Computational Fluid Dynamics and Heat Transfer
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Lecture – 03
Explicit and Implicit Formulations Stability – 1

Good-afternoon everyone. In today's lecture, we will cover Explicit and Implicit Formulations and we will discuss on Stability.

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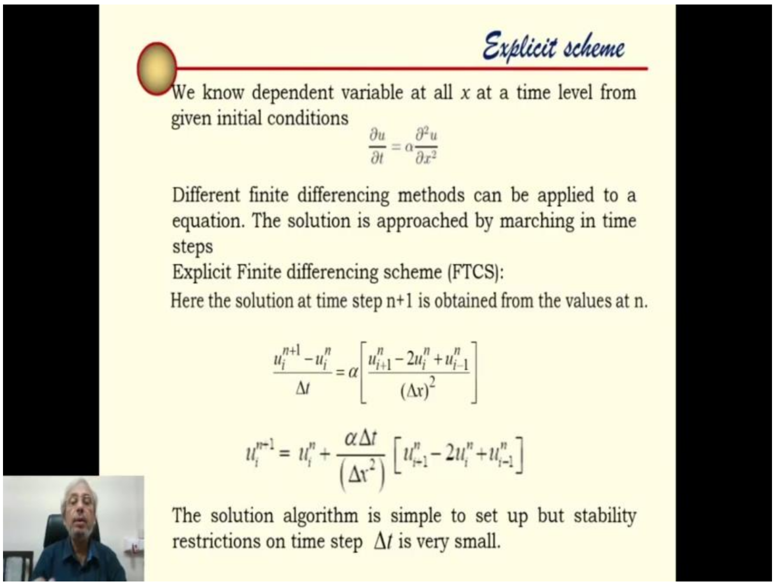

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \right]$$

$$u_i^{n+1} = u_i^n + \frac{\alpha \Delta t}{\left(\Delta x^2\right)} \left[u_{i+1}^n - 2u_i^n + u_{i-1}^n \right]$$

So, again let us consider the model equation that we are using for explaining different aspects of finite difference quotients and formulation.

This is unsteady state heat conduction equation, where u is basically temperature, t is time, alpha is thermal diffusivity and x is the spatial dimension or the space variable on which temperature is varying.

Now, we can apply FTCS scheme. We have already explained what is FTCS, Forward Time and Central in Space. So, the forward differencing in time we can see and then we have taken central difference and space, and all the variables dependent variables we can see are written as n-th level.

Whereas, when we have expressed the del u del t, we have written u i n plus 1 minus u i n by delta t. And here, values of temperature values of u at the entire spatial domain all the points are known at n-th level. We will be determining these values at n plus 1-th level.

So, the obvious outcome of this discretization is u i n plus 1 equal to u i n plus alpha delta t divided by delta x square into u i plus 1 minus u twice u i plus u i minus 1, all are at n-th level.

Very clearly we can see we can find out u i value at n plus 1-th level in terms of all known quantities; u i at n-th level is known, its neighboring value eastern neighbor at n-th level is known, westward neighbor at n-th level is known. So, all the known quantities time step is known delta x is known. So, we will be able to find out u i at n plus 1-th level.

So, this way if we can sweep from i equal to the first point to the last point say i equal to 1 to i max, then all the points we can calculate at n plus 1-th level explicitly from the values at n-th level. So, this is called explicit algorithm. And this is very easy to formulate. Only thing is that delta t and delta x we cannot take arbitrarily, they are related. And they are to be obeyed, they have to obey the stability requirements. We will come to that as aspect little later.

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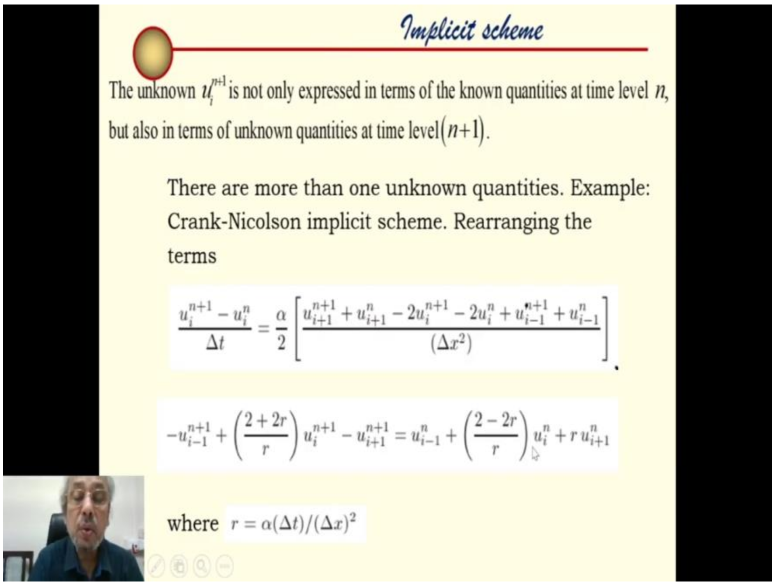

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{\alpha}{2}\left[\frac{u_{i+1}^{n+1}+u_{i+1}^n-2u_i^{n+1}-2u_i^n+u_{i-1}^{n+1}-u_{i-1}^n}{\left(\Delta x^2\right)}\right]$$

$$-u_i^{n+1}+\left(\frac{2+2r}{r}\right)u_i^{n+1}-u_{i+1}^{n+1}=u_{i-1}^n+\left(\frac{2-2r}{r}\right)u_i^n+ru_{i+1}^n$$

$$\text{Where } r=\frac{\alpha\Delta t}{\left(\Delta x^2\right)}$$

Now, the right -hand side, just you know recall this discretization, if at the right -hand side values are all expressed n-th level values. Now, these are changed to average of n plus 1-th level and n-th level values. So, u i plus 1, this is u i plus 1 at n plus 1-th level u i plus 1 at n-th level by 2. So, average of n plus 1-th level and n-th level value by 2.

Similarly, u i also expressed that way and u i minus 1 is also expressed that way, average of n plus 1 and n-th level value. If we write it in this way, we are introducing another strategy, and this strategy is called Crank-Nicolson Implicit Scheme.

What we have to do next? The values at n plus 1-th level are not known, but now we have these unknown values n plus 1-th level at left -hand side and right -hand side both the sides.

So, let us transfer all the values at n plus 1-th level to the left -hand side and all the values at n-th level on the right -hand side.

That means, the values at n-th level are moved to right -hand side and right -hand side can be evaluated and it is known whereas, left -hand side u i at n plus 1-th level is not known and it is expressed in terms of u i minus 1 at n plus 1-th level this is also not known.

And u i plus 1 at n plus 1-th level this is also not known. So, the unknown point u i at n plus 1-th level is expressed in terms of its eastern neighbor and westward neighbor at n plus 1-th time level which are also not known. r is the quantity alpha delta t by delta x square it is known. Now, if this equation we run from i equal to 2 to i equal to say i max minus 1.

Just here itself, if we substitute i equal to 2, we will get the in this equation u 2 n plus 1 this term the first term on the left -hand side will be u 1 n plus 1 and the third term will be u 3 n plus 1. So, u 2, u 3, u 1, all 3 quantities are unknown we will get one algebraic equation in terms of u 1, u 2, u 3 equal to a known quantity, then will vary i equal to 3.

So, then again we will get to another equation u 3 n plus 1 will be the middle term, the first term will be u 2 n plus 1 and the third term will be 4.

So, u 2 at n plus 1, u 3 at n plus 1, u 4 at n plus 1 will appear in the algebraic equation, right -hand side we again be known.

