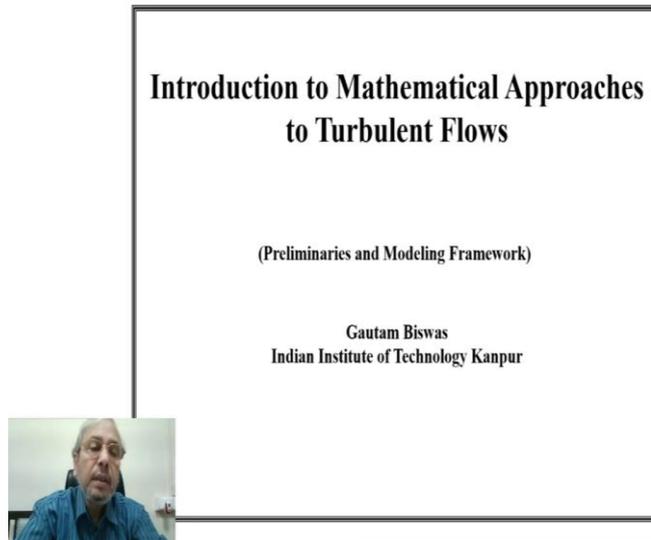**Computational Fluid Dynamics and Heat Transfer**
Prof. Gautam Biswas
Department of Mechanical Engineering
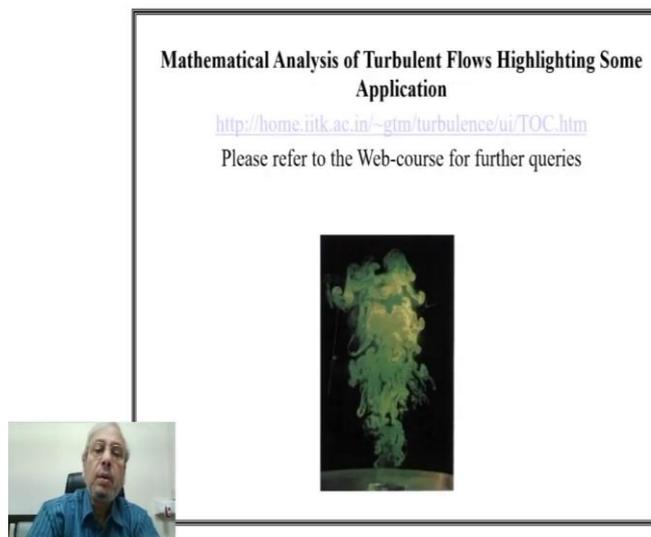**Indian Institute of Technology, Kanpur**

Lecture - 27
Mathematical Approaches to Turbulent Flows-1

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Good morning everybody. Today we will start our discussion on Introduction to Mathematical Approaches to Turbulent Flows.

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Now, I have a full course on turbulent flows and I have given URL of that course; it is also available on NPTEL website; and NPTEL site of IIT Kanpur, NPTEL site of IIT Madras personal homepage.

This is a full course as you know that turbulence is so, vast that you know its salient features cannot be covered in you know one or two lectures; rather I would say theoretical foundation of turbulence that needs one full course. Mathematical approaches modeling approaches that you know usually combine all the aspects and that need one full course.

Obviously, experimental approaches and interpretation of results that need another full course. So, what we are trying to do here, trying to give you know some aspects and how to model turbulence flow in affordable computational initiatives or affordable computational framework.

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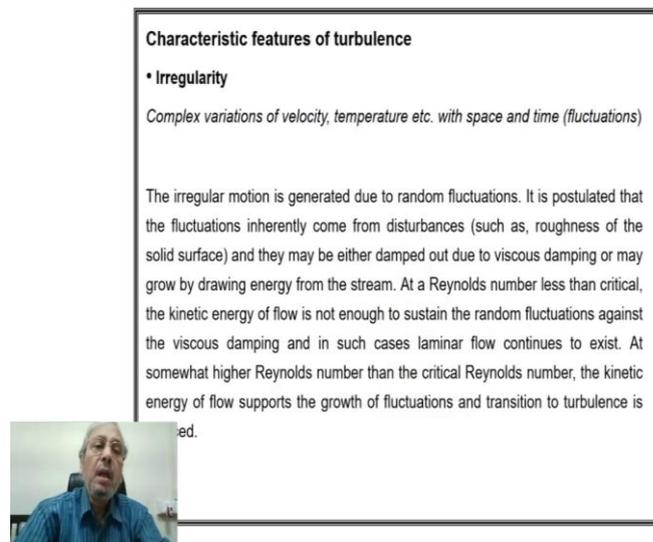
Characteristic features of turbulence

