**Computational Fluid Dynamics and Heat Transfer**
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**Lecture – 05**
**Important Aspects of Flow Modeling - I**

Good afternoon everyone. Today we will discuss some important aspects of flow modeling. Such aspects can be compared with the grammar of a language. If you want to enjoy creation of a given language, poems versus you have to know grammar in order to understand nuances of that language.

Similarly, if you want to model flows and heat transfer problems with greater competence, you required to understand some aspects of flow modeling and we will discuss those aspects today.

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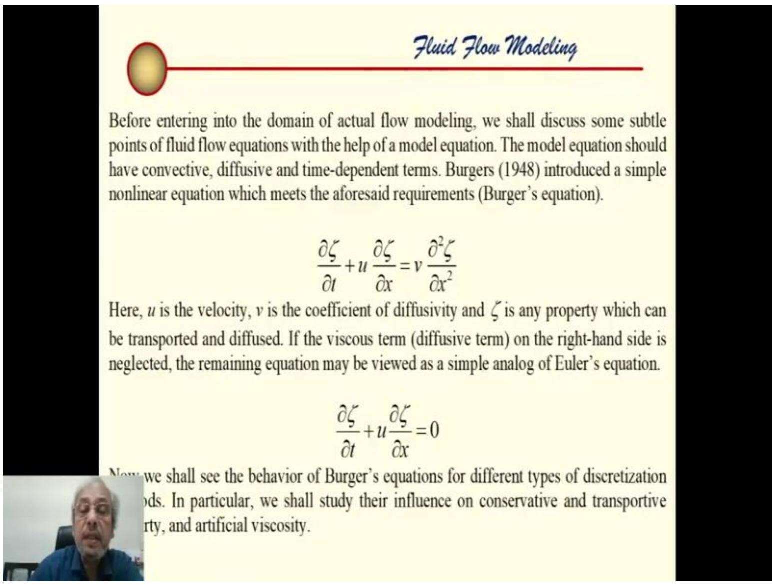


$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}=\nu\frac{\partial^2 \zeta}{\partial x^2}$$

where, u is the velocity, ν is the diffusivity, and ζ is any property which can be transported and diffused.

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}=0$$

Before entering into the domain of actual flow modeling, I have already mentioned about it. So, we will learn this and here, we will take up a model equation, we will apply all our concepts, all our ideas on those model equations and we will see its character, we will see the result. Now, this model equation is called Burger's equation. Jan Burger very very well-known fluid dynamitist develop this equation, it is called one-dimensional Burger's equation.

It has all the components of full-fledged Navier-Stokes equations like this as you can see we have written the equation where u is the velocity and nu is the coefficient of diffusivity or viscosity in case of modeling fluid flow problems and zeta is the property that can be convicted and diffused and this is the temporal term.

Now, if the right-hand side is neglected, then the equation will behave as a simple analog of Euler's equation which will have temporal term and the advective term. So, some cases will include the diffusion, some cases we will only study the advection.

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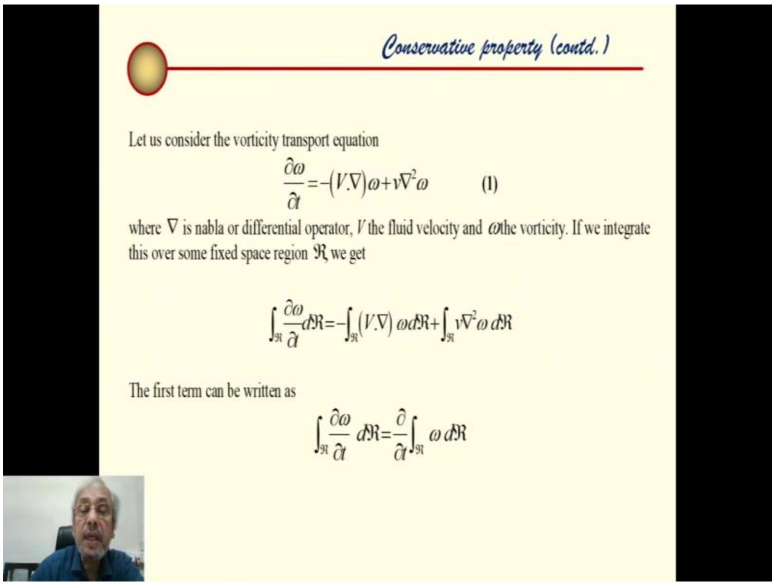


Let us consider the vorticity transport equation

$$\frac{\partial \omega}{\partial t} = -\left(V.\nabla\right)\frac{\partial \omega}{\partial x} + \nu\nabla^2\omega \qquad (1)$$

Where $\nabla$ is nabla or differential operator, $V$ the fluid velocity and $\omega$ the vorticity.

Now, let us consider general vorticity transport equation. This is temporal derivative of vorticity equal to I have taken the convective term to the right-hand side so, minus sign has come, this is V dot nabla which will give in Cartesian coordinate i del u del x plus j del u del y plus k del del z dot product of i into e plus j into v plus k into w.

So, that means, this operator is u del u del x v del u del y w u del del x v del del y w del del z and it can be operated on velocity, temperature or vorticity. So, this is basically the convective term plus nu grad square omega which is deficit term. I have written lambda is differential operator, V is the velocity fluid and omega is vorticity.

$$\int_{\Re} \frac{\partial \omega}{\partial t} d\Re = -\int_{\Re} (V.\nabla)\omega d\Re + \int_{\Re} \nu \nabla^2 \omega d\Re$$

If we integrate it over a fixed region in space given by capital R, this is may be a volume, a fixed region described by a volume through which fluid is coming and you know going through. Now, we can see this is the; I mean we are then integrating the whole equation over the space variable R and we can get simply this equation.

The first term can be written as

$$\int_{\Re} \frac{\partial \omega}{\partial t} d\Re = \frac{\partial}{\partial t} \int_{\Re} \omega d\Re$$

Here, we will go term by term del omega del t integrated over space R, this since if we define a space and we are not changing the space with time so, this time derivative can be taken outside del t of omega over the space R it is integrated.

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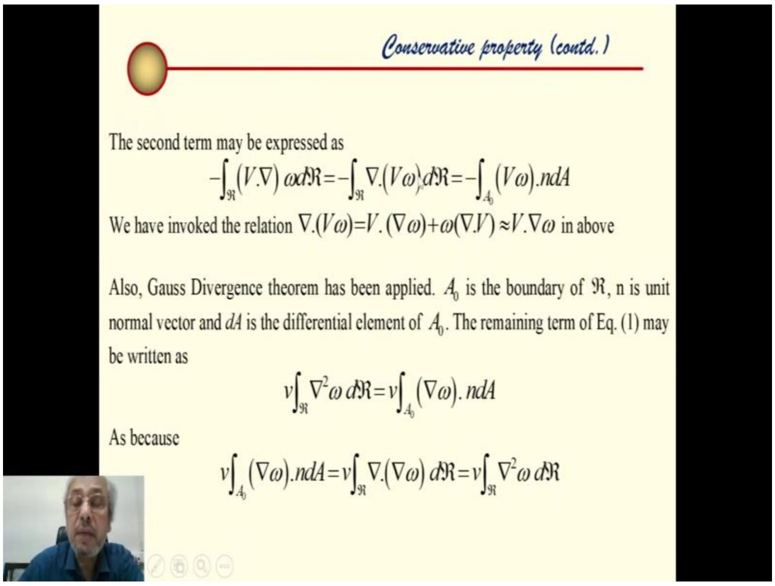


