.

And then we are substituting the difference quotients. Instead of del zeta del t we are writing del zeta i, j at n plus oneth level minus zeta is a at nth level divided by delta t. And since u i is positive we are upwinding it we are involving upstream point; zeta i u is positive that is why it will be i, j minus i minus 1 j.

Please recall that if u is positive, we will involve upstream point. Basically, if i, j is the point of interest then i minus 1 j the upstream point that is what is upwinding and if u is negative, we will involve the downstream point in the finite difference quotient. So, here u is positive. So, we have used i, j minus zeta upstream point i minus 1 j.

Similarly, v is positive. We have written zeta i, j and zeta i, j minus 1 delta y and then we are we have written nu or the viscosity and then right-hand side the central difference. This is simple. Zeta i plus 1 j minus zeta i, j plus zeta i minus 1 j by delta x square; similarly, zeta j plus 1 minus zeta i, j plus zeta i, j minus 1 by delta x delta y square.

And just the way we did in the last example if we write the partial differential equation and substitute the finite difference quotients by these terms and then we bring the truncated terms, and try to reconstruct back the equation, we will get then del zeta del t equal to minus u del zeta del x minus v del zeta del y plus nu artificial in x direction, So, artificial viscosity del 2 zeta del x 2 plus nu this is physical viscosity plus nu ey in artificial viscosity in y direction del 2 zeta del y 2.

So, if we want to write down the expression for artificial viscosity, we will get half u delta x into 1 minus Courant number in x direction. That means, we will involve u velocity and delta x here. And, again artificial viscosity in y direction half v delta y 1 minus Courant number in y direction; that means, v delta t by delta y, and here u delta t by delta x.

So, what we can say that for two-dimensional equation, we if we reconstruct the equation after having discretized through FTCS method we will be able to see generation of artificial viscosity in both the directions. And those artificial viscosities are functions of respective Courant numbers in x and y directions.

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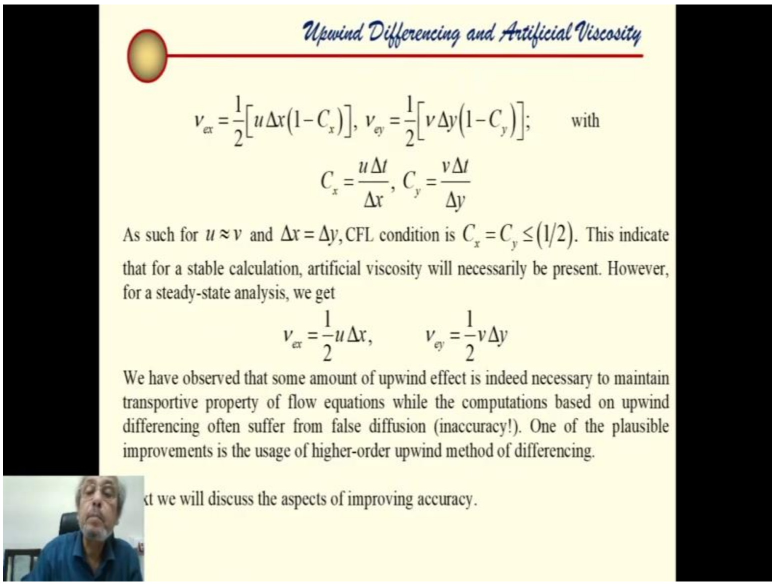

$$\nu_{ex} = \frac{1}{2}\left[u\Delta x\left(1-C_x\right)\right],\ \nu_{ey} = \frac{1}{2}\left[u\Delta y\left(1-C_y\right)\right];\ \text{with}$$

$$C_x = \frac{u\Delta t}{\Delta x}\ ,\ C_y = \frac{v\Delta t}{\Delta y}$$

As such for $u \approx v$ and $\Delta x = \Delta y$, CFL condition is $C_x = C_y \le \left(\frac{1}{2}\right)$. This indicate that for stable calculation, artificial viscosity will necessarily be present. However, for a steady-state analysis, we get

$$\nu_{ex} = \frac{1}{2}u\Delta x \qquad\qquad \nu_{ey} = \frac{1}{2}v\Delta y$$

We have observed that some amount of upwind effect is indeed necessary to maintain transportive property of flow equations while the computations based on upwind differencing often suffer from false diffusion (inaccuracy!). one of the plausible improvements is the usage of higher-order upwind method of differencing.