• Irregularity

Complex variations of velocity, temperature etc. with space and time (fluctuations)

The irregular motion is generated due to random fluctuations. It is postulated that the fluctuations inherently come from disturbances (such as, roughness of the solid surface) and they may be either damped out due to viscous damping or may grow by drawing energy from the stream. At a Reynolds number less than critical, the kinetic energy of flow is not enough to sustain the random fluctuations against the viscous damping and in such cases laminar flow continues to exist. At somewhat higher Reynolds number than the critical Reynolds number, the kinetic energy of flow supports the growth of fluctuations and transition to turbulence is ...ed.

Now, characteristics features of turbulence. First feature I would like to mention is irregularity; that is complex variations of velocity temperature with space and time and these are often called fluctuations. The irregular motion is generated due to random fluctuations; it is postulated that the fluctuations inherently come from disturbances such as roughness of the solid surface.

And they may be either damped out due to viscous damping or may grow by drawing energy from the stream. At a Reynolds number less than critical the kinetic energy of flow is not enough to sustain the random fluctuations against the viscous damping and in such cases laminar flow continues to exist.

At somewhat higher Reynolds number than the critical Reynolds number the kinetic energy of flow supports the growth of fluctuations and transition to turbulence is induced. So, you get a sort of rudimentary idea about some characteristics features of turbulence.

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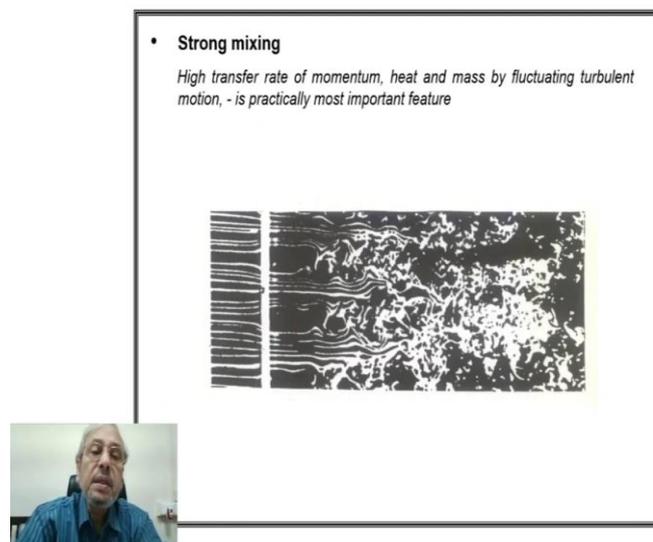
- **Strong mixing**

  High transfer rate of momentum, heat and mass by fluctuating turbulent motion, - is practically most important feature
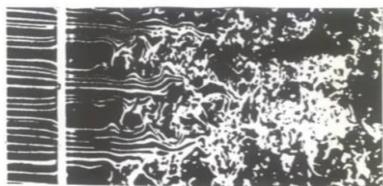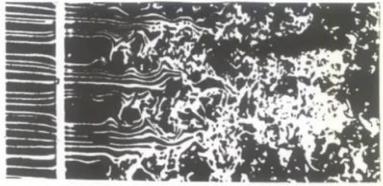
Strong mixing; so, basically high transfer rate of momentum heat and mass by fluctuating turbulent motion is practically most important feature of turbulence.

