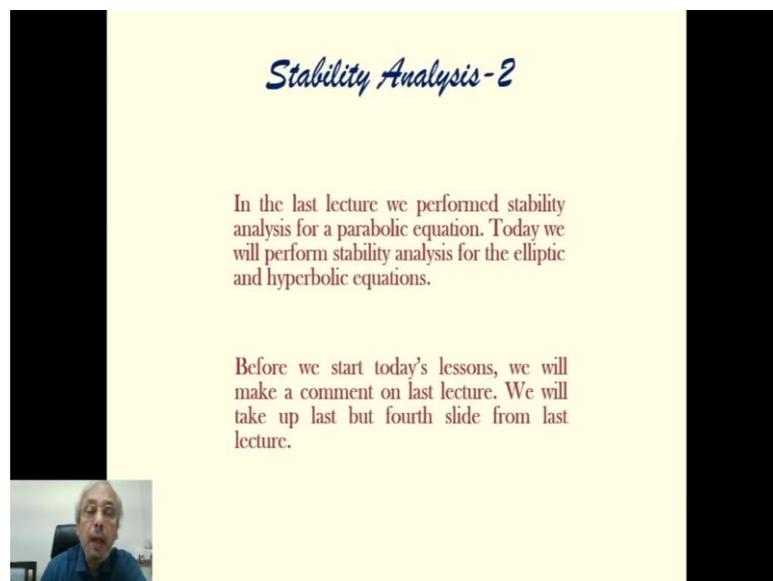**Computational Fluid Dynamics and Heat Transfer**
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Lecture – 04
Stability Analysis-2

Good afternoon everyone. So, we will start our lecture today; basically we will be working on Stability Analysis.

(Refer Slide Time: 00:26)

However, in the last lecture we performed stability analysis for a parabolic equation. Today we will perform stability analysis for elliptic and hyperbolic equations. Now, before we start today's lessons, we will make a comment on last lecture. We will take up last but fourth slide from the last lecture.

Substituting eq.(6) into (4)

$$\frac{e^{a(t+\Delta t)}e^{Ik_m x} - e^{at}e^{Ik_m x}}{\Delta t} = \alpha\left[\frac{e^{at}e^{Ik_m(x+\Delta x)} - 2e^{at}e^{Ik_m x} + e^{at}e^{Ik_m(x-\Delta x)}}{(\Delta x)^2}\right] \quad (7)$$

Divide eq. (7) by $e^{at}e^{Ik_m x}$

$$\frac{e^{at}-1}{\Delta t} = \alpha \left[\frac{e^{Ik_m\Delta x} - 2 + e^{-Ik_m\Delta x}}{(\Delta x)^2} \right] \qquad (8)$$

Recalling the identity $\cos(k_m\Delta x) = \dfrac{e^{Ik_m\Delta x} + e^{-Ik_m x}}{2}$

Can be written as

$$e^{a\Delta t} = 1 + \frac{\alpha(2\Delta t)}{(\Delta x)^2}\left(\cos(k_m\Delta x) - 1\right)$$

$$e^{a\Delta t} = 1 - 4\frac{\alpha(\Delta t)}{(\Delta x)^2}\sin^2\left[(k_m\Delta x)/2\right]$$

If you recall, we were trying to find out amplification factor and that we arrived at equation 8 from the governing equation after substituting the error quantities. And in equation 8 we substituted cos theta equal to e to the power minus ith e to the power i theta plus e to the power minus i theta by 2.

So, this is the identity e to the power i theta plus e to the power minus i theta by 2 makes it cos theta. We substitute cos theta in equation 8 and we get this equation in terms of e to the power a t which is effectively the amplification. Then cos theta minus 1 this is substituted by minus 2 sin square theta by 2 and we get equation 9. And finally, we find out condition for which modulus of e to the power a t is less than or equal to 1.

Stability requirement of one dimensional stability is :

$$\frac{\alpha\left(\Delta t\right)}{\left(\Delta x\right)^{2}} \leq \frac{1}{2}$$

For, two dimensional heat conduction equation, the requirement is:

$$\frac{\alpha\left(\Delta t\right)}{\left(\Delta x\right)^{2}} + \frac{\alpha\left(\Delta t\right)}{\left(\Delta y\right)^{2}} \leq \frac{1}{2}$$

And finally, from there our conclusion is that the stability requirement of one-dimensional parabolic equation is alpha delta t by delta x square less than equal to half. This alpha is thermal diffusivity. In case of momentum transport, it will be molecular diffusivity or viscosity.