The second term can be expressed as

$$-\int_{\Re}(V.\nabla)\omega d\Re = -\int_{\Re}\nabla.(V\omega)d\Re = -\int_{A_0}(V\omega).ndA$$

$$\nabla.(V\omega) = V.(\nabla\omega) + \omega(\nabla.V) \approx V.\nabla\omega$$

$$v\int_{\Re}\nabla^2\omega d\Re = v\int_{A_0}(V\omega).ndA$$

$$v\int_{A_0}(V\omega).ndA = v\int_{\Re}\nabla.(\nabla\omega)\, d\Re = v\int_{\Re}\nabla^2\omega d\Re$$

Similarly, this convective term V dot nabla omega dR integrated over R, this can be written as nabla dot or grad dot V omega into dR which is again can be written as area integral V omega dot ndA. A A naught is the area which is encompassing the volume of interest.

Now, this is a subtle mathematical expression I mean certain operations have gone in like grad if it is operated over V and omega, we can write V dot grad omega plus omega into divergence of V simply expand grad dot V omega.

Now, if we expand this way for incompressible flows, this part divergence of V is 0 so, we can drop this so, that means, we get this as V dot grad omega. Now, V dot grad omega it is integrated over space, over volume if this volume integral is converted into area integral, then we can write area encompassing that is A naught. So, V omega dot n is a unit normal vector dA.

So, in this operation Gauss divergence, this Gauss divergence theorem has been applied and A naught is the boundary of R, n is the unit normal vector and dA is the differential element of A naught. Now, so, two terms temporal term and the advective term, we have handled.

Temporal term we have seen, can be written as del del t of integral over R omega dR and this ah second term can be written as minus integral over area describing the space V into omega dot ndA and the remaining term is the viscous term which is nu grad square omega integrated over R dR.

This we can write as nu into grad omega dot ndA over the space. How? This is also this quantity nu into integral of grad omega over the area A naught is basically can be written as volume integral again applying Gauss divergence theorem nu into again integral over the volume divergence of or grad dot grad of omega dR and then, this grad dot grad if those are operated, we will get grad square omega dR integral over R.

So, integral over R grad square omega dR is basically nu into it comes from nu into area integral of grad omega dot ndA.

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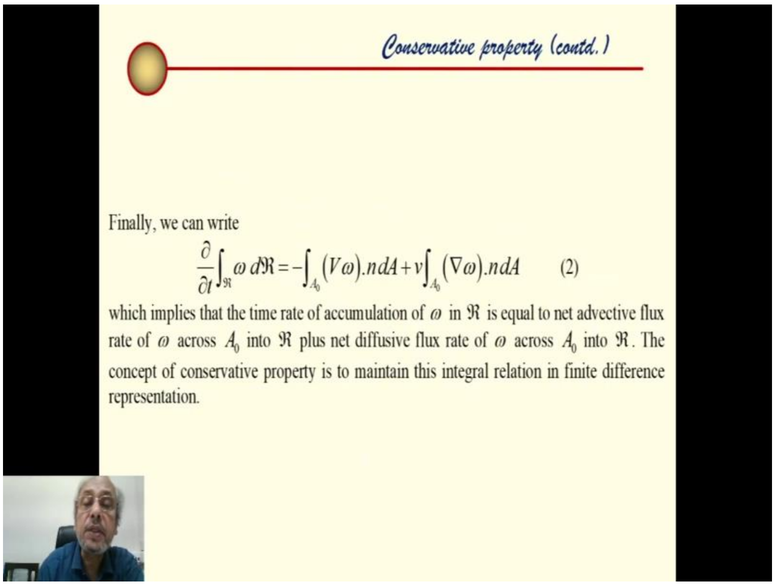

Finally, we can write

$$\frac{\partial}{\partial t}\int_{\Re} \omega d\Re = -\int_{A_0} (V\omega).ndA + \nu \int_{A_0} (\nabla \omega).ndA \qquad (2)$$

So, all three terms now, we have at our disposal. We can write the integration of vorticity transport equation as del del t integral over R omega dR equal to minus integral over the area that is describing are V into omega dot ndA and nu again integral over area grad omega dot ndA.

Now, this way when we can write, we can say the time rate of accumulation of omega in R is equal to net advective flux rate of omega across the area that is describing the region R plus net diffusive flux rate of omega again across R that is across area that is describing R.

The concept of conservative property is to remain this integral relation in the finite difference formulation that means, when the formulation will be done, we will be able to appreciate that convective fluxes are through formulation crossing the entire space smoothly and diffusive for flux also crossing the entire definition of space smoothly.

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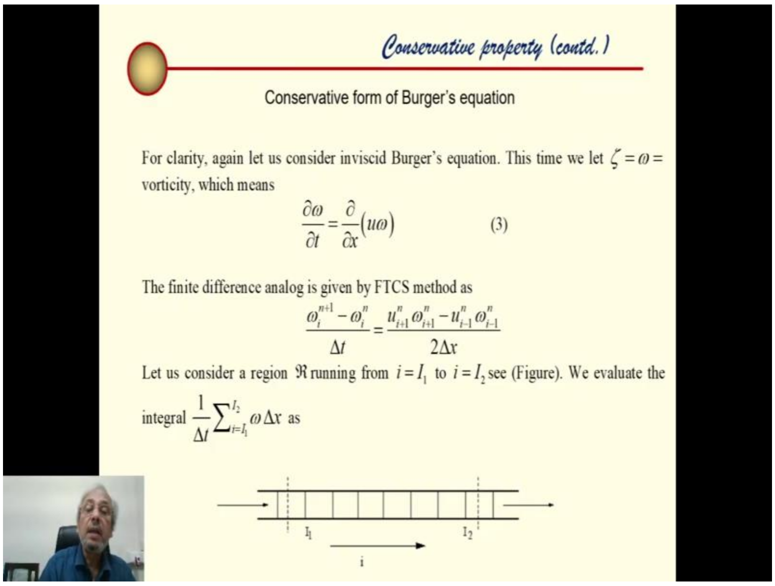


Conservative form of Burger's equation

For clarity, again let us consider inviscid Burger's equation. This time we let $\zeta = \omega =$ vorticity, which means

Let us consider a inviscid burger's equation

$$\frac{\partial \omega}{\partial t} = \frac{\partial (u\omega)}{\partial x} \qquad (3)$$

$$\frac{\omega_i^{n+1} - \omega_i^n}{\Delta t} = \frac{u_{i+1}^n \omega_{i+1}^n - u_{i-1}^n \omega_{i-1}^n}{2\Delta x}$$

Let us consider a region $\Re$ running from i=$I_1$ to i=$I_2$ see (Figure). We evaluate the integral

$\frac{1}{\Delta t}\sum_{t=t_1}^{t_2} \omega \Delta x$ as

Now, how typically that can be shown or proved. We will take a recourse to again Burger's equation. This is conservative form of Burger equation. As we said that now we are saying conservative form that means, this convective term, this original form has to be convoluted to a conservative form and how that is done? That is done in this way del omega del t equal to del del x of u omega.