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*Turbulent motion always 3D

For a parallel flow it can be written that the axial velocity component is

$$u(y,t) = \bar{u}(y) + u'(\Gamma, t)$$

y is the normal direction and r is the any space variable

Even if the bulk motion is parallel, the fluctuation u' being random varies in all direction. Now let us look at the continuity equation

$$\frac{\hat{\partial}\bar{u}}{\hat{\partial}x} + \frac{\partial u'}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Since $\dfrac{\partial u'}{\partial x} \neq 0$ the above equation depicts that y and z components of velocity exists for the parallel flow if the flow is turbulent. We can write

$$\left.\begin{array}{lcl} u(y,t) = \bar{u}(y) + u'(\Gamma, t) \\ v = 0 \quad + \quad v'(y,t) \\ w = 0 \quad + \quad w'(y,t) \end{array}\right]$$

Turbulent motion is always 3D even for parallel flow, we have rate parallel flows you know like plane Poiseuille flow, Eigen Poiseuille flow. Some such flow where u the flow direction velocity is function of only cross normal direction and time there for turbulent

flow we can say it is a mean flow plus fluctuation. So, y is the normal direction and capital gamma is a space variable even if the bulk motion is parallel the fluctuation u prime being random varies in all directions.

Now, let us look at the continuity equation; where del u del x u is substituted by u mean plus u fluctuation as we have written here u equal to u mean plus u fluctuation this is u instantaneous. So, we have plugged in that and then we have gotten this equation, but in this equation since u prime is present. So, del u prime del x is not equal to 0, which demands that you know there is some v prime and there is some w prime.

Even though mean velocity of v is 0 mean velocity of w is 0, but there is time mean velocity is 0, time mean velocity of v is 0, but at any instant there is fluctuation of v and w since fluctuation in u is present. And we will see later separately continuity equation has to be satisfied or u is satisfied with fluctuating components; that means, del u prime del x plus del v prime del y plus del w prime del z equal to 0; that is why for non-zero value of del u prime del x del v prime del y and del w prime del z will exist.
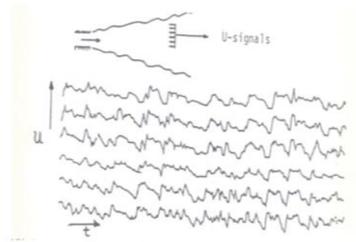
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*Turbulent motion carries vorticity- is composed of eddies interacting with each other
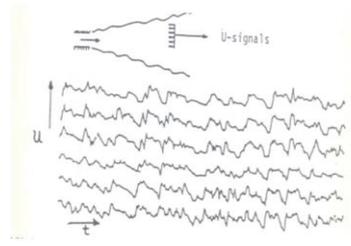
*wide spectrum of eddy sizes and corresponding fluctuation frequencies

Turbulent motion carries vorticity it is composed of eddies interacting with each other. So, this is a turbulent jet and you can see the velocities at different points are being sensed and that signal is reproduced here or that signal is traced here. So, this is basically you can see that at different positions the u velocity together with its fluctuations how does it vary?

This is variation of u at maybe this location then u at this location and then u at this location how does it vary with time? And at any instant of time, if you try to see you will not be able to find any coherence between this instantaneous use.

So, every instant it is changing although there is a possibility to having a mean value which at every point probably will time mean value will be same. We will come to this aspect again. So, we can say from here wide spectrum of eddy sizes and corresponding fluctuation frequencies are present in a turbulent flow.

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Also one can say the motion is not organized it is chaotic and it has random behavior. So, random behavior means, non repeatability although a finer analysis of turbulence by sophisticated numerical and experimental tools one can show presence of a coherent component in seemingly random turbulent motion.

This is again you know a very special topic and to be discussed separately. Many of you may be knowing the name coherent structure in turbulent flows; this was a discovery of a famous scientist professor Fazley Hussein and you know this is even today a topic of research.

Enhance diffusion and dissipation, three dimensionality and vertical structures I have already mentioned about it and intermittency in space and time these are also random fluctuations in space and time. So, these are you know characteristics feature and from these whatever we have just discussed till now it appears that it is impossible to apply any systematic computational tool to analyze turbulence.