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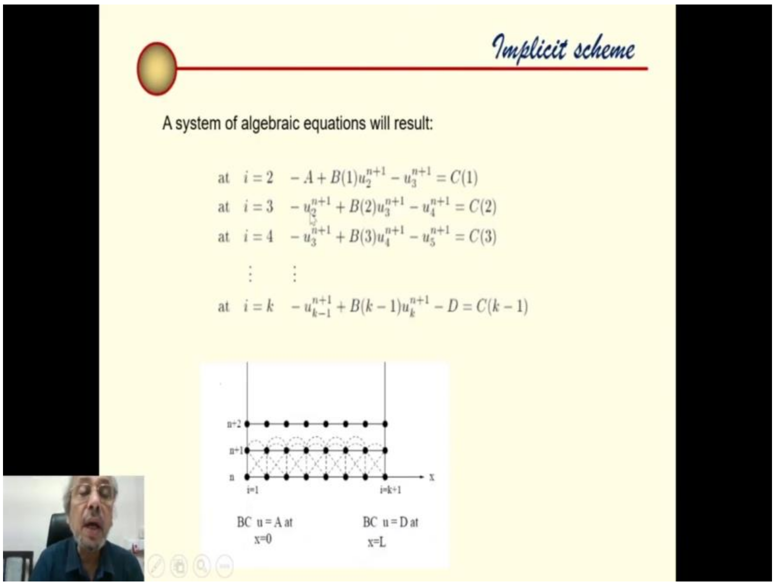

$$i=2 \quad -A+B(1)u_2^{n+1}-u_3^{n+1}=C(1)$$

$$i=3 \quad -u_2^{n+1}+B(2)u_3^{n+1}-u_4^{n+1}=C(2)$$

$$i=4 \quad -u_3^{n+1}+B(3)u_4^{n+1}-u_5^{n+1}=C(3)$$

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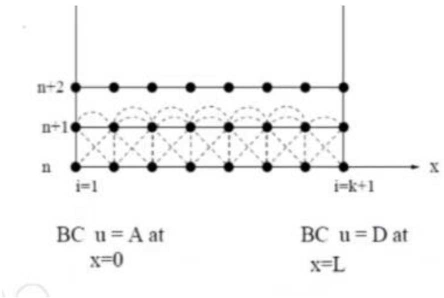
$$i=k \quad -u_{k-1}^{n+1}+B(k-1)u_k^{n+1}-D=C(k-1)$$



So, this way if we keep on varying i which we have done here i equal to 2, i equal to 3, i equal to 4, so for each i we will get one algebraic equation. And in this algebraic equation as we have seen that this coefficient we can write as B and entire right -hand side we can write as C.

So, we will see for i equal to 2, u 2 with the coefficient u 3, both are at n plus 1-th level not known equal to right -hand side known C. The first point which is u 1 this is falling on the boundary. So, for example, if this is a domain in the figure if you focus i equal to 1 to i k plus 1, k plus 1 is a n point, i equal to 1 is a starting point.

Now, boundary condition means at two extremes i equal to 1 and k equal to 1 are the boundary conditions and we have given here boundary condition for the dependent variable that is u at i equal to 1 we have written the value as A, and i equal to k plus 1 we have written the value as D. So, boundary conditions are u equal to A here and u equal to D at k plus 1-th point.

So, in the first equation the quantity u 1 at n plus 1 this was with minus sign we can look into it and then this will be minus A, because u at 1 or u 1 is known and that is basically the boundary condition A. So, then we can this way substitute i equal to 2, i equal to 3, i equal to 4, and i equal to k then we will get u k minus 1 again the coefficient u k and u k plus 1.

Now, again u k plus 1 this value is basically the boundary condition D and right -hand side is known. So, we get algebraic equations for each i value. And this set of algebraic equation a system of algebraic equation. We can express in this way in matrix form.

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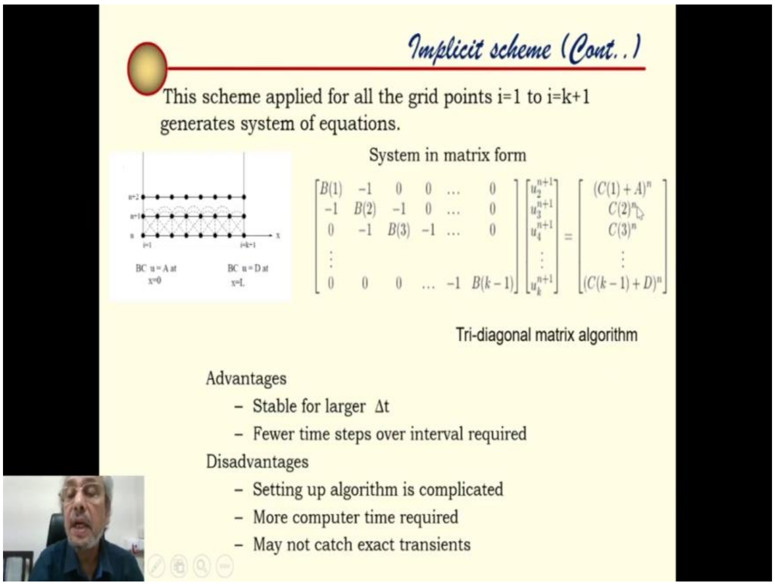

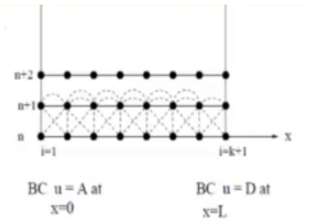

$$\begin{bmatrix} B(1) & -1 & 0 & 0 & \cdots & 0 \\ -1 & B(2) & -1 & 0 & \cdots & 0 \\ 0 & -1 & B(3) & -1 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & & & B(k-1) \end{bmatrix} \begin{bmatrix} u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ \vdots \\ u_k^{n+1} \end{bmatrix} = \begin{bmatrix} (C(1)+A)^n \\ C(2)^n \\ C(3)^n \\ \vdots \\ (C(k-1)+D)^n \end{bmatrix}$$

So, this is A x equal to B, usual matrix equation, A is a matrix which is a known matrix, x is a vector which is unknown vector equal to the known quantity, known vector.

So, if we can convert this matrix into a triangular matrix and you know upper triangular matrix we will be able to evaluate all these quantities through back substitution. And in this matrix we can see, A matrix the diagonal elements are present, immediate sub diagonal elements are present, immediate super diagonal elements are present.

All other elements are 0. This is called tridiagonal matrix. And inversion of the matrix or converting in it into a upper triangular matrix is very simple. We have standard algorithm. We will learn about that algorithm and we will apply that.

So, if we can apply that on this matrix then we can easily find out all these variables, that means, u 2, u 3, u 4 up to u k. u 1 at n plus 1-th level is known the boundary condition, u k plus 1 at n plus 1-th level is known that is also the boundary condition shifted to the right -hand side. This is A also shifted to the right -hand side, and u 2 to u k are not known and this will be now known.

So, all these values from n-th level to n plus 1-th level we can proceed. We can find out all these values by solving this matrix equation and these values will be evaluated in one row, just after inverting this matrix we will be able to get u 2 to u k as an array immediately.

So, individually now what is the value of u 2, u 3, u 4, we need not to evaluate. We will come to know all the values just in one go. And this formulation is called implicit formulation. And this is Crank-Nicolson implicit formulation, since the way we wrote the right -hand side following the Crank-Nicolson method we wrote it.

So, once n plus 1-th level all the values are known we can go for next step, that means all the values of n plus 2 time level. And this way we can do the time marching from n to whatever whatever level we want to reach.

Now, this is a very efficient methodology we can see, very efficient formulation. we have no stability restriction, delta t can be large. And wherever we are trying to get basically the final desired level value quickly. We can go for implicit formulation, I just am giving an example that at t equal to 0 we know the basic distribution of temperature maybe on a plate at different points and then we want to get it after 10 seconds.

So, my delta t here can be 5 seconds or straightaway 10 seconds, whereas we will see in explicit formulation it has to be very small, we have to progressively reach the final desired value. In implicit formulation in relatively fewer time steps we can reach the final desired level.