And then, if we discretize using FTCS forward time central space, we can write omega at ith point n plus 1 minus omega at ith point n divided by delta t equal to so, you are writing expressing this using central difference centered in space u i plus 1 omega i plus 1 minus u i minus 1 omega i minus 1 all are at nth level.

And let us consider a region which is R one-dimensional volume running from and we are dividing it by number of compartments running from i equal to I 1 to i equal to I 2. This is i running through.

So, now, we evaluate this integral which in discretized form if we write it is 1 by delta t, then summation of since we are integrating over this summation of omega delta x over i equal to 1 to i equal to 2. So, left-hand side becomes this. Then, right-hand side simply will become summation of this entire space. So, summation of this quantity delta x over i equal to 1 to i equal to 2.

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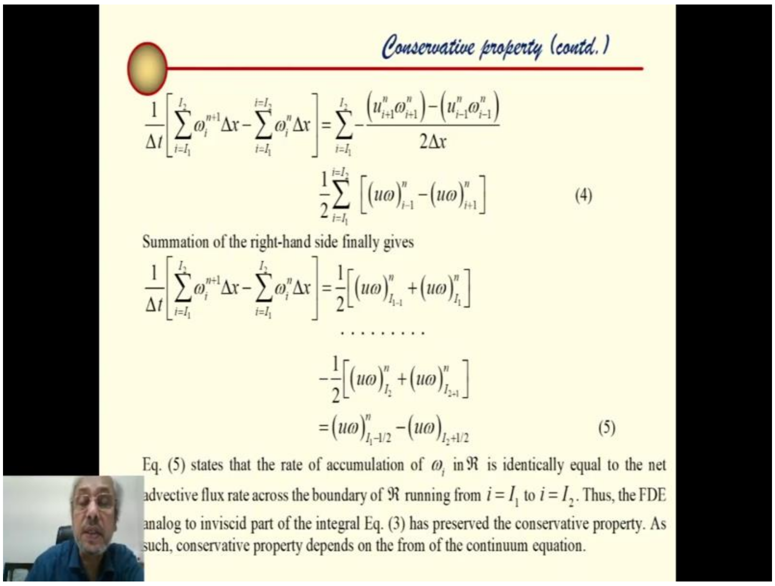


$$\frac{1}{\Delta t}\left[\sum_{i=I_1}^{i=I_2}\omega_i^{n+1}\Delta x-\sum_{i=I_1}^{i=I_2}\omega_i^{n}\Delta x\right]=\sum_{i=I_1}^{i=I_2}-\frac{\left(u_{i+1}^{n}\omega_{i+1}^{n}\right)-\left(u_{i-1}^{n}\omega_{i-1}^{n}\right)}{2\Delta x}\Delta x$$

$$=\frac{1}{2}\sum_{i=I_1}^{i=I_2}\left[\left(u\omega\right)_{i-1}^{n}-\left(u\omega\right)_{i+1}^{n}\right] \qquad \text{....(4)}$$

Summation of the right -hand side finally gives

$$\frac{1}{\Delta t}\left[\sum_{i=I_1}^{i=I_2}\omega_i^{n+1}\Delta x-\sum_{i=I_1}^{i=I_2}\omega_i^{n}\Delta x\right]=\frac{1}{2}\sum_{i=I_1}^{i=I_2}\left[\left(u\omega\right)_{I_{i-1}}^{n}-\left(u\omega\right)_{I_1}^{n}\right]$$

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$$-\frac{1}{2}\sum_{i=I_1}^{i=I_2}\left[\left(u\omega\right)_{I_2}^{n}-\left(u\omega\right)_{I_{2+1}}^{n}\right]$$

$$=\left(u\omega\right)_{I_1-1/2}^{n}-\left(u\omega\right)_{I_{2+1}+1/2}^{n} \qquad (5)$$

And exactly we are doing that 1 by delta t omega i n plus 1 delta x minus omega i n delta x i equal to I 1 to I 2, i equal to I 1 to I 2 this is equal to what we did here basically this quantity will be integrated over the space i equal to 1 to i equal to 2 and I have done a small mistake delta x will come here, delta x has come on the left-hand side, delta x will appear on the right-hand side too.

So, it is half i equal to 1 to i equal to 2 u omega i minus 1 n minus u omega i plus 1 n this minus sign has been absorbed inside and then, we can write this way. So, please you know be careful.

I have done two small mistakes here. I should have written a delta x here and then, as a result of this entire operation, this quantity summation over i equal to I I 1 to I 2 delta x is equal to this half summation of i equal to I 1 to I 2 u omega i minus 1 minus u omega i plus 1 at nth level. So, it will run over this control volume, this quantity from i minus 1 to we will get it up to i plus 1.

So, left-hand side we are not changing, right-hand side that means, this fluxes if we now vary i, if we write i equal to I 1, then it is I 1 minus 1 and this will produce I 1. Next term will be basically we can write i equal to 2 so, then, it will be i say i I 1 minus 1 to I 1 plus 1 that means, I 1. So, I 1 plus to I 2 we will run. So, next term will be I 1 plus 1. So, I 1 plus 1 if we substitute here, that will be I 1 and then, this will be minus quantity I I 2; I 2; I 2; I 2; I 2 I 1 plus 2 at nth level.

So, we will be able to form an array, just substitute i equal to first I 1, then i equal to I 1 plus 1, i equal to I 1 plus 2, i equal to I 1 plus 3 ultimately i equal to I 2 minus 1 i equal to I 2 we write and then, if you do that simply express these two terms by substituting i equal to first I 1, then i equal to I 1 plus 1, then i equal to I 1 plus 2 and this way if you go up to i equal to I 2 minus 1 i equal to I 2 you will get you know array of these two quantities and you will see always you know one term will get cancelled with the another term in the next expression.

And all the terms will be this way intermediate terms will be cancelled out and the final expression which we will get will be basically so, I have given dot dot dot. So, finally, i equal to I 2 and then, this small i equal to I 2 so, I 2 and this is I 2 plus 1. So, then you will see that we can, we will be able to write u omega I 1 minus half minus u omega I 2 plus half.

So, again let me repeat that we will write i equal to I 1 to I 2, i will vary here and then, first term will be I 1 minus 1 I 1, second term will be again I 1 plus 1 will come here so, it will be i it will be I 1 plus 1, then I 1 plus 2 will substitute small i I 1 plus 2 will come here. So, we will get the expressions and the expressions can be arranged in such a way that all intermediate terms will get cancelled and finally, we will be able to write from here u omega I 1 minus half at nth level minus u omega I 2 plus half at nth level.

So, basically these two last terms, this is the last term, this will constitute this, and this is a first term, this will constitute this. Basically, u omega we have written at I 1 and I 1 minus 1 average of that, that means, whatever is entering through this dotted line and finally, u omega at I 2 and u 2 omega at I 2 plus 1 so, by 2 and this is basically whatever is crossing through this dotted line, we will get that. So, whatever is entering that is going out.

So, equation 5 states that the rate of accumulation of omega i in R is identically equal to the net advective flux rate across the boundary of R running from i equal to I 1 to small i equal to I 2. Thus, the FDE analog, finite difference analogue to inviscid part of integral equation 3, Burger’s equation has preserved the conservative property.