We take it up later, but right now as an example we took up one dimensional heat conduction equation and we also discussed two-dimensional conduction equation which is also parabolic equation. And for that, we found out alpha delta t by delta x square plus alpha delta t by delta y square less than equal to half. So, this was the conclusion of our last lecture.

(Refer Slide Time: 04:00)

Second order wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Here, c is the wave speed and u is the wave amplitude. This can be written as system of two first-order equations. If $v = \dfrac{\partial u}{\partial t}$ and $w = c\dfrac{\partial u}{\partial x}$, then we may write $\dfrac{\partial v}{\partial t} = c\left(\dfrac{\partial w}{\partial x}\right)$ and $\dfrac{\partial w}{\partial t} = c\left(\dfrac{\partial v}{\partial x}\right)$

As system of first order equations (1D)

$$\frac{\partial u}{\partial t} + [A]\frac{\partial u}{\partial x} = 0$$

$$u = \begin{Bmatrix} v \\ w \end{Bmatrix} \text{ and } A = \begin{Bmatrix} 0 & -c \\ -c & 0 \end{Bmatrix}$$

Now, we will take up second order wave equation. Here c is the wave speed and u is the wave amplitude. This can be written also as a system of two first order equations. If we substitute v equal to del u del t and if we bring in a function w equal to c into del u by del x then it is possible to write del v del t equal to c del w del x or del w del t equal to c del v del x, which is same.

Now, if we can write that then possibly from there we can form a first order equation which is del u del t plus A del u del x equal to 0; where u is now, u means an array of v and w and A means a matrix 0 minus c minus c 0. These are the elements. So, this equation second order wave equation can be represented as a system of first order equation.

(Refer Slide Time: 05:38)

Eigen values of matrix $[A]$, λ are given by

$$\det|A - \lambda I| = 0 \text{ or } \lambda^2 - c^2 = 0$$

$$\lambda_1 = +c \text{ and } \lambda_2 = -c$$

Representing two travelling waves with speeds given by

$$\left(\frac{dx}{dt}\right)_1 = c \text{ and } \left(\frac{dx}{dt}\right)_2 = -c$$

And then we can find out Eigen values from the matrix which is lambda and simple way is to find out determinant A minus lambda I equal to 0. This I of course, unit matrix, not unit complex number. One has to be little careful and so, lambda square minus c square equal to 0. We get two roots lambda 1 equal to plus c and lambda 2 equal to minus c.

Representing two traveling waves with the speed; one is del x del t plus c, another is del x del t minus c. This can also be called this del x del t 1 and del x del t 2 this can also be called characteristic lines for the wave equation. One is from here if we find out one will be x equal to ct left running passes through a point left from the point of interest. If i is a point of interest delta x left i minus 1.

So, c x equal to ct runs through i minus 1 and x equal to minus ct, this runs through i plus 1 point. We will show this graphically in one of the subsequent slides. So, i, i minus 1, i plus 1 are separated by delta x.

(Refer Slide Time: 07:26)

First order wave equation(contd.)

The system of equations in this example is hyperbolic and it has also been seen that the eigenvalues of the **A** matrix represent the characteristic differential representation of the wave equation. Euler's equation may be treated as a system of first-order wave equations. For Euler's equations, in two dimenssions, we can write a system of first order as

$$\frac{\partial E}{\partial t} + [A]\frac{\partial E}{\partial x} + [B]\frac{\partial E}{\partial y} = [S]$$

where

$$E = \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad A = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix},$$

$$B = \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} \quad and \quad S = \begin{bmatrix} -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{bmatrix}$$

$$\frac{\partial E}{\partial t} + [A]\frac{\partial E}{\partial x} + [B]\frac{\partial E}{\partial y} = [S]$$

where

$$E = \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad A = \begin{Bmatrix} u & 0 \\ 0 & u \end{Bmatrix},$$

$$B = \begin{Bmatrix} v & 0 \\ 0 & v \end{Bmatrix} \text{ and } S = \begin{Bmatrix} -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{Bmatrix}$$

Now we are talking about system of equations. Now, system of equation in this example is hyperbolic and it has also been seen that the eigenvalues of the A matrix represent the characteristics differential representation of wave equation. Now, Euler's equation may be treated as a system of first order wave equations.