So, now this is the conclusion. Again, I am repeating that for the next slide also that artificial viscosity in x direction, artificial viscosity in y direction. In x direction artificial viscosity is having Courant number in x direction u delta t by delta x and artificial viscosity in y direction is having Courant number in y direction.

Now, if u equal to v, delta x equal to delta y then CFL condition gives Cx equal to Cy equal to C and both are less than half. This indicate that for a stable calculation artificial viscosity will unnecessarily be present. And for a steady state analysis again Courant number will not be there. We get artificial viscosity in x direction as half u delta x and in y direction as half v delta y.

We have observed that some amount of upwind effect is indeed necessary to maintain transportive property of flow equations, while the computations based on upwind differencing. Because this upwinding is needed for retaining transportive property that we examined in last class.

Today we have not done that. But today we can observe that if we do upwinding then there will be some amount of numerical viscosity. For unsteady problem we can have seen the expressions for numerical viscosity, steady flow problems also we have seen the expressions for numerical viscosity.

So, what viscosity does? The physical viscosity it diffuses. So, numerical viscosity also will diffuse the error and this is called sometimes false diffusion. And obviously, this is source of inaccuracy. One of the plausible improvements is the usage of higher order of upwind differencing method. So obviously, we can see that transportive property is needed. And for transportive property upwinding is essential, but if we do upwinding we introduce artificial viscosity or false diffusion.

So, these are the issues we have to learn in whatever way we approach a physical problem either by finite difference or finite value or finite element these issues will always be there like retaining consistency, retaining conservative properties, those we have discussed in detail. Retaining transportive property also we have discussed, but now we can see that for having transportive property we must have some amount of artificial viscosity.

Yet another you know a dichotomy we are entering into that is if we have artificial viscosity then that will introduce a false diffusion inaccuracy or whatever you call error due to artificial viscosity. So, our target should be to reduce that.

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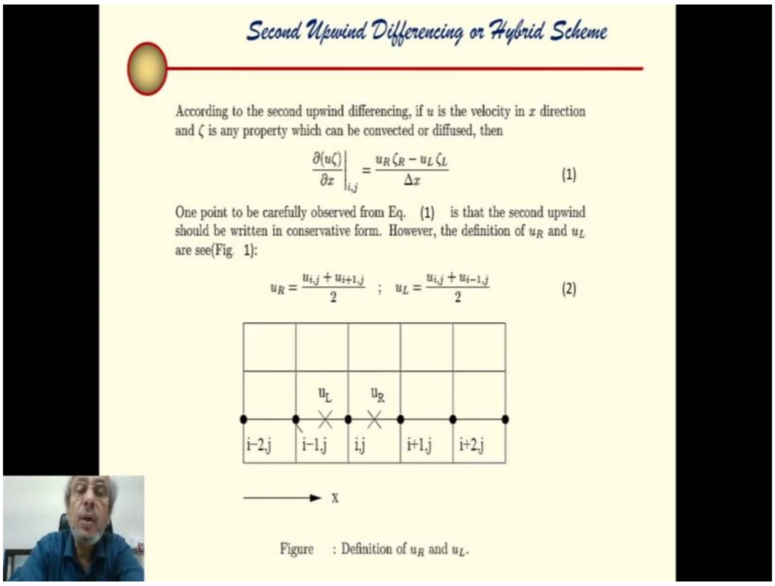

$$\left.\frac{\partial(u\zeta)}{\partial x}\right|_{i,j} = \frac{u_R\zeta_R - u_L\zeta_L}{\partial x}$$

$$u_R = \frac{u_{i,j} - u_{i+1,j}}{2};\ u_L = \frac{u_{i,j} + u_{i-1,j}}{2};$$

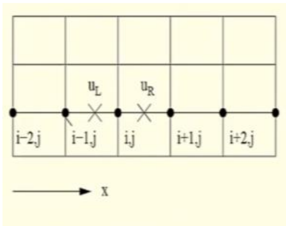

Figure : Definition of u_R and u_L.