So, you can see if this way we write all intermediate fluxes will get cancelled and whatever is coming in that will be going out, final expression will be this. So, by formulation we are preserving the conservative property.

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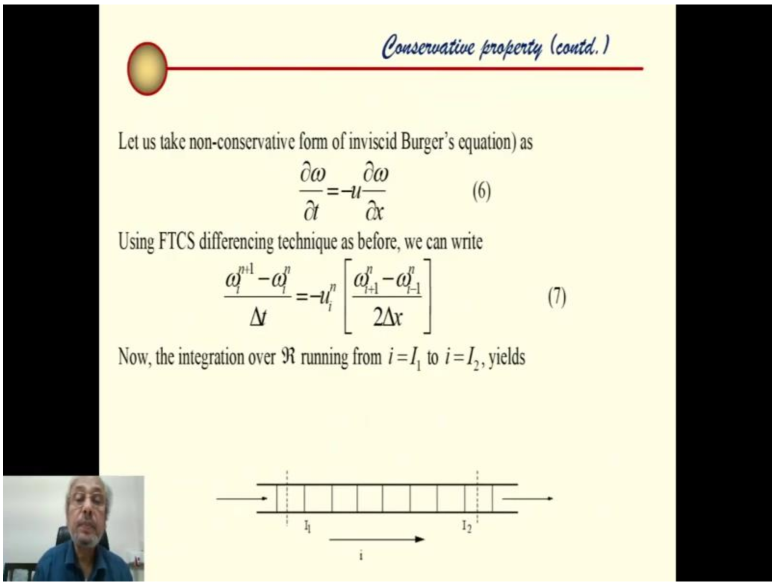


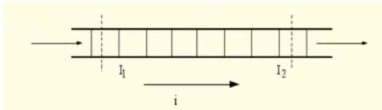

Consider a non-conservative form of inviscid Burger’s equation as

$$\frac{\partial \omega}{\partial t} = u \frac{\partial (\omega)}{\partial x} \qquad (6)$$

Using FTCS differencing technique as before, we can write

$$\frac{\omega_i^{n+1}-\omega_i^n}{\Delta t}=u_i^n\left[\frac{\omega_{i+1}^n-\omega_{i-1}^n}{2\Delta x}\right] \qquad (7)$$

Now the integration over $\Re$ running from i=$I_1$ to i=$I_2$, yields




But non-conservative form, if we write minus u del omega del x, then and then integrate it, we will differentiate it first ah, we will get omega i at n plus 1 omega i at n divided by delta t equal to minus u at nth level here omega i plus 1 minus omega i minus 1 centered in space at nth level divided by twice delta x and again we will run i equal to I 1 to I 2.

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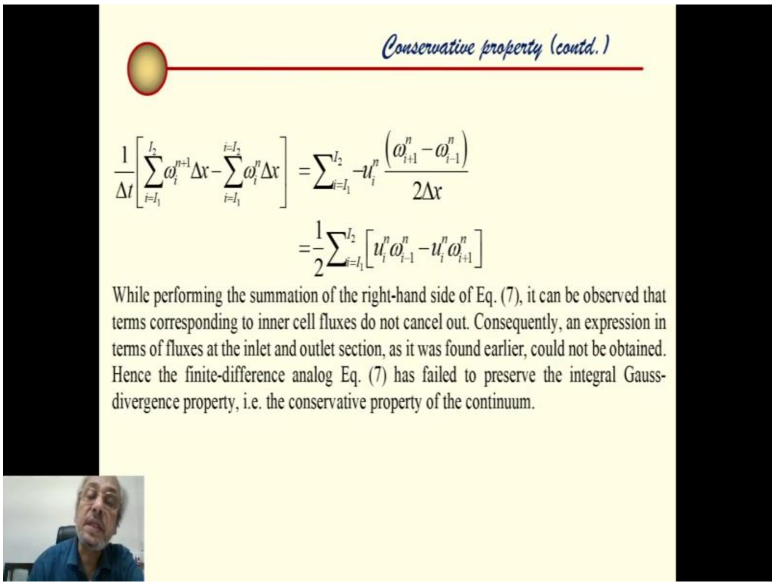


$$\frac{1}{\Delta t}\left[\sum_{i=I_1}^{i=I_2}\omega_i^{n+1}\Delta x-\sum_{i=I_1}^{i=I_2}\omega_i^n\Delta x\right]=\sum_{i=I_1}^{i=I_2}-u_i^n\frac{\left(\omega_{i+1}^n-\omega_{i-1}^n\right)}{2\Delta x}$$

$$=\frac{1}{2}\sum_{i=I_1}^{i=I_2}\frac{\left(u_i^n\omega_{i-1}^n-u_i^n\omega_{i+1}^n\right)}{2}$$

Then, if we write down the expressions of this integral, here also we have to write delta x here, I have miss that please be careful, I will try to represent these two slides in my next lecture again so that you can correct that. So, and then, you express in this way.

But in this way, now, if you substitute i equal to I 1 and then i equal to I 1 plus 1, i equal to I 1 plus 2 up to I 2, you will see you form the array taking two-three terms that means, i equal to 1, i equal to 1 i equal to I 1, i equal to I 1 plus 1, i equal to I 1 plus 2 and then, last three terms may be i equal to I 2 minus 1, i equal to I 2. You will see all intermediate terms are you know not cancelling at all.

So, basically fluxes are by definition not eventually balanced. So, while performing the summation of the right-hand side of you have written equation 7, it can be observed that terms corresponding to inner cell fluxes do not cancel out.

Consequently, an expression in terms of fluxes at inlet and outlet section, as it was found earlier, could not be obtained. Hence finite difference analog of 7 has failed to preserve the integral Gauss divergence property that is conservative property of the continuum.

So, again little repetition. This is conservative formulation and conservative formulation final expression of left-hand side is equal to this I I said that I am missed equal to sign in front of this expression. Here, if you keep on i equal to keep on substituting i equal to I 1, then I 1 plus 1, I 1 plus 2 may be three terms and then, i equal to I 2 minus 1, i equal to I 2 plus 2 or 3 terms, you will see fluxes can be arranged such a way that all intermediate fluxes will get cancelled. The in the domain, the inlet flux will just go out as out let out flux, in flux will just go out as out flux.

So, basically it will by formulation itself or by writing itself in this way conservative form, we are retaining conservative property whereas, by writing this way which is equation 6, we are not retaining conservative property non-conservative form.