For Euler's equations in two dimensions we can write a system of first order equations as del uppercase c t by del t plus A matrix del E del x plus B matrix del E del y equal to S, where E is an array u and v. E means u and v. A is a matrix elements are u 0 0 u. B is another matrix elements are v 0 0 v and S is a source term in Euler's equation which are basically pressure gradients; minus 1 by rho del p del x and minus 1 by rho del p del y.

So, Euler's equation in two dimensional Euler equations can be represented as system of first order equations.

(Refer Slide Time: 09:06)

The first order wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial x} = \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \; , \; \frac{\partial u}{\partial t} = \frac{u_i^{n+1} - \frac{1}{2}\left(u_{i+1}^n - u_{i-1}^n\right)}{\Delta t}$$

Discretized form

$$u_i^{n+1} = \frac{u_{i+1}^n - u_{i-1}^n}{2} - c\frac{\Delta t}{\Delta x}\frac{\left(u_{i+1}^n - u_{i-1}^n\right)}{2} \quad .....1$$

Now first order wave equation if we consider we can write del u del t plus c del u del x equal to 0. Now, del u del x if we go for finite difference equivalent del u del x we can write as u i plus 1 n minus u i minus 1 n by twice delta x. This is second order accurate, second order accurate derivative of the first order term; first order derivative is expressed through second order accuracy.

Then del u del t can be expressed as u i n plus 1 and minus u i at nth level, this i at nth level is taken at nth level, but average between two neighboring points; i plus 1 and i minus 1. So, u i plus 1 plus u i minus 1 by 2 is u i and that is at nth level divided by delta t.

Then if we feed in del u del t here and del u del x here, we will get an expression for u i n plus 1 which will be simply as you can see u i plus 1 plus u i minus 1 at nth level by 2 minus c delta t by delta x u i plus 1 at nth level minus u i minus 1 at nth level by 2. We are giving a number.

These numbers are arbitrarily problem specific. I am not maintaining number for all the slides in a sequence. So, this particular problem this is equation 1 and the time derivative the way we expressed it is, it was first done by Peter Lax and it is called Lax method.

Assume error of the form

$$\varepsilon_m(x,t) = e^{at} e^{Ik_m x} \qquad (2)$$

As done previously, and substitute this in Eq. (1), following the same arguments as applied to the analysis of heat conduction equation, the amplification factor becomes

$$e^{at} = \cos(k_m \Delta x) - IC \sin(k_m \Delta x)$$

where $C = c\left(\dfrac{\Delta t}{\Delta x}\right)$. The stability requirement is $|e^{at}| \leq 1$. Finally, the condition culminates in $C = c\left(\dfrac{\Delta t}{\Delta x}\right) \leq 1$

C is the courant number.

Famous mathematician Peter Lax developed it; there was a purpose we will come to know later in subsequent lessons. So, we define our error at any arbitrary element m is as you know is a Fourier series. Any arbitrary element of a Fourier series has been taken which is epsilon m equal to basically e to the power a t e to the power unit complex number I km x. As done previously and we substitute equation 2 in equation 1.

First order wave equation

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial x} = \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \ , \ \frac{\partial u}{\partial t} = \frac{u_i^{n+1} - \frac{1}{2}\left(u_{i+1}^n - u_{i-1}^n\right)}{\Delta t}$$

Discretized form

$$u_i^{n+1} = \frac{u_{i+1}^n - u_{i-1}^n}{2} - c\frac{\Delta t}{\Delta x}\frac{\left(u_{i+1}^n - u_{i-1}^n\right)}{2} \quad .....1$$

That means in this equation the error quantity because if u is substituted by d plus error then this equation is valid in d also. So, if we subtract that we get an equation in error and there we substitute error quantity and just the way we did it in case of parabolic equation, again we will get an expression for e to the power a t cos k m delta x minus I C sin k m delta x.

This C has been formed by clubbing together c delta t and x. So, this C delta t and x this combination has been substituted by uppercase C and we get this expression. And all of

us know that stability requirement is modulus of these; that means, e to the power a t; that means, modulus of cos theta minus IC sin theta while theta is k m delta x has to be less than or equal to 1.