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WHY DO WE NEED TURBULENCE MODELS-1

Is direct simulation of turbulence (DNS) possible?

*Exact equation are known:

$$\frac{\partial U_i}{\partial x_i} = 0$$

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_i \partial x_j}$$

*They can be solved numerically in principle

*Flow field must be discretized

*Main problem at high Re, turbulence consists of wide spectrum of eddy sizes

But scientists have put in effort it is not so, it can be done. First I will mention about a technique which is called direct numerical simulation or DNS. The exact equations of any flow situation are basically continuity equation and Navier Stokes equations in three orthogonal directions. So, four equations and four unknowns U v w and P. Ideally this can be solved without any problem even for turbulent flows and that is what is ideally direct numerical simulation.

But what one would need number one discretization scheme should be absolutely error free, errors you know like truncation error diffusion error, artificial viscosity dispersion error all these errors should be tending to 0 almost 0. So, these are special techniques discretization techniques. Then also for we know the turbulent fluctuations can vary or turbulent motion can vary over a wide scale.

Its variation fluctuations spatially in a domain may be you know in the scale of meter in the flow direction in the scale of centimeters, in the cross flow direction. But, there the minimum level of fluctuations that mean given by time and length scale these are expressed by Kolmogorov scale can be fraction of a millimeter.

So, you can see in order to simulate a domain directly without applying for high Reynolds number flows; any turbulence model you need a grid size which is able to capture the smallest scales of fluctuations.

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* $N_{max} \approx 100$ for computers used these days for numerical solutions. This results in $N_{total} \approx 10^6$ for 3- dimensional flows.

*This values yield: $Re_{max} \leq 20000$. Only for Reynold numbers around this range, direct numerical simulation are possible.

*For higher Reynolds number direct numerical simulation (DNS) of the stokes equation is difficult.

$$\frac{\Delta u \Delta L}{\nu} \leq 2; \frac{U_0}{N} \frac{L}{N} \frac{1}{\nu} \leq 2; \frac{U_0 L}{\nu} \leq 2N^2; \mathrm{Re} \leq 20000; \text{ for } \left(100 \times 100 \times 100\right) \text{ domain}$$

Now, I will give you a very rudimentary example this is just to give you an idea, I mean this is not exactly the a reason of you know not going for direct numerical simulation always. But, why direct numerical simulation is difficult this is one of the reasons I have tried to explain see maybe a quite flow type of flow we are discussing.

So, this is the bottom plate this is the top plate top plate velocity is U naught; obviously, bottom plate velocity is 0. And this is the slope of velocity so, velocity is increasing and the domain is divided by number of grids.

So, velocity is increasing delta U is the velocity increase from you know in the normal direction across normal direction perpendicular direction. And this is the grid size delta x so, delta x is the grid size delta U is the velocity change and N is a number of grid points per coordinate direction.

Usually most of the affordable computers, we can take for some flow simulation if it is three dimension it is considered to be a very powerful computer. If we are able to take 100 by 100 by 100 grids so; that means, 1 million grid points.

Now, one can easily show that if you take 1 million grid points what would be the upper limit of the Reynolds number? Here we have done sort of you know very rudimentary calculation del u into del L if this direction is l small del x or del L whatever you call it del L by nu we know the computational state stable computational requirement is should be less than equal to 2. So, del u we can say is U naught by N and del L or del x is basically L by N multiplied by 1 by nu viscosity less than equal to 2.

So, this makes U naught into L by nu is less than equal to 2 N square and N is 1000. So, basically Reynolds number less than equal to 20000. So, if we take 100 by 100 by 100 grid our Reynolds number based on characteristics dimension has to be less than 20000.

So, there is a limitation for which direct numerical simulation for all the flows is not always possible. And you will need very fine grid size you will need very highly accurate discretization scheme and also in the time direction, you must be having huge data every small time interval to finally, draw a conclusion about the flow field.