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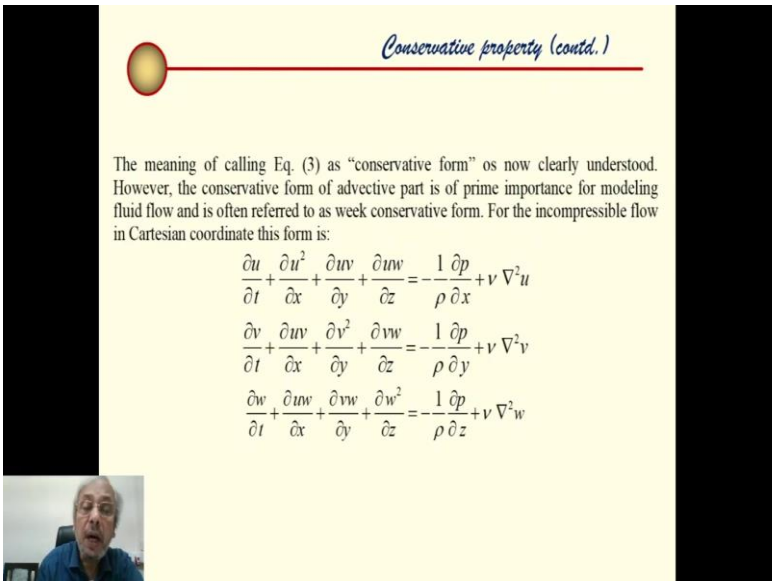


For the incompressible flow in Cartesian coordinate the conservative form is:

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\Delta^2 u$$

$$\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}+\frac{\partial vw}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\Delta^2 v$$

$$\frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial vw}{\partial y}+\frac{\partial w^2}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\Delta^2 w$$

So, from here that means, the calling equation 3 as conservative is now clearly understood. However, the conservative form of advective part is of prime importance for modeling fluid flow and is often referred to as week conservative form. So, instead of writing del u del t plus u del u del x plus v del u del y plus w del u del z equal to 1 by rho del p del x plus nu grad square u we can write it in this way del del x of u square del del y of uv del del z of uw.

Similarly, y momentum equation our non-conservative form is del v del t plus u del v del x plus v del v del y plus w del v del z instead of that we will write del v del t plus del del x of uv del del y of v square and del del z of uw right-hand side, we are not touching. So, all the advective terms, the convective terms can be written in conservative form. Now, if we only write the advective terms or advection in conservative form, then that is called week conservation.

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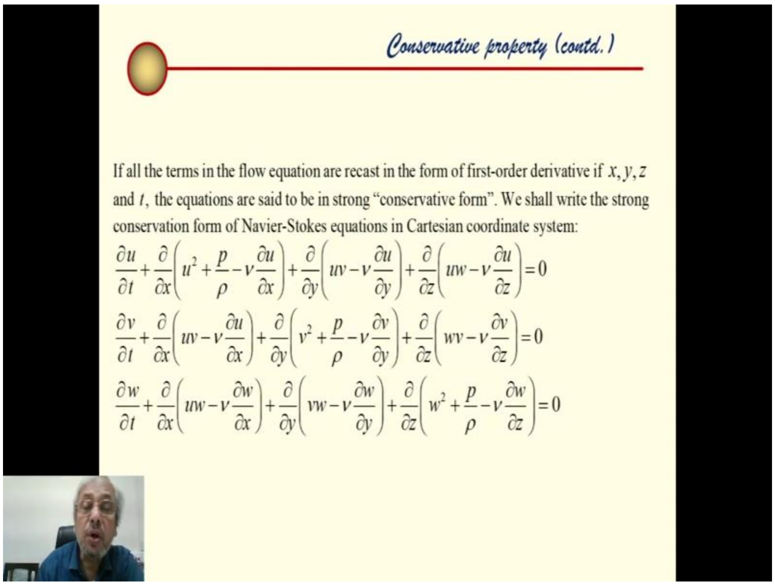


The strong conservation form of Navier-Stokes equations in Cartesian coordinate system:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(u^2+\frac{p}{\rho}-\nu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(uv-\nu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(uw-\nu\frac{\partial u}{\partial z}\right)=0$$

$$\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}\left(uv-\nu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(v^2+\frac{p}{\rho}-\nu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(wv-\nu\frac{\partial v}{\partial z}\right)=0$$

$$\frac{\partial w}{\partial t}+\frac{\partial}{\partial x}\left(uw-\nu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(vw-\nu\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(w^2+\frac{p}{\rho}-\nu\frac{\partial w}{\partial z}\right)=0$$

In a similar manner, if we try to write the all the terms as first order derivative of x, y and z and t, then we will be able to get something which is called strong conservative form. You have seen that what we have done now. Now, right-hand side also enters into the expression in terms of wherever it fits in del del xy or del del x or del del y or del del z.

You can see del u del t plus del del x of u square plus p by rho minus nu del u del x. If you move into the right-hand side, it will be del del x minus 1 by rho del p del x that has been brought here and if you again shift this to right-hand side, this will be nu del 2 u del x2 that has come here.

So, similarly del del u of uv and right-hand side from Laplacian u, if you bring del 2u del y2 on this side, you will be able to get this and if you bring nu del 2u del z2 on left side, you will get left-hand side, you will get this. Similar way v momentum equation or y momentum equation can be written, or the z momentum equation can be written. When you write in this way, it is called conservative form, strong conservative form.

Strong conservative form is not a mandatory requirement because of ease of formulation at times strong conservative form is preferred. In quite a few well-known algorithms, the equations are discretized in non in strong conservative form. This way in strong conservative form, but as I mentioned earlier, week conservative form that means, writing convective derivatives or advection terms in conservative form is often very very helpful and in almost all the well-known algorithms, it is done this way.

So, we have learned first conservative form of convective term, it always balances convective flux by through its definition. Second is this conservative form of equation can be again written in two different ways; week conservative form and strong conservative form.

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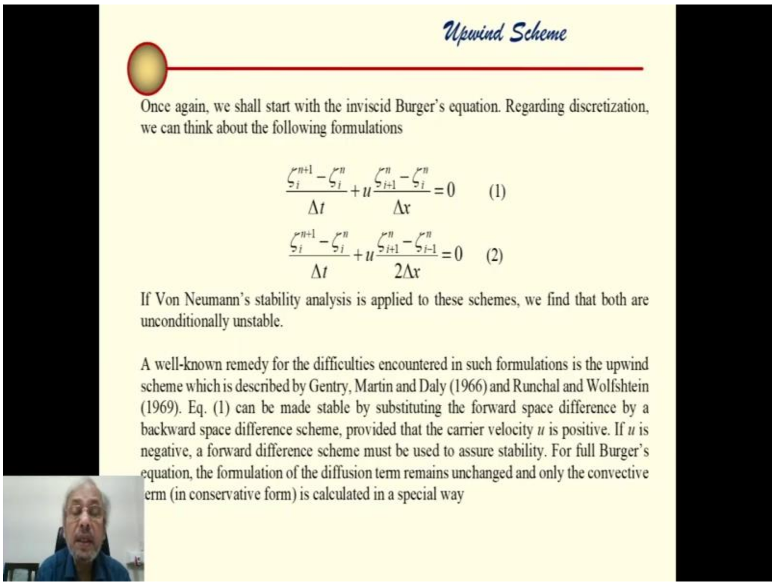

$$\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}+u\left[\frac{\zeta_{i+1}^n-\zeta_i^n}{\Delta x}\right]=0 \qquad (1)$$

$$\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}+u\left[\frac{\zeta_{i+1}^n-\zeta_{i-1}^n}{2\Delta x}\right]=0 \qquad (2)$$

Now, again I will go back to Burger's equation in terms of zeta. So, del zeta del t plus u del zeta del x and now, we will write two different finite difference schemes. One is you can very well appreciate equation 1 forward in time and a non-conservative form also forward in space not centered in space, but forward in space.