Now, modulus of this is basically cos square theta plus C square sin square theta, and root over that quantity basically. We will see that in order to satisfy the condition for the amplitude be less than 1 we have to have C less than equal to 1.

That means, small c delta t by delta x less than equal to 1. Uppercase C or less than equal to 1 or lowercase c delta t by delta x less than equal to 1. This uppercase C is called Courant number. This equation restricts actually delta t. If c del t by del x has to be less than 1, it poses a condition on delta t; that means, delta t has to be less than delta x by c.

And that will make this solution of this equation stable if we choose delta t less than delta x by c that is the stability requirement. So, equation 3 is called Courant Friedrichs and Lewy condition. In my introductory lecture, I mentioned about these three scientists and their enormous contribution on partial differential equations and this condition is called CFL condition that Courant number is less than equal to 1.

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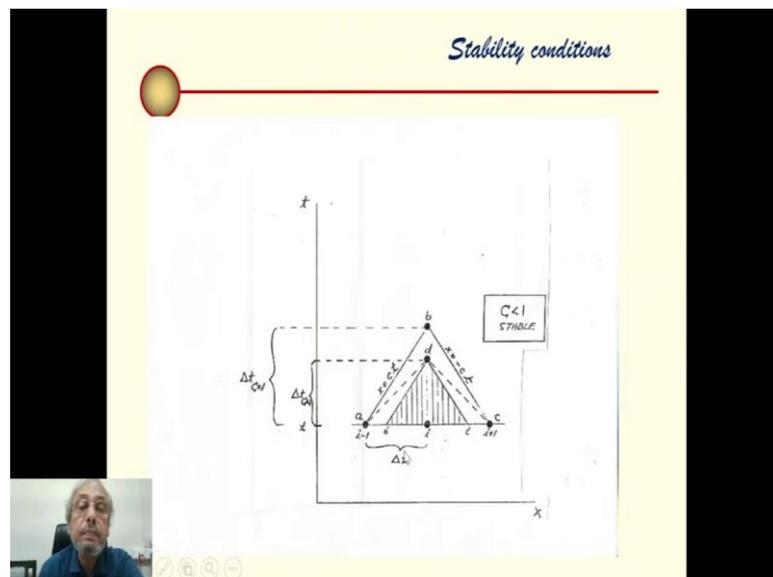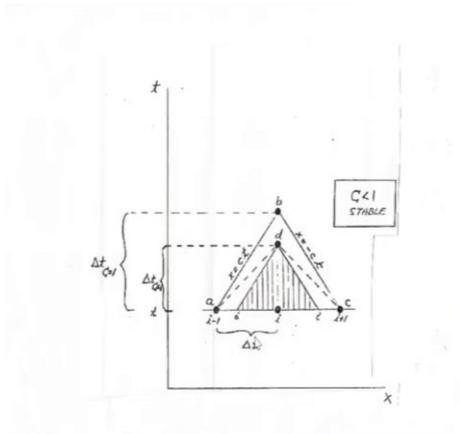
And I was mentioning about xt plane, left running characteristics line, right running characteristics line and through one is though i minus 1 another is through i plus 1, you can see for unit Courant number this is basically delta x by c. Our delta t for the 1D wave equation has to be less than that. So, this is the you know the limitation we have to the numerical space has to be less than the analytical expression of delta t equal to delta x by c.

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$$\text{Courant number } C = c\left(\frac{\Delta t}{\Delta x}\right)$$

$$C = c\left(\frac{\Delta t}{\Delta x}\right)$$

So, Courant-Friedrichs-Lewy condition is based on Courant number, where Courant number is C. Here it is wave velocity delta t by delta x, where c is a wave velocity, delta t is a time step, delta x is the spatial step. For one dimensional first order hyperbolic equation CFL condition obtained by the above procedure is C less than 1.

Different stability conditions for different equations can be obtained from appropriate conditions and then by substituting c because everywhere velocity plays a role, so, restriction on velocity over a time step that is obtained from this CFL condition. And if we substitute in the stability requirement expression, combination of c delta t by delta x or velocity into delta t by delta x we will get uppercase C or basically Courant number.