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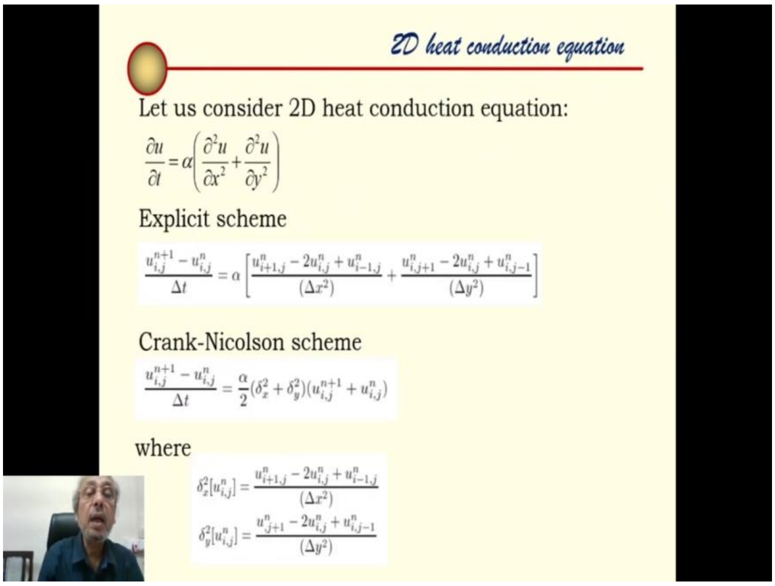

Let us consider 2D heat conduction equation

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Explicit scheme

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \alpha\left[\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\left(\Delta x^2\right)} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\left(\Delta y^2\right)}\right]$$

Crank-Nicolson scheme

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \frac{\alpha}{2}\left(\delta_x^2 + \delta_y^2\right)\left(u_{i,j}^{n+1} + u_{i,j}^{n}\right)$$

$$\text{Where } \delta_x^2\left[u_{i,j}^{n}\right] = \frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\left(\Delta x^2\right)}$$

$$\delta_y^2\left[u_{i,j}^{n}\right] = \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\left(\Delta y^2\right)}$$

So, now if we look at two dimensional heat conduction equation. So, the space variable; that means, alpha we have second derivative in x and secondary derivative in y in x and y direction may be on a plane and then we are progressing in time direction.

Now, here also we can write explicit scheme; that means, that central differences in the x direction of the variable involving eastern neighbor and western neighbor u i, j is the point of interest at n-th level divided by delta x square, again u i j its northern neighbor is u i j plus 1 and u i j minus 1 southern neighbor, divided by delta y square simple central difference. And here u i, j at n plus 1-th level is not known.

We can easily write now u i, j at n plus 1-th level if we move all other terms on the right -hand side, entire right -hand side will be known that will be this bracketed terms into alpha delta t by delta x square multiplied by u i plus 1 u i, j u i minus 1 j, again plus alpha delta t multiplied by u i, j plus 1 u i, j u i, j minus 1 divided by delta y square. So, all the terms are known plus u i, j at n-th level that will give me as u i, j at n plus 1-th level and this is again explicit scheme.

If we apply Crank-Nicolson scheme here also what we will have to do? We have to write the variables on the right -hand side in terms of average quantities between n and n plus 1-th level. And then we will apply central difference operator on this variable which is u at i, j at n plus 1-th level and n-th level by 2.

So, u ui, j at n plus 1 plus u i, j at n divided by 2 on this quantity we will apply central difference operator delta x square and delta y square, and this operator means we have clearly written it, you can follow that.

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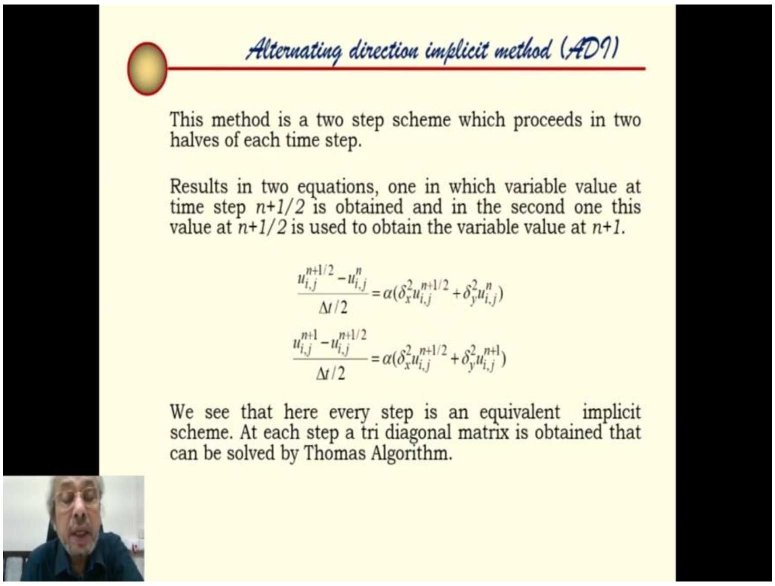

$$\frac{u_{i,j}^{n+1/2}-u_{i,j}^{n}}{\Delta t/2}=\alpha\left(\delta_x^2 u_{i,j}^{n+1/2}+\delta_y^2 u_{i,j}^{n}\right)$$

$$\frac{u_{i,j}^{n}-u_{i,j}^{n+1/2}}{\Delta t/2}=\alpha\left(\delta_x^2 u_{i,j}^{n+1/2}+\delta_y^2 u_{i,j}^{n+1}\right)$$

So, if we do that we have to further formulate a strategy that we will not go from u i, j at n-th level to u i, j at n plus 1-th level directly. What we will do? We will go from u i, j at n-th level to u i, j at n plus half level. We will divide the time step exactly by 2.

So, if delta t is the time step first we will progress delta t by 2. And in this step, we will progress delta t by 2 and u i, j at n plus half level is unknown and the central difference operator will apply to the variable at n plus half level in x direction. In y direction, we will apply it on n-th level. All n-th level values are known, but n plus half level values at this stage are not known.

So, we will be able to write an algebraic equation, where n plus half level values are not known and this will be expressed in terms of right -hand side which is known. Having done that we will progress from n plus half to n plus 1 level. Then, what we will do?

We will write i, j at n plus 1 minus i, j at n plus half divided by delta t by 2 equal to alpha, central difference operator we will apply on the variable u i, j at n plus half level which is known and in x direction and central difference operator in y direction we will apply now at n plus 1 level. And this all these values are unknown.

And then these unknown at this stage n plus 1 is unknown, so we will move all the variables at n plus 1 to left -hand side and all the variables at n plus half level on the right -hand side, right -hand side is known. Again, we will get explicit equation of u i, j in terms of u i, j plus 1 u i, j minus 1 because we are applying central difference operator in y direction on n plus 1. So, at n plus 1-th level u i, j plus 1 u i, j minus 1 and u i, j are not known at n plus 1-th level, right -hand side everything is known.