Second one, again forward in time zeta n plus i at ith location n plus 1 minus zeta at ith location n divided by delta t plus u zeta i plus 1 at n minus zeta i minus 1 at n divided by twice delta x. Now, this is called again non conservative form central difference.

Now, if you apply Von Neumann stability analysis which we learned earlier that formulation is stable or not from numerical stability point of view, although we do not have this custom here, but if we try to perform for Neumann stability analysis, we will find both this formulation 1 and 2 are unconditionally unstable that means, whatever you do, it will blow off, the calculation will not converge.

So, obviously, you know we have to look for the reason and look for the remedy. I have given name of few scientist; one is Gentry, Martin and Daly again, this group is from Los Alamos and Runchal and Wolfshtein again both of them were at that time at Imperial college. Although Runchal is an Indian, many of you might be knowing him.

He was at Imperial College. Wolfshtein, he is from Israel basically very well-known two students of Professor Spalding and their this we are co-workers of Professor Harlow at Los Alamos anyway. To cut the long story short that they experimented in those days people were trying to evolve Navier-Stokes solver and you know several other experiments with full Navier-Stokes equations. So, these were very relevant and very important.

Now, they found out that the very efficient way to get you know such derivatives stabilized is the following that is we have written equation 1 can be made stable by substituting the forward space difference by a backward space difference scheme, provided the carrier velocity is positive means if u is positive and then instead of forward difference scheme, if we write backward difference scheme or rearward difference scheme, if this will be stable.

If u is negative, then if u is a negative velocity, then forward difference scheme must be used to assure stability. For full Burger's equation, the formulation of the diffusion term remains unchanged and only the convective terms will matter. So, what we will do?

We only consider convective term, number 1 and number 2 we have already sort of tested that conservative form is more effective or more useful. So, we will apply this philosophy, that means, if u is positive, go by rearward difference, backward difference. If u is negative, go by forward difference of the dependent flow variable.

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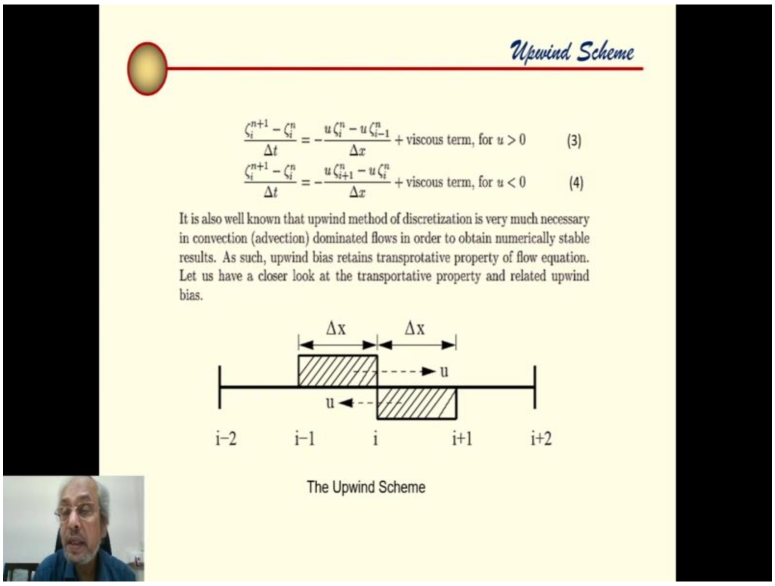
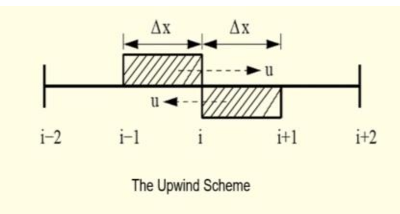



$$\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t} = -\frac{u\zeta_i^n - u\zeta_{i-1}^n}{\Delta x} + \text{viscous term, for u} > 0 \tag{3}$$

$$\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t} = -\frac{u\zeta_{i+1}^n - u\zeta_i^n}{\Delta x} + \text{viscous term, for u} < 0 \tag{4}$$



Now, we have exactly done that zeta i n plus 1 minus zeta i at nth level by delta t equal to minus zeta i n u zeta i n conservative form u zeta i minus 1 n plus viscous term for u greater than 0 and minus zeta i plus 1 minus u zeta i for u less than 0, that means, for u negative and this is i minus i minus 1 for u positive, zeta i plus 1 minus zeta i minus 1 for u negative and since, conservative formulation u and zeta are together.

It is also well-known that upwind method of discretization is very much necessary in convection dominated flows. It is basically what is mean meant is high speed flows and I will discuss later that this is this was a major bottleneck for solving full Navier-Stokes equations for higher Reynolds number and this was specifically a problem with finite element formulation.

Anyway, we will a not confuse with those information now, what we will do that we will take it up equation 3 and equation 4 and we will test its effect on again a one-dimensional domain running from i minus 2, i minus 1, i, i plus 1, i plus 2 and once we will take u as positive so, in the positive direction and in their occasion, we will take u as negative, u running in the negative direction, that means, in the opposite direction of the coordinate system.

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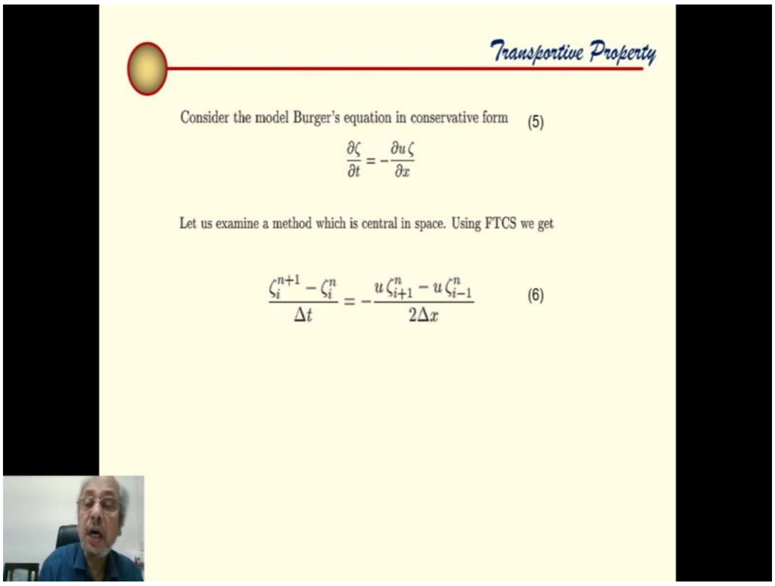


$$\frac{\partial \zeta}{\partial t} = -\frac{\partial u \zeta}{\partial x}$$

$$\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}=-\frac{u\zeta_{i+1}^n-u\zeta_{i-1}^n}{2\Delta x} \qquad (6)$$

So, consider again model Burger equation as I said. So, it and only we will consider temporal term and the convective term del zeta del t equal to minus del del x of u zeta and if we go for FTCS so, u and zeta, u I am not writing specifically i plus 1 it is understood that u is with zeta whatever is the zetas coordinate, u is with it.

So, zeta i plus 1 minus u i zeta i minus 1 by twice delta x this is we know forward time central space otherwise you know consistent good formulation.