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Velocity transport equation,

$$\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = v \left[\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right] \quad \text{.......1}$$

Discretization using FTCS scheme,

$$\frac{\omega_{i,j}^{n+1} - \omega_{i,j}^{n}}{\Delta t} = -u \left[\frac{\omega_{i+1,j}^{n} - \omega_{i-1,j}^{n}}{2\Delta x} \right] - v \left[\frac{\omega_{i,j+1}^{n} - \omega_{i,j-1}^{n}}{2\Delta y} \right] + v \left[\frac{\omega_{i+1,j}^{n} - 2\omega_{i,j}^{n} + \omega_{i-1,j}^{n}}{\left(\Delta x^2 \right)} \right]$$

$$+\nu\left[\frac{\omega_{i,j+1}^{n}-2\omega_{i,j}^{n}+\omega_{i,j-1}^{n}}{\left(\Delta y^{2}\right)}\right] \qquad \dots\dots(2)$$

Now, we take up another very popular transport equation. As I said that we will take all the all possible equations that means, we have taken parabolic, we have just discussed about hyperbolic. And now let us discuss elliptic equation and I have chosen vorticity transport equation.

I am sure all of you know what is vorticity transport equation. If the x momentum equation is differentiated with respect to y, and y momentum equation is differentiated with respect to x and then former is subtracted from the later we get basically vorticity transport equation.

So, this is as you can see omega is del v del x minus del u del y. So, that quantity omega lowercase omega which is vorticity into dimension given by del v del x minus del u del y that quantity is transported in this way. Del omega del t plus u del omega del x plus v del omega del y equal to nu coefficient of viscosity del 2 omega del x 2 plus del 2 omega del y 2.

And again we apply FTCS; that means, Forward in Time and Central in Space. The time derivative del omega del t has been discretized, omega at i, j point; omega i, j at n plus 1th level minus omega i, j at nth level by delta t forward in time minus u; here u the velocity will act as coefficient, v will act as coefficient.

This is how we get rid of non-linearity and then omega is discretized by central difference. Omega i plus 1 j minus omega i minus 1 j divided by twice delta x omega at nth level. Then again omega j plus 1 minus i, j plus 1 minus omega i, j minus 1 by twice delta y both omega has been written as nth level; then nu into central difference of omega in x direction at nth level; again plus nu into central difference of omega in y direction at nth level.

So, all the spatial variables are discretized through central differencing scheme and temporal term discretized through forward difference scheme and we have yet another problem. The first problem we handled today was wave equation which is hyperbolic equation, this is elliptic equation.

Let us consider N=D+ε with

$$\varepsilon\left(x,y,t\right)=e^{at}\sum_m e^{\left(Ik_m x+Ik_m y\right)} \qquad \text{........(3)}$$

Where N is the numerical solution obtained from computer, D the exact solution of the FDE and ε error. substituting Eq. (3) into Eq. (2) and using the trigonometric properties, we finally obtain

$$\frac{\varepsilon_{i,j}^{n+1}}{\varepsilon_{i,j}^{n}}=\frac{e^{a\left(t+\Delta t\right)}e^{Ik_m\left(x+y\right)}}{e^{at}e^{Ik_m\left(x+y\right)}}=e^{a\Delta t}=G$$

where $G = 1 - 2\left(d_x+d_y\right) + 2d_x \cos\left(k_m\Delta x\right) + 2d_y \cos\left(k_m\Delta y\right)$

$$-I\left[C_x \sin\left(k_m\Delta x\right)+C_y \sin\left(k_m\Delta y\right)\right]$$

where $d_x = \frac{v\Delta t}{\left(\Delta x\right)^2}$, $d_y = \frac{v\Delta t}{\left(\Delta y\right)^2}$, $C_x = \frac{u\Delta t}{\Delta x}$, $C_y = \frac{u\Delta t}{\Delta y}$

The obvious stability condition $\left|G\right| \leq 1$, finally leads to

$$d_x + d_y \leq \frac{1}{2} \, , \; C_x + C_y \leq 1 \qquad \qquad ....4$$

So, we have started giving nu numbers. So, this is 1, this is 2 and then again usual Von Neumann analysis. The error quantity has to be substituted by a Fourier series and exponent in time domain and then take up any representative number any representative expression from the Fourier series.