One of the very effective way of doing this is basically going for a method which is called second upwinding. The second upwind differencing, this is you will see built in all the commercial codes. Like if you use fluent star city, CFX everywhere you will see the options of discretizations are given one of the options is second upwinding. Now, what is second upwind method? I will discuss that. According to second upwind differencing then again, we are writing u into zeta.

Zeta is a property being transported. It may be even u velocity, v velocity, w velocity or temperature with the u component of velocity. So, u zeta del del x of that at i, j point. We will first write this is equal to u and then u R zeta R minus u L zeta L by delta x. Here R and L these are right point and left point you know taking i, j as the reference.

Look at the figure. i, j is the point of interest, u R is right word point that will be considered called u R. This is between i and i plus 1 j and u L is the leftward point which is between i and i minus 1 j. You can write east and west, whatever basic idea is this is between i, j and i plus 1 j another is between i, j and i minus 1.

Now, we have taken uniform grid. So, we can write u i, j plus u i plus 1 j by 2 and u i. u L is u i, j plus u i minus 1 j by 2. These are the right point I mean right to the point of interest; this is left to the point of interest point of interest is i, j. And we will write the finite difference quotient at this point. i, j point del del x of u zeta. And u is carrier velocity which is defined at R and L points we have explained already this way.

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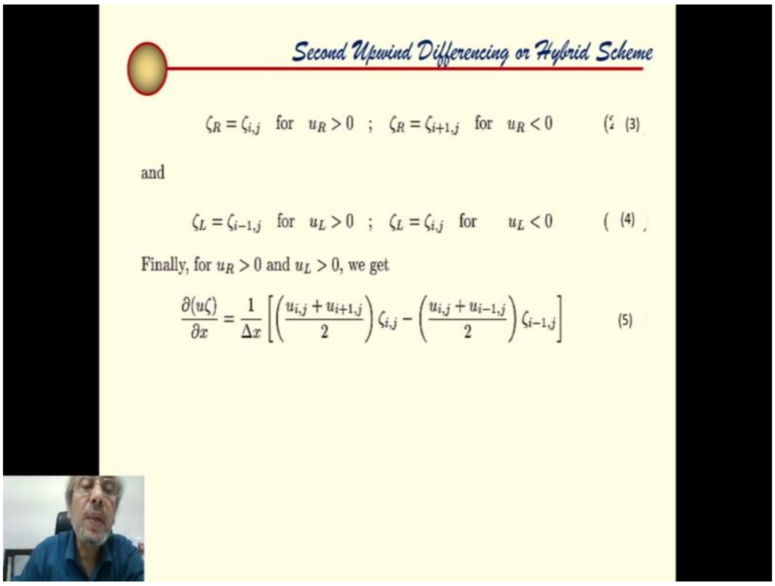

$$\zeta_R = \zeta_{i,j} \quad \text{for } u_R > 0; \quad \zeta_R = \zeta_{i+1,j} \quad \text{for } u_R < 0 \qquad (3)$$

$$\zeta_L = \zeta_{i-1,j} \quad \text{for } u_L > 0; \quad \zeta_L = \zeta_{i,j} \quad \text{for } u_L < 0 \qquad (4)$$

Finally, $u_R > 0$ and $u_R > 0$, we get

$$\frac{\partial(u\zeta)}{\partial x}=\frac{1}{\Delta x}\left[\left(\frac{u_{i,j}+u_{i+1,j}}{2}\right)\zeta_{i,j}-\left(\frac{u_{i,j}+u_{i-1,j}}{2}\right)\zeta_{i-1,j}\right] \qquad (5)$$

Now, we will apply the philosophy of upwinding. Zeta is being transported. So, if u R is positive then zeta will be a point which is basically i, j point, the point in the upstream direction. So, u R is positive. So, the point which is upstream point related to u R is i, j point itself. So, if I mean u R is positive then zeta R is i, j.

Similarly, zeta R is zeta i plus 1 j if u R is negative u u R is negative then we will involve i minus 1 point, but if u R is positive, we will involve upstream point which is i, j point. And similarly, if u L is positive if u L is positive then we will involve upstream point u i minus 1 j u zeta i minus 1 j. And if again u L is negative, we will involve zeta at zeta i, j. So, zeta L will be zeta i, j.