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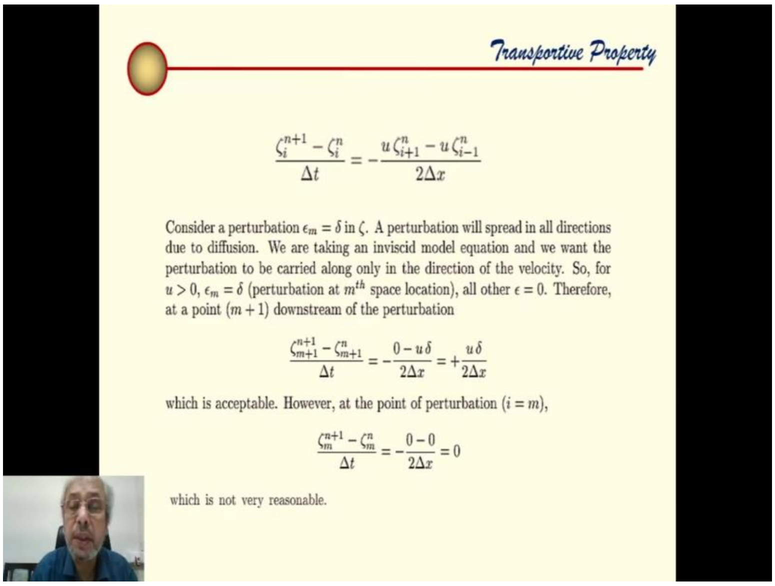


$$\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}=-\frac{u\zeta_{i+1}^n-u\zeta_{i-1}^n}{2\Delta x}$$

$$\frac{\zeta_{m+1}^{n+1}-\zeta_{m+1}^n}{\Delta t}=-\frac{0-u\delta}{2\Delta x}=+\frac{u\delta}{2\Delta x}$$

$$\frac{\zeta_m^{n+1}-\zeta_m^n}{\Delta t}=-\frac{0-0}{2\Delta x}=0$$

So, we have rewritten that because we will very soon do the analysis on this template, on this equations, on this equation with this terms. Consider a perturbation epsilon m equal to delta in zeta. A perturbation will spread in all directions due to diffusion. This is a very nice concept and very natural concept.

If you take a you know for example, steel plate and place a small heated object at the center of it, you will see slowly you know entire plate in all direction equally it will be heated up, temperature will increase or if you throw a stone in a steel water in a small pond so that, you can observe you will see repels are going with the same speed on all directions and this is very natural way of describing diffusion. So, if we give a perturbation, diffusion will propagate on all direction.

So, since we are taken in an inviscid model so, a perturbation going on all direction that party is eliminate as if it is not diffusing. So, you have switched off the diffusion term and perturbation to be carried along, then only in the direction of velocity it will go. So, it is like coloring a spot in a flow stream. So, if flow stream is not very gently going in one direction, that color will keep spreading in other direction.

So, for u greater than 0 and this epsilon m equal to delta perturbation a perturbation at mth space location, all other epsilons are 0. Now, if we substitute here having perturbed it at mth location i equal to m, we are trying to see what happens when i at the next time step at i equal to m plus 1. So, here we will simply substitute i by m plus 1.

So, this will be zeta m plus 1 n at n plus 1th level minus zeta m plus 1 at nth level by delta t equal to minus u into then m if we substitute i by m plus 1, it is zeta m plus 2.

So, nothing is there if you perturb at m; mth level not the next point, but next to next point and also at i minus 1, that means, m minus 1 if we substitute here that means, at mth point, mth point we have disturbed, delta is there at mth point.

So, we are we can see this is plus it is becoming twice delta by u delta by twice delta x which is acceptable right because at the next point, you will fill at m plus 1th m plus 1; 1th point in the space you will get some effect of perturbation giving perturbation at mth point that is why we have written which is acceptable.

However, at the point of perturbation what is happening? If we substitute i by m here also if we substitute i so, this will be zeta m at n plus 1 minus zeta m at n divided by delta t equal to here i is substituted by m. So, at m plus 1 and m minus 1. Now, perturbation is not there at m minus 1 or m plus 1.

Here at mth point earlier case, perturbation we substituted by perturbation, but here at m plus 1th point, no perturbation, m minus 1th point no perturbation 0 which is not very reasonable because after giving perturbation, the next time step itself you are not able to fill anything there you know no quantity there is not very reasonable.

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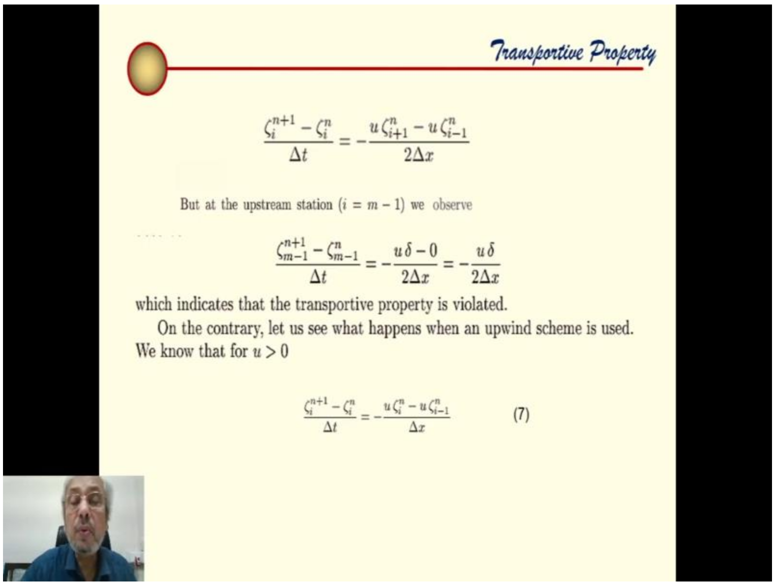


$$\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}=-\frac{u\zeta_{i+1}^n-u\zeta_{i-1}^n}{2\Delta x}$$

But at the upstream location ($i$ = m - 1) we observe

$$\frac{\zeta_{m-1}^{n+1}-\zeta_{m-1}^n}{\Delta t}=-\frac{u\delta-0}{2\Delta x}=-\frac{u\delta}{2\Delta x}$$

$$\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}=-\frac{u\zeta_i^n-u\zeta_{i-1}^n}{\Delta x} \qquad (7)$$

Again, I have written in the same form of equation. At m minus 1th point so, again you know i is; i is substituted by m minus 1 and this is u zeta and u this u zeta, i should be substituted by m minus 1. So, m minus 1 means this is m and m minus 1 means this is m minus 2. So, at m minus 2 point, nothing is there. At if we give m minus 1, this is mth point. So, mth point delta perturbation.

So, we can see this is becoming equal to minus u delta by twice x which indicates that at m, but finally, we are observing at m minus 1th level m minus 1th point sorry at the point m minus 1 special point.

Now, u velocity is in the positive direction. So, i, i plus 1, i plus 2 all are points in the positive direction, u velocity is moving in the positive direction. After giving a perturbation, if we get effect of the perturbation in the i minus at i minus 1 point or at i minus 2 point which are in fluid mechanical language upstream points.

So, when u is positive, upstream point should not get perturbed, but we can see upstream point is getting perturbed. So, this is not really acceptable, we can say transportive property is violated.