So, here we have taken mth quantity or e to the power I km x plus I km y; this is an e to the power a t. This is basically the representative term of the error quantity. This is substituted again in equation 3 and then you have to do little bit of algebra here after substitution. So, that you can get an expression in epsilon i, j n plus 1th level divided by epsilon i, j nth level. You know mentioning this e to the power a t e to the power I km delta k m x plus e to the power I k m x, this is error quantity epsilon.

So, epsilon is substituted in the governing equation and then through some algebraic steps we will get epsilon i, j n plus 1 divided by epsilon i, j at nth level; that means, this is amplification is e to the power a t plus delta t e to the power I k m x plus y divided by e to the power delta t e to the power I k m x plus y. And which is basically e to the power delta t or this amplification factor.

And then if again as I said after substitution if we do little algebraic calculations, we will be able to express G, I we did it in detail while doing it for parabolic equation. Here we are just similar steps have to be followed. We will get 2 dx plus d y plus 2 dx cos k m delta x 2 dy cos k m delta y minus I imaginary parties C x sin k m delta x plus C y sin k m delta y.

And here delta x and delta y's; we can see delta x is delta x is nu delta t by delta x square, dx is nu delta x by delta y square, d y is nu delta t by delta y square and C x is u into delta t by delta x, C y is v into del t by delta y. And again modulus of G has to be less than equal to 1.

This will then pose condition which is modulus of 1 minus 2 dx plus dy plus 2 dx cos k m delta x plus 2 dy cos k m delta y plus this imaginary quantity entire expression under modulus operation or modular sign has to be less than equal to 1.

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The obvious stability condition $|G| \leq 1$, finally leads to

$$d_x + d_y \leq \frac{1}{2}, \quad C_x + C_y \leq 1 \qquad ....4$$

when $d_x = d_y = d \left(for \ \Delta x = \Delta y \right)$, $d \leq \frac{1}{4}$

which means $\dfrac{v\Delta t}{(\Delta x)^2} \leq \dfrac{1}{4}$

This is twice as restrictive as the one-dimensional diffusive3 limitation. Again for the special case $\left(u=v \ and \ \Delta x = \Delta y \right)$

$$C_x = C_y = C, \text{ hence } C \leq \frac{1}{2}$$

If we apply that, we will get these two conditions. dx plus d y less than equal to half, C x plus C y less than equal to 1. Now, d x equal to if we say d y; because difference

between dx and dy only the grid size. This is grid size delta x in x direction this is grid size in y direction.

So, if we take equal grid then we can substitute delta x and delta y as d, d has to be then less than equal to one-fourth which means nu delta t by whatever is the grid size it is same in now x and y direction, grid size squared is less than equal to 1 by 4.

This is twice as restrictive as the one-dimensional diffusion limitation. Again, the special case we can say the velocity in u direction, x direction and y direction if we assume those are same. And again, delta x equal to delta y then it is possible to write C x equal to C y equal to C and if C x equal to C y then we can see this is less than equal to half.

So, this CFL condition is also twice as restrictive as one- dimensional case. So, diffusion limitation is twice as restrictive as one-dimensional case in two dimension and CFL condition is also twice as restrictive as the one-dimensional equivalent one-dimensional case.

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Second order wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Discretized form

$$\left[\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} \right] = c^2 \left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \right] \ldots\ldots\ldots\ldots(5)$$

Again assume $N = D + \varepsilon$ …………(6) $\varepsilon_i^n = e^{at} e^{Ik_m x}$ …………(7)

So, now we will go for stability analysis of second order wave equation. We will not go to construct a system of first order wave equations rather we will solve second order wave equation which is del 2 u del t 2 equal to c square del 2 u del x 2. Here we since this is second order derivative we are substituting this with a finite difference quotient which is central difference.

So, u i at n plus 1 minus twice u y at n plus u i at n minus 1; 3 time levels divided by delta x square equal to c square then del 2 u del x 2 again we apply central differencing between i plus 1 and i minus 1 point in between i. So, all are at nth level divided by del x square. Then again if this n is a solution of this equation then equation 5 is satisfied by n.

If u is replaced by N then if N is replaced by D plus e; D is the ideal solution of the finite difference equation without any rounding of error, which is physically not possible when you get a solution from machine from computer this rounded up error has to be there rounding error; so, D plus e epsilon.

So, we substitute D plus epsilon then this equation holds good for D. We subtract equation in D from equation in D plus epsilon we get an equation for error in terms of epsilon. And then again epsilon is substituted by any term of Fourier series, which is e to the power a t into e to the power I k m x.