So, this is whether positive or negative we can take our own choice and we can easily perform this exercise for having u R positive u R negative u L positive u L negative. Now, in this example which we have worked out we have taken u R positive and u L positive. Then what will be del del x of u zeta?

1 by delta x u zeta, u R is given by u i, j plus u i plus 1 j by 2 and if u R is positive then zeta R should be zeta i j minus then u L is u i, j plus u i minus 1 j by 2 and if u L is positive then zeta L will be zeta i minus 1 j zeta i minus 1 j, we have written it here substituted.

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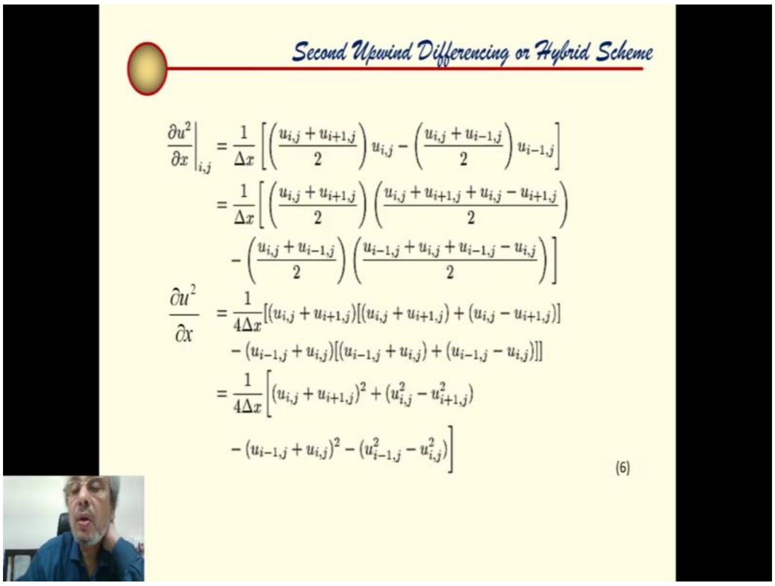

$$\left.\frac{\partial u^2}{\partial x}\right|_{i,j} = \frac{1}{\Delta x}\left[\left(\frac{u_{i,j}+u_{i+1,j}}{2}\right)u_{i,j} - \left(\frac{u_{i,j}+u_{i-1,j}}{2}\right)u_{i-1,j}\right]$$

$$= \frac{1}{\Delta x}\left[\left(\frac{u_{i,j}+u_{i+1,j}}{2}\right)\left(\frac{u_{i,j}+u_{i+1,j}+u_{i,j}-u_{i+1,j}}{2}\right)\right.$$

$$\left.-\left(\frac{u_{i,j}+u_{i-1,j}}{2}\right)\left(\frac{u_{i-1,j}+u_{i,j}+u_{i-1,j}-u_{i,j}}{2}\right)\right]$$

$$\frac{\partial u^2}{\partial x} = \frac{1}{4\Delta x}\begin{bmatrix}\left(u_{i,j}+u_{i+1,j}\right)\left[\left(u_{i,j}+u_{i+1,j}\right)+\left(u_{i,j}-u_{i+1,j}\right)\right] \\ -\left(u_{i-1,j}+u_{i,j}\right)\left[\left(u_{i-1,j}+u_{i,j}\right)+\left(u_{i-1,j}-u_{i,j}\right)\right]\end{bmatrix}$$

$$= \frac{1}{4\Delta x}\left[\left(u_{i,j}+u_{i+1,j}\right)^2 + \left(u_{i,j}^2 - u_{i+1,j}^2\right)\right.$$

$$\left.-\left(u_{i-1,j}+u_{i,j}\right)^2 - \left(u_{i-1,j}^2 + u_{i,j}^2\right)\right] \qquad (6)$$

So, now del del x of conservative form if we write u square. So, we have written here the expression involving u and zeta, but zeta if it if it becomes velocity itself. So, then we can write del del x of u square at i, j.