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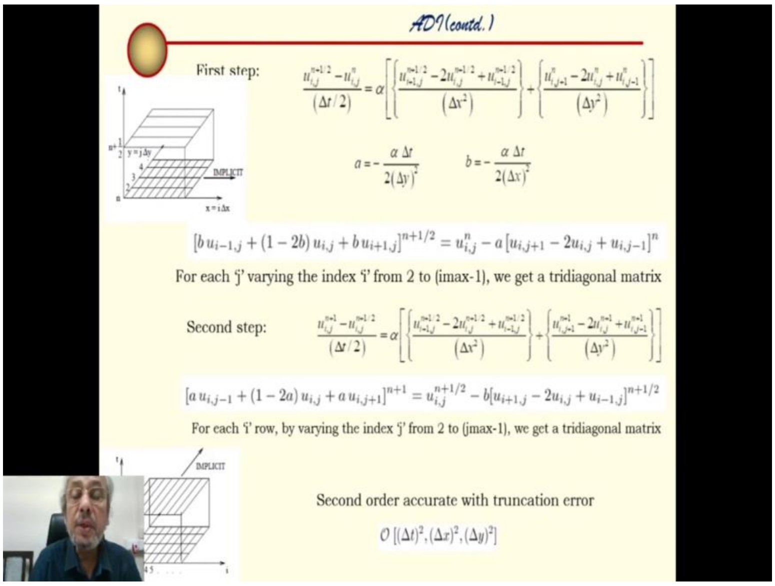

$$\text{First step}\ \frac{u^{n+1/2}_{i,j} - u^{n}_{i,j}}{\Delta t/2} = \alpha\left(\left\{\frac{u^{n+1/2}_{i+1,j} - 2u^{n+1/2}_{i,j} + u^{n+1/2}_{i-1,j}}{\left(\Delta x^2\right)}\right\} + \left\{\frac{u^{n}_{i,j+1} - 2u^{n}_{i,j} + u^{n}_{i,j-1}}{\left(\Delta y^2\right)}\right\}\right)$$

$$a = \frac{\alpha\ \Delta t}{2\left(\Delta y^2\right)}\ b = \frac{\alpha\ \Delta t}{2\left(\Delta x^2\right)}$$

$$\left[bu_{i-1,j} + (1-2b)u_{i,j} + bu_{i+1,j}\right]^{n+1/2} = u^{n}_{i,j} - a\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right]^{n}$$

$$\text{Second step:}\ \frac{u^{n+1}_{i,j} - u^{n+1/2}_{i,j}}{\Delta t/2} = \alpha\left(\left\{\frac{u^{n+1/2}_{i+1,j} - 2u^{n+1/2}_{i,j} + u^{n+1/2}_{i-1,j}}{\left(\Delta x^2\right)}\right\} + \left\{\frac{u^{n+1}_{i,j+1} - 2u^{n+1}_{i,j} + u^{n+1}_{i,j-1}}{\left(\Delta y^2\right)}\right\}\right)$$

$$\left[au_{i,j-1}+\left(1-2a\right)u_{i,j}+au_{i,j+1}\right]^{n+1}=u_{i,j}^{n+1/2}-b\left[u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right]^{n+1/2}$$

$$O\left[\left(\Delta t\right)^{2},\left(\Delta x\right)^{2},\left(\Delta y\right)^{2}\right]$$

So, again we will get this algebraic equation and then we will just be able to evaluate values at n plus 1-th level. To explain it little more, the first step what we wrote here, progressing from n-th level to n plus half level that is the first step. So, u i, j n plus half minus u i,j n divided by delta t by 2 equal to we can see the right -hand side and the central differencing operator in x direction has been applied to the variables at n plus half level.

Whereas, at n-th level central differencing operator in y direction we have applied here, these terms are known n plus half terms are not known. They are moved to left -hand side known terms are moved to right -hand side and then we will we get this equation, which is b u i minus 1 j plus 1 minus twice b u i, j plus b u i plus 1 j whereas, b is given by alpha delta t by twice delta x square.

And at n plus half level these are all unknown. On the right -hand side, n-th level values are known we can now rearrange them u i, j at n-th level minus a into u i, j plus 1 minus twice u i, j plus u i, j minus 1, these are all at n-th level and these are known.

Now, if we vary i from 2 to i max 1 at any given j, for example, this is the y direction and this is the x direction. So that means, these divisions will be counted by i index, these divisions will be counted by j index, and we are progressing in time direction.

So, at n-th level everywhere value is known, we are calculating at n plus half level. At n plus half level for j equal to 1, all these values are known because they are boundary values. So, this is known. At j equal to 2, if we vary i, we will get u at 1 j, at 2 j, at 3 j as an equation.

Similarly, i equal to 3, that means, u 2 j, u 3 j, u 4 j, and here j is fixed. So, you are varying i from 2 to i max minus 1, because why 2 to i max minus 1? when we apply i equal to 2 we get u at 1 j and that is boundary condition. Similarly, when we substitute i equal to i max minus 1, then we will get one term i max minus 2, another is i max u i max minus 1 and another is u i max. And u i max value is falling on the boundary that is known.

So, this way at j equal to 2, if we apply i equal to 2 to i max minus 1, we will get a tridiagonal matrix and from that matrix, after the solution is obtained we will get all these values of i at j equal to 2. Similarly, at j equal to 3 we will get.

Similarly, at j equal to 4 we will get all the i values. So, this is called implicit in i direction. We will vary j equal to 1 to 2, 2 to 3, 3 to 4 and for each j we will get a tridiagonal matrix and through that we will get values at generate values at all the i's. Then, we will complete all the values at n plus 1-th level.

Having done that we will come at n-th level that meant second step. And what we will do at the second step? Second step u i, j at n plus 1 this is not known, but at n plus half all the values are known, up to this we have progressed. Now, we will progress from here to n. So, this is from n plus half to n plus 1.

And then what we are doing? We are applying central differencing operator in x on n plus half level, and at n plus 1 level we are applying central difference operator in y direction. Again n plus 1 these values are not known.

We move all the variables that are not known to the left -hand side and all the variables that are known to the right -hand side and we will be able to follow this equation a u i, j minus 1 plus 1 minus 2 a u i, j plus a u i, j plus 1, all are at n plus 1-th level all are unknown, equal to the values at n plus half level. And a and b, we have written what are the expressions for a and b.

And here what we will do? We will take up one i and then we will vary j equal to 2 to j max minus 1. For example, I mean i equal to 1, I am not applying because i equal to 1 all the values are falling on the boundary they are known.

At i equal to 2 then we will we will vary j equal to 2 to j max minus 1, j equal to 2 means here u 1 j minus 1 u 2 j u 3 j plus 1.

So, i is fixed, i is 2 for example, and then j when j is 2 here, this is 1, this is 3. Then again j is 3, this is 2, this is 4. So, basically fixing i equal to 2 j equal to 1 to j max from beginning to the end all the values will be expressed in terms of unknown quantities at n plus 1-th level and right -hand side is known.

So, for a fix i we will vary j 2, 3, 4 up to j max minus 1. So, we will get a tridiagonal matrix and by inverting the tridiagonal matrix we will be able to find out all the values at i equal to 2, all j values. So, entire this row will be filled up.

Similarly, now i equal to 3 we will get all the values i equal to 4, we will get all the values of j. So, at n plus 1-th level then all the values on the domain at all the points will be known. And this time j direction is implicit, i we are fixing at i and going marching in j direction, implicit direction.So, for going from n equal to n to n plus half we made i direction implicit, and for going from n plus half to n plus 1 we are making j direction as implicit. That is why it is called alternate direction implicit scheme. And in this alternate direction implicit scheme, we can see order of accuracy in the spatial dimension is second order. And while doing this we have achieved again second order accuracy in temporal direction.