Now, so, but it was FTCS formulation, right, equation 6 is FTCS formulation. So, again conservative form of equation that means, equation 5 we will go for the formulation for u positive. Now, what we said if u is positive, then it will be backward difference.

So, for positive u, we have to involve a point which is at the behind which is basically upstream point. So, u zeta i n minus u zeta i minus 1 n by delta x for u greater than 0. So, for positive u, we involve the upstream point or the backward difference.

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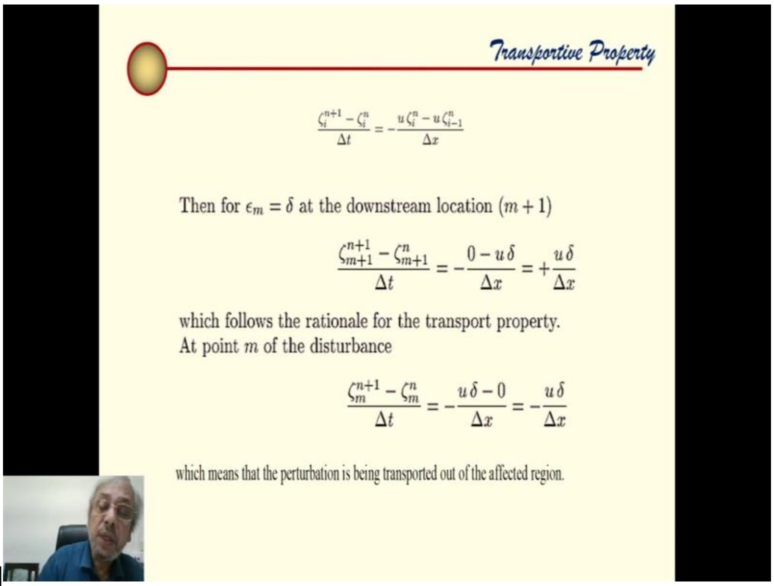

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$$\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}=-\frac{u\zeta_i^n-u\zeta_{i-1}^n}{\Delta x}$$

Then for $\varepsilon_m=\delta$ at the downstream location ($m$+1)

$$\frac{\zeta_{m+1}^{n+1}-\zeta_{m+1}^n}{\Delta t}=-\frac{0-u\delta}{\Delta x}=+\frac{u\delta}{\Delta x}$$

which follows the rationale for the transport property.

At point $m$ of the disturbance

$$\varepsilon_m=\delta\ \frac{\zeta_m^{n+1}-\zeta_m^n}{\Delta t}=-\frac{u\delta-0}{\Delta x}=-\frac{u\delta}{\Delta x}$$

So, then, we can write this form that means, we know now that u is positive so, that difference is i minus u u zeta i minus u zeta i minus 1. Again, we are perturbing at mth location and then, we can we will see what is happening to m plus 1. So, we substitute i by m plus 1 and this is at m plus 1, we did not give any perturbation at m plus 1 so, we will write 0 perturbation value is 0, but if m plus 1 is substituted to this I, this is m, perturbation value was given there.

So, this quantity is at the next time level at the next point is becoming u delta by delta x which follows the rational for the transportive property that means, in the forward point, it must move. At point m of the disturbance, again in this equation you substitute all is by m's so, we have done that.

Now, since at mth point disturbance was given, but m minus 1 point there was no disturbance so, we can see at mth point at the next time level what is happening? You know some effect of perturbation we are able to see that means, a perturbation is you know being transported out, it has a negative sign also.

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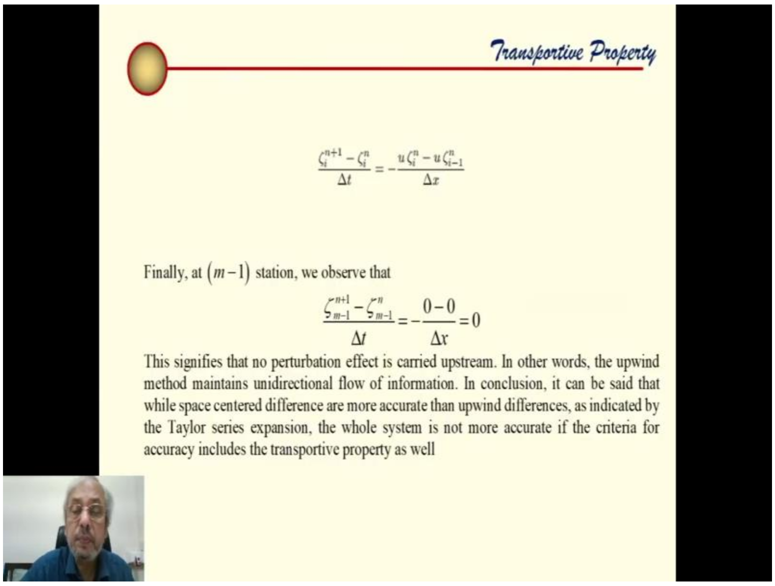


$$\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}=-\frac{u\zeta_i^n-u\zeta_{i-1}^n}{\Delta x}$$

Finally, at ($m$-1) station, we observe that

$$\frac{\zeta_{m-1}^{n+1}-\zeta_{m-1}^n}{\Delta t}=-\frac{0-0}{\Delta x}=0$$

Again, you know if we observed at m minus 1 station, we have repeatedly written this equation so that we can easily substitute m minus 1 and if it is m minus 1, there is no perturbation at m minus 1. If we substitute i by m minus 1, then it is u zeta m minus 2 there is no perturbation there, so these are 0 equal to 0. So, at the upstream point that means, a point behind there is no effect.

So, this signifies that no perturbation effect is carried upstream. In other words, upwind method maintains unidirectional flow information. In conclusion, it can be said that while space centered difference are more accurate, space centered difference schemes they are more accurate than upwind differences, as indicated by Taylor series expansion that we have seen earlier in our first few lectures.

The whole system is not more accurate if the criteria for accuracy includes transportive property because here, there is a bias, the fluid velocity is inducing that bias on the space. So, this is not a stagnant space, we are not solving heat conduction problem. Here, the medium is moving, and flow velocity is affecting. So, the finite difference portion will be very appropriate if this effect is inducted, if this effect is mimicked in formulation and that is what is done if we do or introduce up winding.

Up winding means; again we will go back by the definition given by Gentry, Martin and Daly and Runchal and Wolfshtein that you know you go for first order a difference quotient specially del u del x del v del x such quantities, you discretize it using backward difference in that means, using upstream point, point which is behind if the flow velocity is positive. If the flow velocity is negative, then you involve downstream point that means forward point.

So, go for forward differencing for locally negative velocity. Go for backward differencing for positive velocity. This philosophy is called up winding and we will see that when we will formulate a complete problem, this information or this guideline helps us a lot although things may not be the so much straight forward.

But the hidden idea will remain the same effectively, we will involve upstream points for positive velocity and downstream points for the negative velocity which we will see be very stable and also will be very accurate, but to maintain the accuracy, we have to involve some additional technique, additional care.

We will do in the subsequent classes. We will stop here today. Thank you very much. This was the conclusion that we will in transportive property is one-directional flow property maintaining that we have to go for up winding. So, we have progressed up to this. We will take up from here in the next class.

Thank you very much.