So, 1 by delta x that we have written here equation 5 1 by delta x, we will substitute this value of u that we have already taken. u i plus 1 plus u u i, j plus u i plus 1 j by 2 and then zeta i, j will be u i, j. Similarly, u i, j plus u i minus 1 j by 2 zeta i minus 1 j we will write u i minus 1 j. Exactly we have written that 1 by delta x this is u R into i, j minus this is u L into u i minus 1 j.

Then we have done some algebraic manipulations for bringing about some specific concepts, but slowly we will discuss 1 by delta x. This term we have written u i, j what we have done? We have added u i plus 1 j and subtracted u i plus 1 j and divided by 2. So, u i, j plus u i plus 1 j plus u i, j minus u i plus 1 j by 2, value is written u i, j. Similarly, minus of this quantity u i, j plus u i minus 1 j by 2 multiplied by u i minus 1 j.

So, instead of u i minus 1 j we have what we have done is i minus 1 j plus u i, j; again, plus u i minus 1 j minus u i, j. If we write this and divide it by 2 effectively value is u i minus 1 j, but we are writing in this way in order to get some specific form.

So, then all these 2s will you know 2 into 2 4 that will come out 1 by 4 delta x and within the bracket now we can write u i, j plus u i plus 1 j and this term we can regroup. One group is u i, j plus u i plus 1 j another group is u i, j minus u i i plus 1 j, just we are regrouping this.

Similarly, minus u i, j plus u i minus 1 j we are retaining this term, but the multiplication is being done with this bracketed term. So, this bracketed term now we can write; u i minus 1 j plus u i, j plus u i minus 1 j minus u i j. So, if we write it in this way clearly we can see 1 by 4 delta x within the bracket. Here u i, j plus u i plus 1 j whole square and this term will be obviously, a plus b into a minus b; so, a square minus b square. So, u i, j square minus u i u i plus 1 j square minus quantity now look at the minus quantities.

So, this is u i minus 1 j plus u i, j multiplied by this means u i minus 1 j plus u i j whole square and then this is again a plus b being multiplied with a minus b. So, it will be minus u i minus 1 j square minus u i, j square. So, equation 6 we get.

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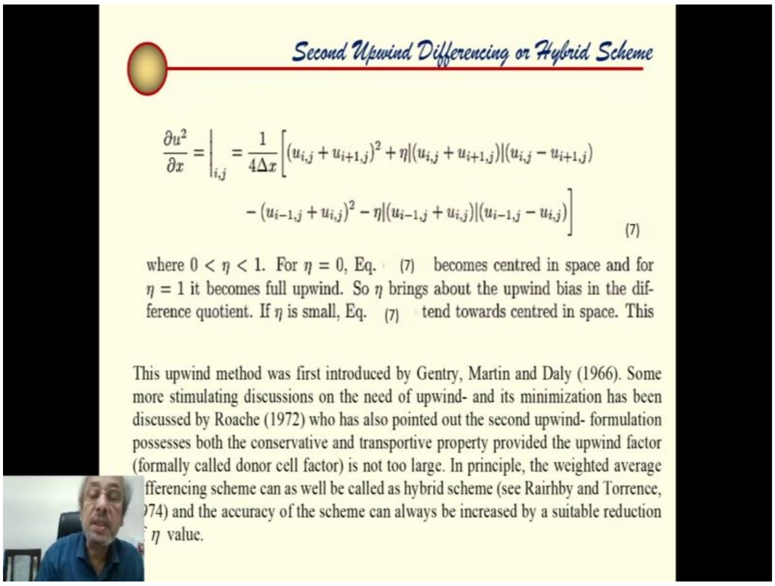

$$\left.\frac{\partial u^2}{\partial x}\right|_{i,j} = \frac{1}{4\Delta x}\Big[\left(u_{i,j}+u_{i+1,j}\right)^2 + \eta\left[\left(u_{i,j}+u_{i+1,j}\right)\right]\left(u_{i,j}-u_{i+1,j}\right)$$

$$-\left(u_{i-1,j}+u_{i,j}\right)^2 - \eta\left[\left(u_{i-1,j}+u_{i,j}\right)\right]\left(u_{i-1,j}-u_{i,j}\right)\Big] \qquad (7)$$

Now, we will apply another philosophy on equation 6 itself, we will introduce a modulus sign. How? First term we do not touch, u i, j plus u i plus 1 j square, but the second term this term essentially, we retain the same term, but we place u i, j plus u i plus 1 j, this is basically u i, j plus u i plus 1 j into u i, j minus u i plus 1 j. So, this u i, j plus u i plus 1 j that part we place under a modulus sign. And u i, j minus u i plus 1 j this part which is contributed this, this part again we bring here.