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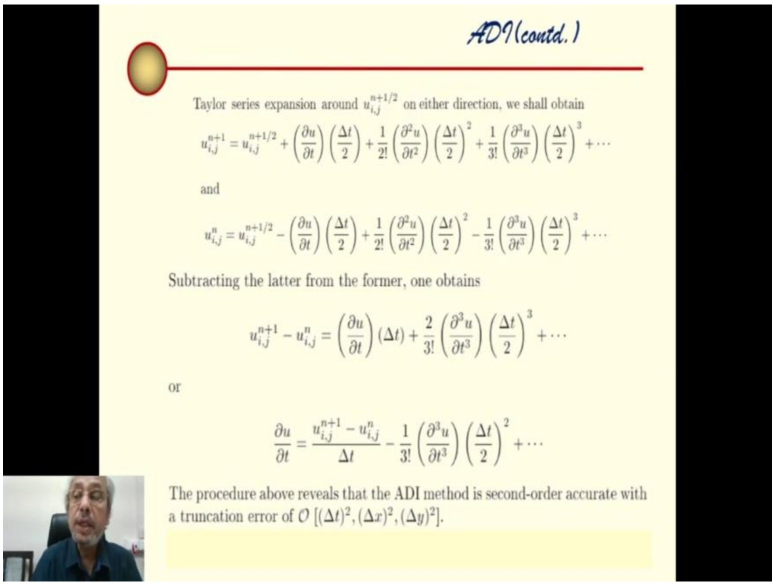

$$u_{i,j}^{n+1} = u_{i,j}^{n+1/2} + \left(\frac{\partial u}{\partial t}\right)\left(\frac{\Delta t}{2}\right) + \frac{1}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)\left(\frac{\Delta t}{2}\right)^2 + \frac{1}{3!}\left(\frac{\partial^3 u}{\partial t^3}\right)\left(\frac{\Delta t}{3}\right)^3 + \cdots$$

and

$$u_{i,j}^{n} = u_{i,j}^{n+1/2} - \left(\frac{\partial u}{\partial t}\right)\left(\frac{\Delta t}{2}\right) + \frac{1}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)\left(\frac{\Delta t}{2}\right)^2 - \frac{1}{3!}\left(\frac{\partial^3 u}{\partial t^3}\right)\left(\frac{\Delta t}{3}\right)^3 + \cdots$$

Subtracting the latter from the former, one obtains

$$u_{i,j}^{n+1} - u_{i,j}^{n} = \left(\frac{\partial u}{\partial t}\right)(\Delta t) + \frac{2}{3!}\left(\frac{\partial^3 u}{\partial t^3}\right)\left(\frac{\Delta t}{2}\right)^3 + \cdots$$

or

$$\left(\frac{\partial u}{\partial t}\right) = \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} - \frac{1}{3!}\left(\frac{\partial^3 u}{\partial t^3}\right)\left(\frac{\Delta t}{2}\right)^2 + \cdots$$

If you look at this discretization, you will be readily appreciating it. Here we have expressed n plus 1-th level values simply by Taylor series, n plus 1 u i n plus 1 in Taylor series expansion, the u i n plus half plus del u del t delta t by 2, so on and so forth. Similarly, u i, j at n-th level, again this is backward difference u i, j at n plus half minus del u del t del t by 2 plus 1 by factorial 2, del 2 u del t 2 del t del x square so on and so forth.

And from these two Taylor series expansion of ui, j n plus 1 and u i, j n, if we subtract second from the first, we will get this expression from where we can write down the del u del t and we can see if del u del t now we can write as u i, j n plus 1 minus u i, j n by delta t, the highest term of the truncated series is a second order term, delta t square. So, eventually we have reached second order accuracy of discretization in the time direction.

So, alternate direction implicit scheme. It is not only second order accurate in space, but also second order accurate in time. So, you have learned through this explicit formulation, Implicit formulations, simply Crank-Nicolson implicit scheme and Alternate Direction Implicit scheme which is ADI with higher order accuracy in temporal direction as well.

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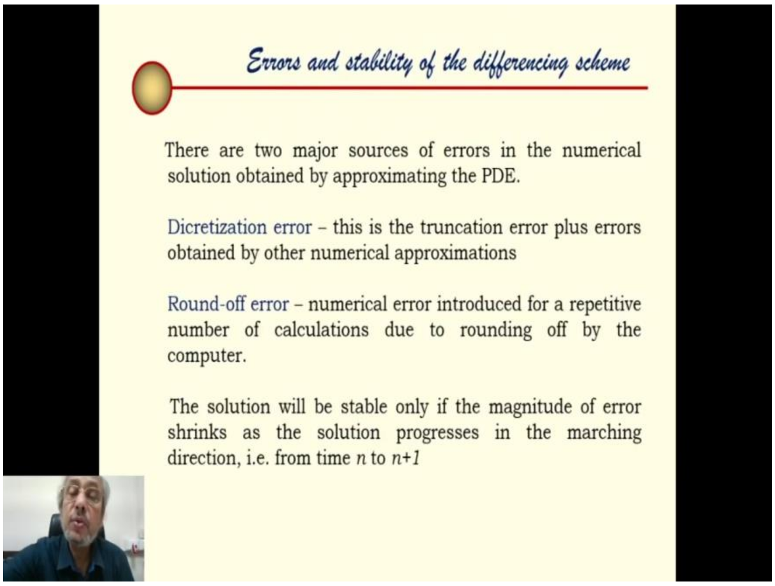

So, having done that, since so far we have been mentioning you know about the difficulty in explicit formulation, which is basically restriction of you know grid size in spatial and temporal direction. And this is some form of restriction otherwise explicit formulation will not work, whereas in implicit formulation this restriction in the temporal direction is relaxed that means, in the marching direction is relaxed.

So, now how can we say whether it is stable or it is unstable. We go for that analysis. There are two major sources of errors in the numerical solutions obtained by approximating the partial differential equation. Discretization error, this is the truncation error plus errors obtained by other numerical approximation. Obviously, numerical scheme.

Rounded off error, these are numerical error introduced for a repetitive number of calculations due to round off by the computer. The solution will be stable only if magnitude of error shrinks, as a solution progresses in the marching direction, that is from n to n plus 1. The errors should reduce or in the worst case it should stay at the same level, otherwise error will blow up in repetitive calculations.

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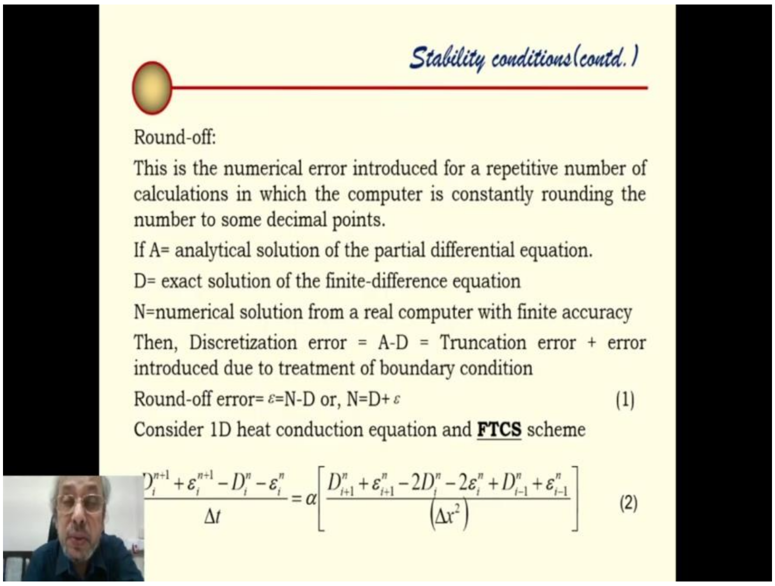

$$\frac{D_i^{n+1}+\varepsilon_i^{n+1}-D_i^n-\varepsilon_i^n}{\Delta t}=\alpha\left[\frac{D_{i+1}^n+\varepsilon_{i+1}^n-2D_i^n-2\varepsilon_i^n+D_{i-1}^n+\varepsilon_{i-1}^n}{\left(\Delta x^2\right)}\right]$$

Now, what is this round? This is a numerical error introduced for repetitive number of calculations in which computer is constantly rounding the number to some decimal points. All of us know that depending on the computer architecture sometimes it rounds, I mean rounds up at 32nd decimal place, sometimes it rounds up at 64th decimal place, but that is called double precision.

So, even if it rounds off at 64th decimal place. It will introduce some error. Now, in repetitive calculations with different quotients, this error will keep on progressing. and this error distribution can be such variation can be such that through repetitive calculations these errors will accumulate and finally this will really disturb the solution. So, how to -handle this situation?

Now, if A is analytical solution of the partial differential equation, D is the exact solution of the finite difference equation. That means, in the finite difference equation whatever differencing method we have adopted for that the solution that was to be obtained, if we get exactly that solution we call it D.