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WHY DO WE NEED TURBULENCE MODEL-2

*Prospects of large eddy simulation
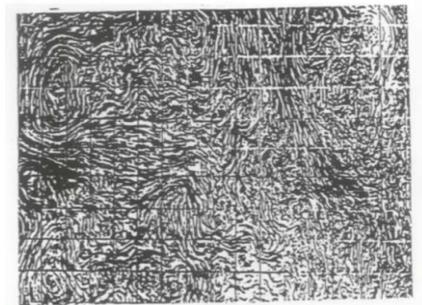
Now, but direct numerical simulation is a very vibrant area of research in turbulence. Then I will mention about another very exciting area of research in turbulence is a large eddy simulation. Here the grids can be larger than the direct numerical simulations. So, the fluctuations that are scale wise falling below the spatial scale defined by the grid size those fluctuations can be handled by a model.

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- Effect of eddies smaller than mesh size simulated by subgrid-scale model
- Can be applied to high-*Re* flows

  especially when viscous sublayer not resolved but bridged by wall functions
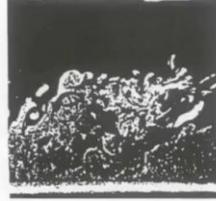
...ll be method for practical calculations in 5 years.

...ge computing times to get statistics (~100000 time steps/ ...les of instantaneous flow fields)

*Effects of eddies smaller than mesh size simulated by sub grid-scale model

 *Can be applied to higher Re flows

Especially when viscous sublayer not resolved but bridged by wall functions
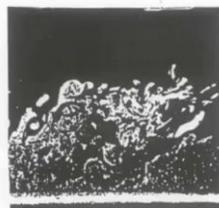
Be method for calculations in 5 years.

 Computing times to get statistics (-100000 time steps/tables of instantaneous flow fields)

And these are called sub grid scale models. And one can go for high and reasonably high Reynolds number flows, but there is yet another difficulty when viscous sub layer is not resolved; because as you know turbulent fluctuations as one approaches the confining walls. The because of the no slip boundary condition at the wall and the gradual damping of the fluctuating components very close to the wall we call it viscous sub layer where flow is like laminar flow.

And since there is no fluctuation, so describing velocity and other turbulent flow related parameters within the viscous sub layer; I mean those are to be very accurately determined or one can also apply something called wall function we will discuss about it little later. So, that sub layer and the turbulent regime can be breached through some functions which are called wall functions.

So, with the help of wall functions, one can go for even slightly higher Reynolds numbers, but even then one needs reasonably high number of grids. So, grid resolution or the fineness of grid the requirement is not as much as direct numerical simulation.

But here also grid size has to be quite small and usually for laminar flows when you go for grid independence test and all you know by reducing the grid size increasing the grid numbers, for large eddy simulation techniques definitely we need much larger grids than those you know type of grid requirements.

It is not as high as I would say I have already mentioned it is not as high as the requirement of DNS, but here also grid requirement of number of grids is quite high. So, with all these for usually flows of engineering importance where we want to calculate pressure drop or screen friction or if it is heat transfer.

Finally, the average Nusselt number or average heat transfer the for those requirements we often cannot use LES or DNS. Maybe LES is being implemented for all complex flows including combustion where you know uncertainties are quite significant.

But you know even now it is not really used for all engineering flow calculations for very special purpose flows it is used. Maybe in another 5 years it will be a method of calculating engineering flows. And here also in order to derive the correct statistics one required to you know progress in time and maybe I have mentioned it here that you can see.

That 100000 you know time steps may be needed instantaneous flow fields may be needed we need not store in all 100000, but there are ways of you know calculating from such instantaneous flow fields the time average flow then flow field etcetera. So, large number of time steps or time instance are needed to derive the correct statistics.
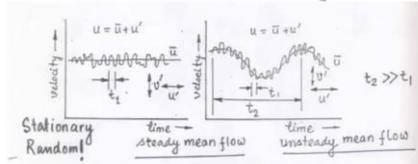
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WHY DO WE NEED TURBULENCE MODEL -3

*Statistical calculations methods

*Details of turbulent fluctuation usually not of interest to engineer for design

*Hence statistical approach is taken and turbulence is averaged out
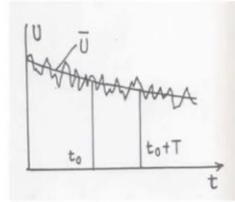
*Statistical quantities:

$$U_i = \bar{U}_i + u_i', P = \bar{P} + p', \bar{U} = \frac{1}{T} \int_{t_0}^{t_0+T} U dt$$

So, arising out of you know these difficulties although DNS and LES both are very important and very elegant scientifically very elegant methods, but for all engineering flow calculations these are not really used in practice. So, third method what we have suggested here is statistical calculation methods.