Similarly, what we do the minus quantities? So, here we do not touch this, we retain this as such. So, u i minus 1 j plus u i, j whole square minus again sorry minus again what we do? Because how did we form this term? This term was formed by multiplying u i minus 1 j plus u i, j with this term. So, we separate out this term under modulus sign and separate out and write this term separately.

So, u i minus 1 j plus u i, j under modulus sign u i minus 1 j minus u i, j. And not only we introduce this modulus sign we also introduce a factor eta. Now, these eta and modulus sign they have their own role. Modulus sign has been put to make this equation you know free of this bias because we started as u R greater than 0, u L greater than 0. So, u R, u L this can be 0 greater than 0, this can be less than 0. Both can be positive locally positive or locally negative.

So, if you do a little exercise on your pen and paper you will see that to introduce that you know sort of versatile character that you know it does not matter if you write this way and program this way. Locally anywhere velocity may become positive may become negative because of flow separation several other things it will not affect. And it will eventually take the right value positive or negative if we introduce modulus sign here. And this makes us independent of whether u R is positive or negative locally and this makes us independent of whether u L is positive or negative locally.

And then this entire term you can see is multiplied by eta. Eta is a small numerical factor. If eta is 0, you can see this discretization is equivalent to then central difference discretization; just what here this minus this quantity. So, it is exactly the central difference discretization. And if eta is 0 it is central difference. If eta is 1 then it is first order upwinding. And if eta is a fraction, it is being be in between first order upwinding and central differencing.

So, eta brings about the upwind bias in the finite difference quotient if eta is small equation 7 tends towards centered in space. So, it will be like central difference. So, basically this formulation we can see that it is called second upwind, it is basically an expression while locally the velocity is becoming positive or negative it get eventually sensitized the discretization expression.

And if eta is retained at a small level say 0.1 or 0.2 usually it is kept at a very small level either 0.1 or 0.2 or maybe below 0.3 definitely. And then you know it is quasi second order accurate, it is basically you know tending towards second order accuracy, but it has some you know numerical error artificial viscosity.

That is what I have written here. This upwind method was first introduced by Gentry, Martin and Daly in 1966. Some more stimulating discussions on the need of upwind and its minimization has been discussed by Roache in his very famous textbook on CFD is probably the first formal text book on CFD.

Roache's book 1972 who was first pointed out the second upwind formulation possesses both the conservative and transportive property, provided in the upwind factor. That is eta it is formerly called donor cell factor is not too large.

In principle the weighted average this is also called weighted average differencing scheme as well be called hybrid scheme. Rairhby and Torrence again in 1974 it is a very well-known paper they have explained this as hybrid scheme and as I mentioned that its accuracy can be enhanced by reducing the value of eta.

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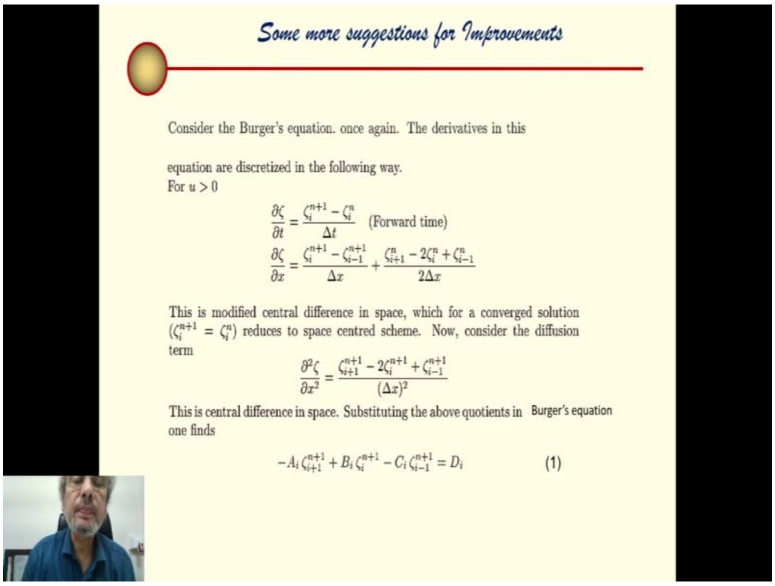