Substituting eq. (7) and (6) in (5) and dividing by e^{at} and $e^{Ik_m x}$, we get

$$e^{at} - 2 + e^{-\Delta at} = C^2 \left[e^{Ik_m x} - e^{Ik_m x} - 2 \right] \tag{8}$$

where C, the courant number $= \dfrac{c(\Delta t)}{\Delta x}$

$$e^{at} + e^{-at} = 2 - 4C^2 \sin^2 \left(\frac{k_m \Delta x}{2} \right) \tag{9}$$

$$G = \left| \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} \right| = \left| e^{a\Delta t} \right|$$

$$e^{2a\Delta t} - 2 \left[1 - 2C^2 \sin^2 \left(\frac{(k_m \Delta x)}{2} \right) \right] e^{a\Delta t} + 1 = 0 \tag{10}$$

Now that means, basically 6 and 7 substituted in 5 and then again we do some algebraic operation and we get this equation. e to the power a delta t minus 2 plus e to the power minus a delta t equal to C square e to the power this is unit complex number I k m x plus e to the power minus I k m x minus 2.

And C is Courant number which you know while doing the algebraic operation we have clubbed together c delta t and delta x in this way and that is capital C. So, we will get

from here e to the power delta t plus e to the power minus a delta t equal to we just have to transferred 2 on the right hand side and then substituted e to the power i theta plus e to the power minus i theta by cos theta then cos theta minus 1 has been substituted by minus 2 sin square theta by 2.

And we have gotten this term this equation which is equation 9 and again the amplification factor is epsilon at ith location n plus 1th level divided by epsilon at ith location at nth level modulus of this is the amplification factor. So, we can write modulus of e to the power a t and that should be less than equal to 1.

Now, if we rearrange equation 9 we will be able to get this quadratic equation, equation 10. Just rearrange this equation divide by e to the power a e to the power minus a delta t and you will get this.

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Stability of second order wave equation (contd.)

Analyzing the discretized equation we arrive at quadratic equation

$$e^{2a\Delta t} - 2\left[1 - 2C^2 \sin^2\left(\frac{k_m \Delta x}{2}\right)\right] e^{a\Delta t} + 1 = 0 \qquad (10)$$

This equation, quite obviously has two roots, and the product of the roots is equal to +1

For stability complex roots are required, so discriminant should be negative

$$e^{2a\Delta t} - 2\left[1 - 2C^2 \sin^2\left(\frac{(k_m \Delta x)}{2}\right)\right] e^{a\Delta t} + 1 = 0 \qquad (10)$$

So, this is equation 10. This equation 10 can have will obviously its quadratic equation will have 2 roots. One route if it is greater than 1 then stability condition will not be

satisfied. So, both the routes have to be equal to 1. That is only possible if it is a complex route. If it is a real route it is not possible. So, it has to be a complex root.

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$$\left[1 - 2C^2 \sin^2\left(\frac{(k_m \Delta x)}{2}\right)\right]^2 - 1 < 0$$

$$C^2 < \frac{1}{\sin^2\left(\frac{(k_m \Delta x)}{2}\right)}$$

So, complex root means this discriminant should be less than 0. So, arising out of these we will have to find out the condition. So, you have written everything here in details that e to the power a t is a magnification e to the power a delta t is the magnification factor. If its value exceeds 1, the error will grow exponentially which will lead to an unstable situation.

All these possibilities mean that equation 10 should possess complex roots in order to both have the value of e to the power a t equal to unity. This implies that discriminant of this equation has to be less than 0, has to be negative in order to get a complex root. So, it is simple algebra. We can clearly see in order to satisfy this condition that 1 minus 2 square C square sin square k m delta x by 2 the whole square minus 1 less than equal to 0; sin square value is always positive.

And from there, we can get that C square has to be less than 1 by sin square k m delta x by 2. So, which is always true, if C is less than 1. Hence CFL condition which is Courant number less than 1 again must be satisfied for stability of second order hyperbolic equations.

So, we have seen parabolic equation for which restrictions were posed on the coefficient of diffusion, time step and the rate size. Then we have seen first order wave equation where condition was Courant number less than 1. Then we have seen elliptic equation where conditions are posed to both on diffusive term and Courant number. And now we have obtained the stability condition for second order wave equation where Courant number is less than 1.