But as I said that a machine, a computer will not give me exactly D because if it say a number, a real number, then maybe after 8 decimal place 10 decimal place or 32nd decimal place, 64th decimal place, it will be rounded off depending on the computer. And that is what is called numerical error.

So, if A is a analytical solution and D is the hypothetical or ideal solution of the finite difference equation, A minus D is the truncation error and also error introduced due to treatment of boundary condition.

If boundary condition, if we introduce some approximation or some way you know maybe boundary condition can be Normal condition where derivative has to be defined, directly value of the dependent variable is not applied, so there we do some approximation.

So, due to all this some error can be introduced or maybe introduced there. So, that is the discretization error A minus D, truncation error plus error introduced due to treatment of boundary conditions. And what is round off error? This is N minus D, N is a actual number that machine gives and D is the number we are supposed to get from our finite difference quotient.

And this is valid not only for finite difference finite volume and finite element also this is valid in the same way, so this is from the solution of the discretized equation, and this is the number we are getting.

So, this difference is basically round off error. So, N, if that is the solution we are getting from computer we can say that that is the ideal solution we should get from partial differential equation plus some error. Even though it is very small it is there, and together we get when we add it up we get the number which computer gives to us.

Now, as I said through repetitive calculation this E may grow and now we will see how the there is a possibility of growth of this epsilon and you know if it grows at all then how to reduce that. So, this E or epsilon is basically the round off error. Now, again let us go to one dimensional heat conduction equation and go for FTCS scheme. If we go by FTCS scheme then we can write u i n plus 1 minus u i n by delta t.

Now, instead of u we will write down now the solution which will get from the computer if that is n basically that is the ideal solution plus the error. So, D i at n plus 1 plus epsilon i at n plus 1 these together is you know the solution which is we are getting substituting u i n plus 1, minus u i n, so that means, minus D i n minus epsilon i n by delta t.

Similarly, here on the right -hand side, the thermal diffusivity into we had u i plus 1 at n minus twice u i at n plus u i minus 1 at n. So, those are substituted by D i plus 1 at n plus epsilon i plus 1 at n minus twice D i at n minus twice epsilon i at n plus D i minus 1 at n plus epsilon i minus 1 at n and divided by delta x square. So, equation 2 is the equation which is basically prevalent if the solution is valid.

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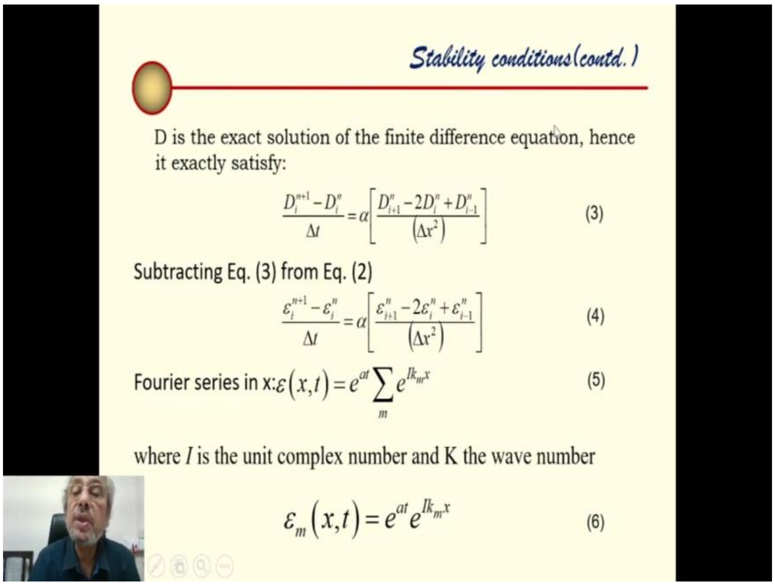

$$\frac{D_i^{n+1} - D_i^n}{\Delta t} = \alpha \left[\frac{D_{i+1}^n - 2D_i^n + D_{i-1}^n}{\left(\Delta x^2\right)} \right] \tag{3}$$

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \alpha \left[\frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{\left(\Delta x^2\right)} \right] \tag{4}$$

$$\text{Fourier in x:} \varepsilon\left(x,t\right) = e^{\alpha t} \sum_m e^{I k_m x} \tag{5}$$

where I is the unit complex number and m is the wave number

$$\varepsilon_m\left(x,t\right) = e^{at} e^{I k_m x} \tag{6}$$

Now, D is the exact solution of the finite difference equation. So, we can straightaway write this; that means, D i at n plus 1 minus D i at n divided by delta t equal to alpha into D i plus 1 minus twice D i plus D i minus 1 all at n-th level divided by delta x square.

So, equation 3 is a valid statement, equation 2 is also a valid statement. Now, if we subtract equation 3 from equation 2, we will get an expression in error, this is an equation involving all error quantities at n plus 1-th level, n-th level at the point of interest at the eastern point of point of interest, at the westward point of point of interest.

So, this gives you know basically the distribution of error and this distribution of error is not known to us. That is why what we will do in x and t domain this distribution of error we will take as a Fourier series in spatial domain and one exponential in the temporal domain it is expressed as e to the power at and then a Fourier series e to the power I kmx, I is a unit complex number and k is the wave number. So, we will just discuss little more about the error.

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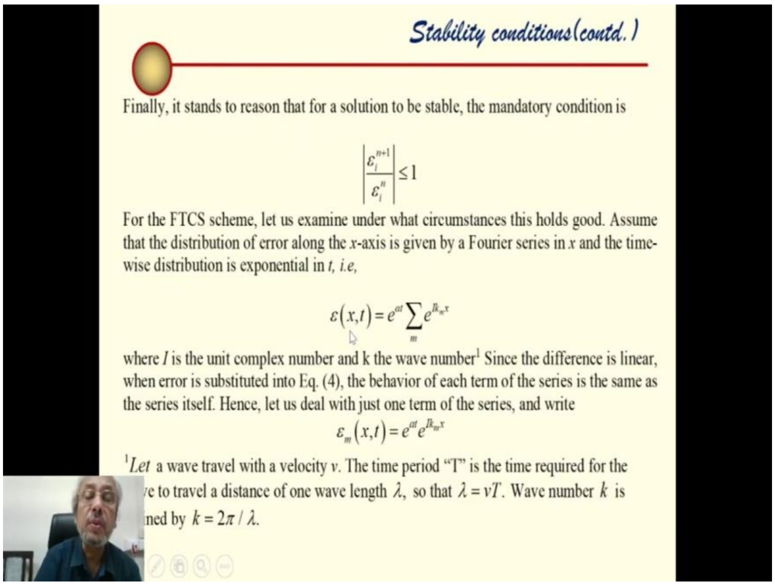

$$\left|\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n}\right| \le 1$$

$$\varepsilon(x,t) = e^{\alpha t}\sum_m e^{Ik_m x}$$

$$\varepsilon_m(x,t) = e^{at}e^{Ik_m x}$$

Now, at i-th location error at n plus 1-th level if that is epsilon i n plus 1 and at nth level if that is epsilon i at nth level, then this ratio modulus of this ratio should be less than 1.

I explained it earlier that error should reduce in each time step. Maybe at some intermediate time level it may stay at the same level, but never it should exceed in order the formulation to be stable.

For the FTCS scheme let us examine under what circumstances this holds good. Assume that the distribution of error along the x axis is given by a Fourier series in x and the time wise distribution is exponential in t. I mentioned about it just in the previous slide while discussing the previous slide. So, this error as I said this is a Fourier series in x, where k is the wave number, I is the unit complex number and e to the power a t that is is exponential in t.

And k is a wave number, I will just explain it conceptually. Let a wave all the errors error propagation are equivalent to wave propagation. Now, a wave let a wave travel with the velocity v. The time period; that means, now time period of the wave if it is T then we can say the time required for the wave to travel a distance one wavelength is T. So, wavelength is equal to wave velocity into T, very simple.