$$\frac{\partial \zeta}{\partial t} = \frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t}$$

$$\frac{\partial \zeta}{\partial x} = \frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta x} + \frac{\zeta_{i+1}^n - 2\zeta_i^n + \zeta_{i-1}^n}{2\Delta x}$$

$$\frac{\partial^2 \zeta}{\partial x^2} = \frac{\zeta_{i+1}^{n+1} - 2\zeta_i^{n+1} + \zeta_{i-1}^{n+1}}{(\Delta x)^2}$$

$$-A_i \zeta_{i+1}^{n+1} + B_i \zeta_i^{n+1} - C_i \zeta_{i-1}^{n+1} = D_i \qquad (1)$$

Now, we will go for yet another very very important discretization strategy. It is just like second upwinding. It is also very very popular and you know this is also built in with several commercial codes. So, you can see that if u greater than 0, this is the del zeta del t, this is forward in time. And del zeta del x the first order derivative is written as basically upwinding is done, we are involving upstream point. Since u is positive zeta i n plus 1 minus zeta i minus 1 n plus 1, but at n plus 1 nth level plus central difference. Zeta i plus 1 minus zeta plus zeta i minus 1 by twice delta x at nth level.

So, at 1 plus n plus oneth level we are involving upstream point since u is positive and we are writing basically real word difference. And we are adding a central difference term again special central difference term for first order derivative in at nth level.

Now, this is modified central difference in space which is a converged solution when the solution finally, in a steady state when zeta i n plus 1 is becomes zeta i n is reduces to a space centered scheme. Now, consider the diffusive term. Diffusive term is written as usual by central difference, but at n plus oneth level.

So, you can see the formulation has very strong implicit character and then you know because we are involving i i is the point of interest we are involving i plus 1 point, i minus 1 point at n plus 1 nth level and the values which are all known values at nth level they are transferred to the right hand side.

So, you can see you know a coefficient. This coefficient after algebra, the values associated with zeta i plus 1 n plus 1 will be A i. The terms associated with zeta i n plus 1 will be b i and terms associated with zeta i minus 1 n plus 1 will be C i.

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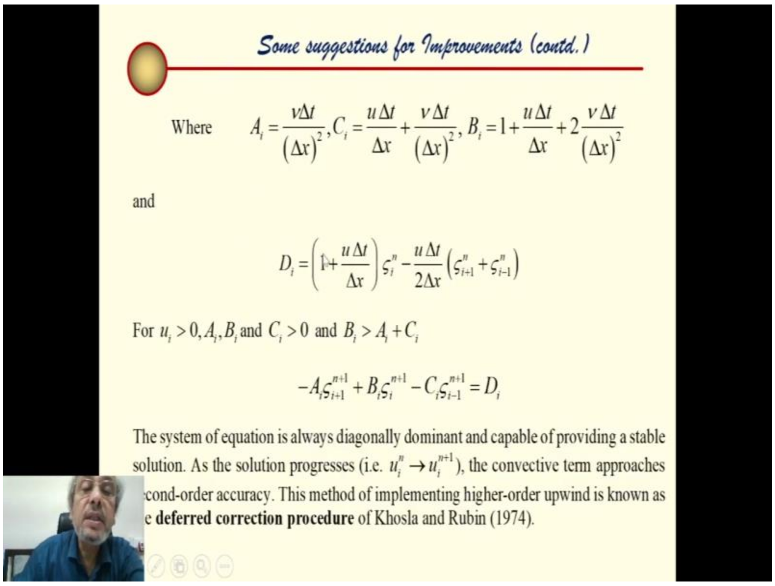