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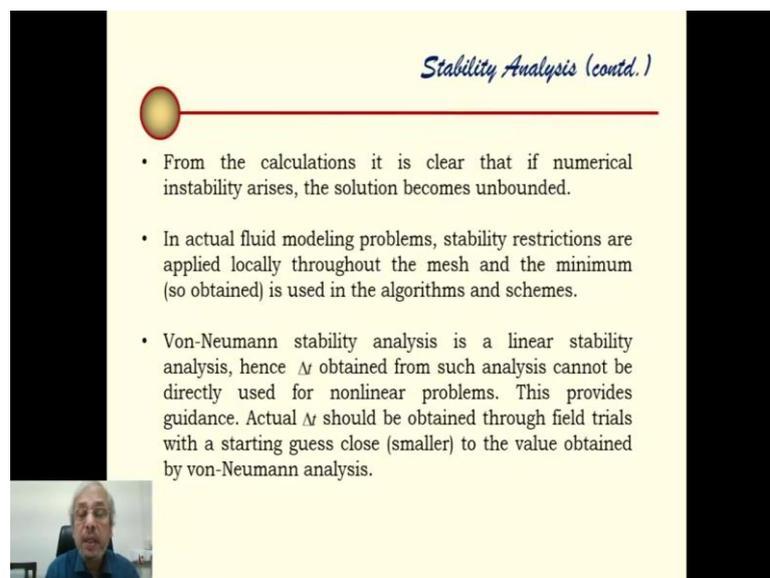
So, from the calculations it is clear that numerical instability arises, I mean if numerical instability arises solution becomes unbounded, we never want that. In actual fluid modeling problem, stability restriction are applied locally throughout the mesh. The mesh and minimum, is obtained that is used for the entire problem that is used for the entire algorithm that is applied to solve the problem.

Now, one thing here the stability analysis which we have done; we have done on linearized equation. Even just slowly I will go to vorticity transport equation which is definitely a non-linear equation, but if you recall I mentioned u and v is are considered here as coefficients which makes the equation linear.

So, these are linearized version of the non-linear equations. So, for Neumann stability analysis is a linear stability analysis. So, hence delta t obtained from such analysis cannot be directly used when we will be solving full Navier-Stokes equations or Euler's equations or energy equation if any nonlinearity there in energy equation.

Of course, energy equation is usually the way we solve is again we it is a quasi linear equation, we find out velocity field first and then feed the velocity field in the energy equation to solve for temperatures. Whatever it is we will discuss when we will take up those specific problems. What I want to mean that, basically fluid flow problems are non-linear problems.

So, this stability analysis whatever delta t we get cannot be directly applied to it, just the value that we all guide the condition that we get from the Von-Neumann analysis, but this is definitely helpful because this gives us the indication. Actually having obtained this indication from the stability analysis, we have to take delta t little less than the restrictive value and then we have to try it for the entire problem and this is sort of numerical experiment.

So, we have to set up a trial run two three trial runs to see whether delta t is functioning well. One may argue that if we go for implicit formation we do not need that, that is true, but when you I mentioned it earlier when we want to capture exact transient; specially, for turbulent flow problems when we calculate for direct numerical simulation or large dissimulation kind of scenario.

Every time step intermediate time step is meaningful then we cannot avoid this explicit advancement rule and there are few other problems also. Some you know a conduction problem while transient is very important. So, you know wherever we have to proceed through explicit formulation, stability analysis is of paramount interest and paramount importance.

So, we have to go through stability analysis to decide on the time step. We have seen earlier also that in order to increase the accuracy we have to make the grids finer, but not only making grid finer we will raise the formal order. You see we have to go also for higher order schemes, but higher order schemes we cannot go arbitrarily we have to obey always the consistency related restrictions.

Having done that we can go for higher order discretization method then we can apply very fine grid size as fine as possible for a given computer, but then it will eventually impose restriction on time step. You have seen that time step is related to spatial grid size delta x or delta y in 3D delta z.

And that is why stability knowing about stability, understanding stability and being guided by the stability requirement is essential to have a stable and correct calculation. We will stop here today. We will take up remaining issues related to discretization, finite difference formulation and other associated problems in the next class.

Thank you very much.