And then wave number is nothing, but twice pi by lambda, twice pi pi by wavelength. So, this k m in the Fourier series is representative of wave number; that means, the error propagation if it is simulated as wave propagation arbitrarily in x direction then k is the representative wave number. Since, the difference is linear when error is substituted in equation 4 equation, 4 means here. So, the behavior of each term in the series this Fourier series is same as the series itself.

Hence, so instead of taking the full Fourier series what we will do? We will just take one term and this is a good representation of the analysis. So, this error E m is function of x and t is e to the power a t in time direction it is time growth is exponential into e to the power Ik m x, k m is the wave number and upper case I is basically the unit complex number.

In many textbooks it is written as lowercase i, but lowercase i and j we are writing for each variable. So, not to create further confusion the unit complex number we have used capital I or uppercase I.

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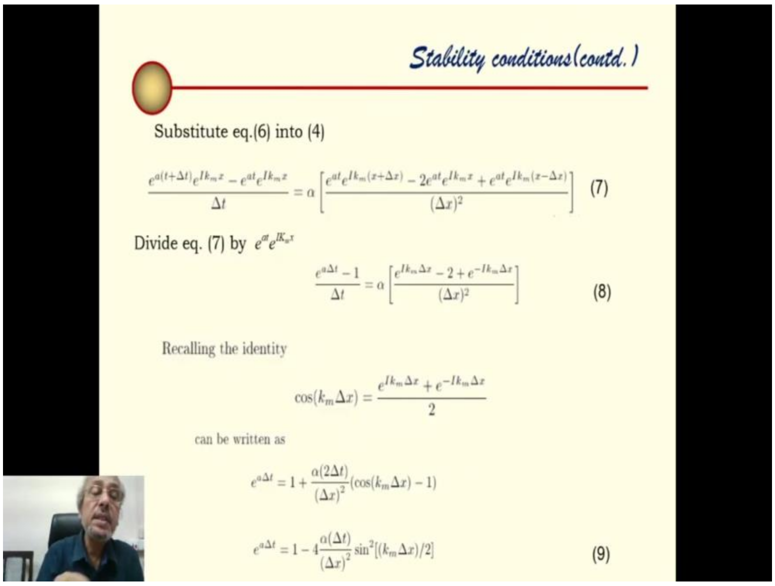

$$\frac{e^{a(t+\Delta t)}e^{Ik_m x} - e^{at}e^{Ik_m x}}{\Delta t} = \alpha \left[\frac{e^{at}e^{Ik_m(x+\Delta x)} - 2e^{at}e^{Ik_m x} + e^{at}e^{Ik_m(x-\Delta x)}}{(\Delta x)^2} \right] \qquad (7)$$

Divide eq.(7) by $e^{at}e^{Ik_m x}$

$$\frac{e^{at} - 1}{\Delta t} = \alpha \left[\frac{e^{Ik_m \Delta x} - 2 + e^{-Ik_m \Delta x}}{(\Delta x)^2} \right] \qquad (8)$$

Recalling the identity $\cos(k_m \Delta x) = \dfrac{e^{Ik_m \Delta x} + e^{-Ik_m x}}{2}$

Can be written as

$$e^{a\Delta t} = 1 + \frac{\alpha(2\Delta t)}{(\Delta x)^2}\left(\cos(k_m \Delta x) - 1\right)$$

$$e^{a\Delta t} = 1 - 4\frac{\alpha(\Delta t)}{(\Delta x)^2}\sin^2\left[(k_m \Delta x)/2\right]$$

if this is the representative of error term at m-th location e to the power a t e to the power Ik m x, then we can substitute equation 4 term by term by this function e m function of x and t and then we will get this.

You can do it just practice it yourself, u a t plus delta t, e to the power Ik m x minus e to the power a t, e to the power Ik m x divided by delta t, a is equal to alpha and again this is epsilon i plus 1; that means, e to the power a t, e to the power Ik m x plus delta x, this is twice epsilon x t.

So that means, e to the power a t to the power Ik m x and this is epsilon i minus 1. so e to the power a t, e to the power Ik m x minus delta x by delta x square. So, this expression 7 if we divide by e to the power a t, e to the power k m x, we will get this, e to the power a t minus 1 divided by delta t equal to alpha into e to the power Ik m delta x minus 2 plus e to the power minus Ik m delta x by x square.

Then, we can recall this identity that cos k m delta x, cos theta equal to e to the power theta plus e to the power minus theta by 2. So, if we invoke this identity here we will be able to write e to the power a t equal to this delta t. Now, go to the right -hand side one plus alpha twice delta t by delta x square and from here it is possible to write cos k m delta x minus 1.

So, again just look at this term. This cos theta minus 1 can be written as twice sin square theta by 2, and we have done that e to the power a t equal to 1 plus minus 4 alpha delta t by delta x square sin square k m delta x by 2. So, basically you know this is trigonometric identity cos theta minus 1 equal to minus twice sin square theta by 2.

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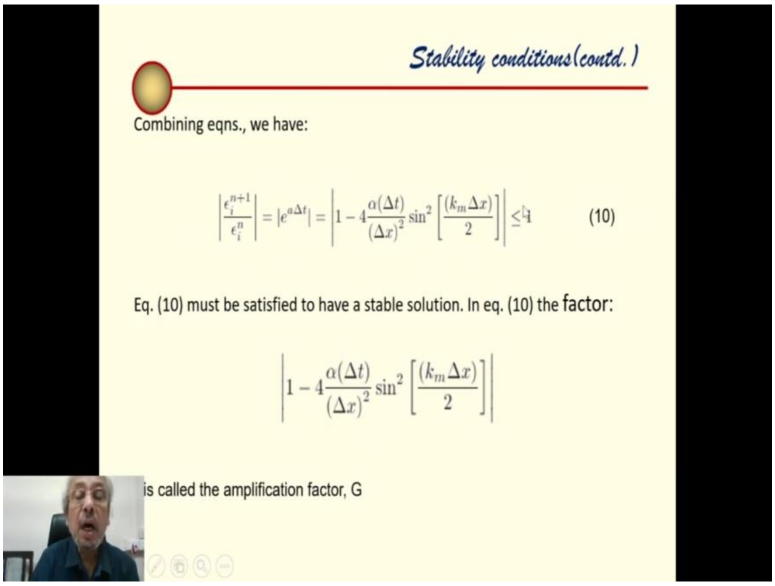

$$\left|\frac{\in_i^{n+1}}{\in_i^n}\right| = \left|e^{a\Delta t}\right| = \left|1 - 4\frac{\alpha(\Delta t)}{(\Delta x)^2}\sin^2\left[\frac{(k_m \Delta x)}{2}\right]\right| \le 1 \qquad (10)$$

$$\left|1 - 4\frac{\alpha(\Delta t)}{(\Delta x)^2}\sin^2\left[\frac{(k_m \Delta x)}{2}\right]\right|$$

Now, from there we can write epsilon i n plus 1 divided by epsilon i at n-th level. And this is the error quantity. Modulus of this should be less than 1, so this is nothing but e to the power a t and modulus of e to the power a t is basically then, if we look at here modulus of e to the power a t is basically modulus of 1 minus this sorry, 1 minus this whole quantity; modulus of e to the power a t is modulus of this entire quantity on the right--hand side of equation 9.

And this should be less than equal to 1. This is the requirement. Now, this can be satisfied and if this is satisfied then our calculations are stable. Error at no condition will grow. And this part is called amplification factor; that means, amplification of error, which is modulus of this quantity minus 4 alpha delta t by delta x square into sin square k m delta x by 2.