$$A_i = \frac{v\Delta t}{(\Delta x)^2},\ C_i = \frac{u\Delta t}{\Delta x} + \frac{v\Delta t}{(\Delta x)^2},\ B_i = 1 + \frac{u\Delta t}{\Delta x} + 2\frac{v\Delta t}{(\Delta x)^2}$$

$$\text{and } D_i = \left(1 + \frac{u\Delta t}{\Delta x}\right)\zeta_i^n - \frac{u\Delta t}{2\Delta x}\left(\zeta_{i+1}^n + \zeta_{i-1}^n\right)$$

for $u_i > 0$, A_i, B_i and $C_i > 0$ and $B_i > A_i + C_i$

$$-A_i\zeta_{i+1}^{n+1} + B_i\zeta_i^{n+1} - C_i\zeta_{i-1}^{n+1} = D_i$$

And we can revisit A i you know we have written, but what are A i? If you calculate it will be nu delta t by delta x square. C i is u delta t by delta x plus nu delta t by delta x square. B i is 1 plus u delta t plus delta x plus twice nu delta t by delta x square.

Now, sorry this is not v nu this equation does not have v in Microsoft Word nu and v while writing you know they look almost same. One has to be very careful in understanding it. So, here of course, there is no view v these are all nu. So, and d i the right-hand side d i is basically 1 plus u delta t by delta x into zeta i n minus u delta i and u delta t by twice delta x into zeta i plus 1 n plus zeta i minus 1.

So, this you know if you do plug in del zeta del t plus u del zeta del x equal to nu del 2 zeta del x 2 in Burger's equation, you know you will get this you know terms as A i C i P i D i and final equation in indicial notation will be a minus A i zeta i plus 1 n plus 1 plus B i zeta i n plus 1 minus C i zeta i minus 1 n plus 1. So, zeta at ith point at n plus 1 time level is not known, but it is expressed in terms of zeta at the location of its eastern neighbor and at the location of its western neighbor.

Right hand side is D i and since these are either property values or velocity or you know the variable value zeta at previous time level at nth level everything is known in D i. Now, if we keep on substituting i equal to 2, i equal to 3, i equal to 4, like i equal to 4 means, this will be at zeta 4, this will be at zeta 3, this will be at zeta 5.

So, we will get again one matrix equation in the form of A x equal to B. A is the coefficient matrix which is known x is the unknown vector which are basically zeta i is at n plus oneth level equal to B equal to right hand side vector which is fully known and if we can invert this matrix, we will get the full solution.

The system of equation is also we can see these conditions from here that for u i positive A i B i and C i all 3 are positive its clearly seen and B i greater than a i plus C i. So, there is a diagonal dominance this obviously, you can well understand that if we keep on substituting i equal to 2, i equal to 3, i equal to 4 up to i equal to i max or whatever this B will constitute the diagonal principle diagonal and A will first row it will be boundary condition, but subsequently it will contribute as sub diagonal element. C will contribute as super diagonal element.

So, we will get a tridiagonal matrix, remaining terms will be 0 and will be if by solving this will be able to know zeta is at next time level. The system of equation is diagonal dominant and capable of providing a stable solution. As the solution progresses; that means, u at u u i at nth level tends to u i at n plus oneth level here u is symbolic q or zeta or v or any variable. The convective term approaches second order accuracy. This method is often called deferred correction procedure.

Because you know this whatever artificial diffusion is introduced that keeps on reducing when we are maybe you know from reaching from n one level to another level we may have to do 2 iterations or 3 iterations, but you know as or even in one iteration finally, when we will get the converge solution we will get basically artificial viscosity minimized. So, that is what is called a deferred flux correction procedure or deferred correction method.

This method was first introduced by Professor Prem Khosla and Stanley Rubin. Stanley Rubin is a very well-known name in CFD. He was a founder editor of the very well-known journal computers and fluids. So, they introduced this scheme in 1974. So, with this I will stop here today. We have already been acquainted with quite a few issues related to fluid flow modeling. We will you know introduce a few more issues and higher order appointing strategies in the next class and probably then the introduction of finite difference method will be over.

Similar way we will introduce finite volume and finite element and as I have said then we will take up real Navier Stokes solvers using such you know finite difference finite volume or finite element techniques. We will stop here today. Thank you very much. We will meet in the next class.

Thank you.