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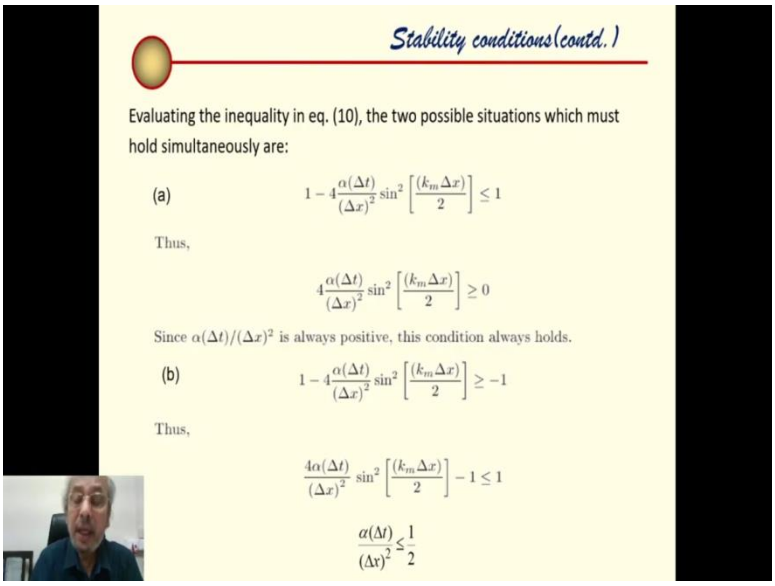

(a) $$\left|1-4\frac{\alpha(\Delta t)}{(\Delta x)^2}\sin^2\left[\frac{(k_m\Delta x)}{2}\right]\right|\leq 1$$

Thus, $$4\frac{\alpha(\Delta t)}{(\Delta x)^2}\sin^2\left[\frac{(k_m\Delta x)}{2}\right]\geq 0$$

Since $\frac{\alpha(\Delta t)}{(\Delta x)^2}$ is always positive, this condition always holds.

(b) $$1-4\frac{\alpha(\Delta t)}{(\Delta x)^2}\sin^2\left[\frac{(k_m\Delta x)}{2}\right]\geq -1$$

Thus, $$4\frac{\alpha(\Delta t)}{(\Delta x)^2}\sin^2\left[\frac{(k_m\Delta x)}{2}\right]-1\leq 1$$

$$\frac{\alpha(\Delta t)}{(\Delta x)^2}\leq\frac{1}{2}$$

Now, this can be done in two different ways. a, we can straight away say if the modulus has to be less than 1, then possibility is there are two possibilities; one is 1 minus 4 alpha delta t by delta x square into sin square k m delta x by 2 this quantity is less than equal to 1. And this is very straightforward.

In order to obey this or in order to satisfy this what has to be done or what has to be satisfied is 4 alpha delta t by delta x square into sin square k m delta x by 2 must be greater than equal to 0. So, we can see that alpha is the diffusivity, it cannot be negative alpha t. Delta t is a time step it is also not negative delta x square. So, this quantity is always positive. And this condition always holds good.

b, is another possibility because our condition was modulus of this has to be less than 1. So, another possibility is 1 minus 4 alpha delta t by delta x square sin square k m delta x by 2 is greater than equal to minus 1.

Now, if that has to be satisfied we can do simple algebra here, 4 alpha delta t by delta x square sin square k m delta x by 2 minus 1, just change the sign, is less than equal to 1. From here the as value of sin theta is 1, so we can see that the restrictive condition is alpha delta t by delta x square is less than equal to half.

So, as I said that in order this amplification factor which is modulus of this quantity given by equation 10, we have two possibilities a and b, combining these two we can conclude that alpha delta t by delta x square has to be less than half.

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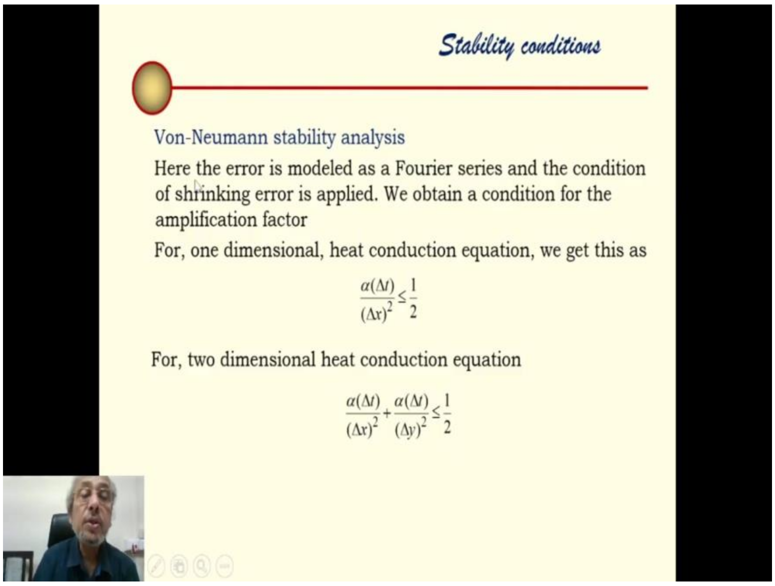

$$\frac{\alpha(\Delta t)}{(\Delta x)^2} \leq \frac{1}{2}$$

For, two dimensional heat conduction equation

$$\frac{\alpha(\Delta t)}{(\Delta x)^2}+\frac{\alpha(\Delta t)}{(\Delta y)^2}\leq\frac{1}{2}$$

So, Von-Neumann stability analysis says this the error is modeled as a Fourier series and the condition of shrinking error is applied. We obtain a condition for amplification factor.

For, one dimensional, heat conduction equation we get this as alpha delta t by delta x square less than equal to half. This, for Neumann stability analysis can easily be extended for two dimensional heat conduction equation. You can do it as an exercise. It is very simple exercise.

I will also give indication how to do it in other examples when we will be discussing further, but even now you can do it very easily. If you do it you will see that the restriction for two dimensional heat conduction equation will be alpha delta t by delta x square, alpha delta t by delta y square is less than equal to half. So, these are the restrictions.

As such I have not mentioned it here, again I will do it little later that alpha delta t by delta x square is called grid Fourier number. So, grid Fourier number in x direction and grid Fourier number in y direction together less than equal to half.

And you can see if delta x equal to delta y grid size is same in 2D it is less than equal to one-forth. However, in 1D alpha delta t by delta x square is less than equal to half that we have derived here itself and all of us know about it how to go, how to do, or how to go about this problem.

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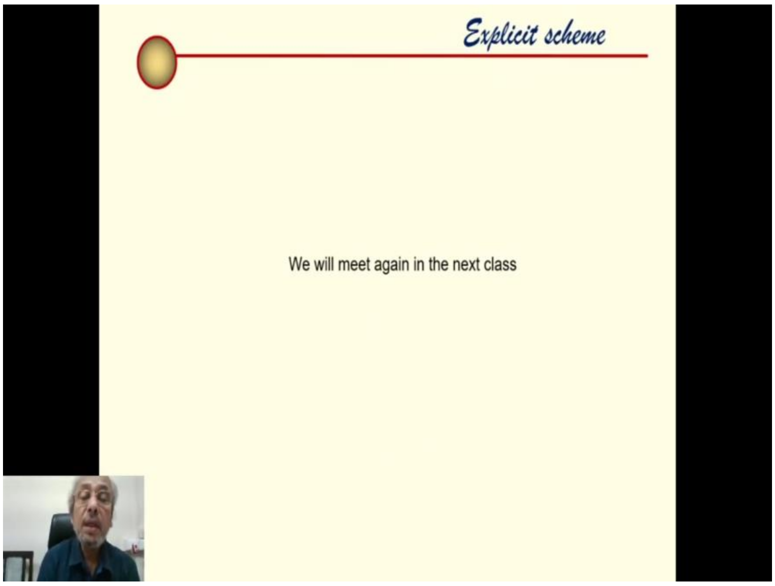

So, with this we will finish our lecture today. And we will meet again in the next class for moving forward taking up issues with more complexity and issue more involved issue issues.

Thank you very much.