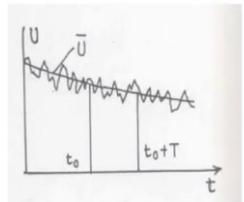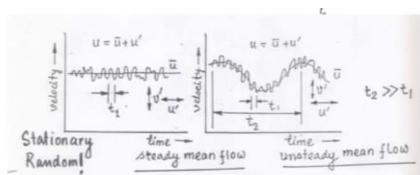Now details of turbulent fluctuations usually not of interest to engineers for design purpose hence statistical approach is taken and turbulence is averaged out. So, we have shown three sample signals here you can see how the velocity is varying and if we average it out this is the U average signal.

Here also we can see instantaneous velocities and if we average out we have found out u average it is not a function of time at all. Here also again instantaneous signals versus time we can see, but if we average out we can see the average velocity is a function of time. Here the time average velocity is not a function of time such flows are called stationary random.

It is random, but time average value is constant stationary here the random fluctuations are seen, but time average value is again function of time, but this time period is quite large and much greater than the time period of turbulent fluctuations. So, turbulent fluctuations time period is so small that frequencies may be as high as several kilohertz.

Time period is small it is 1 by time period is the frequency and frequency is of the order of maybe several thousand hertz. Here time period is large and the frequency may be several hertz much smaller frequency than the turbulent frequencies. So, time period is much bigger much larger than the time period of turbulent fluctuations. So, one can average out the signal over this time period. And here if we take a large time and average it out we will get average quantities.

So, instantaneous U i; that means, it is a vector quantity representing U v w equal to U i bar plus u i prime pressure is P bar plus p prime. Now average velocity is over a time period this was this is what I was saying here a large time period can be taken. Here also you can take a large time period, but at least the time period should be greater than t 2. So, and over during this time window if we perform the integral divided by T we will get the average velocity.
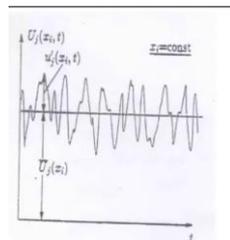
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Turbulent flows can only be '**mentally digested** 'if time averaged properties are given. These are denoted with a cross bar over the variable $\overline{U}_j$ = time averaged velocity

$u_i^{'}$ = turbulent velocity fluctuation in the j- direction

The following decomposition is valid: $\hat{U}_j\left(x_i,t\right)=\overline{U}_j\left(x_i\right)+u_i^{'}\left(x_i,t\right)$

The averaged values is: $\overline{U}_j=\lim_{T\to\infty}\dfrac{1}{T}\int_0^{tT}\hat{U}_j\left(x_i,t\right)dt$

The fluctuations: $\overline{u_j^{'}}=\lim_{T\to\infty}\dfrac{1}{T}\int_0^{tT}u_j^{'}\left(x_i,t\right)dt=0$

Now, here again we have represented this is the average velocity these are all instantaneous values. So, U j bar is the time average velocity and u j prime is turbulent velocity fluctuation in j direction. Turbulent flows can only be mentally digested; if time average properties are given. These are denoted with a cross bar over the variable and we have defined U j bar and u j prime.

The following decomposition is valid U j instantaneous function of space and time equal to U j average function of space plus u j prime function of space and time. The averaged values then U j bar is limit T tends to infinity 1 by T time period 0 to T is the time window instantaneous U d t.

This integral will be divided by 1 by T and for fluctuations u j prime over bar which is basically again integral of u j prime d t divided by 1 by T this is 0. Every instant it is there, but when it is averaged over the time window it is 0.

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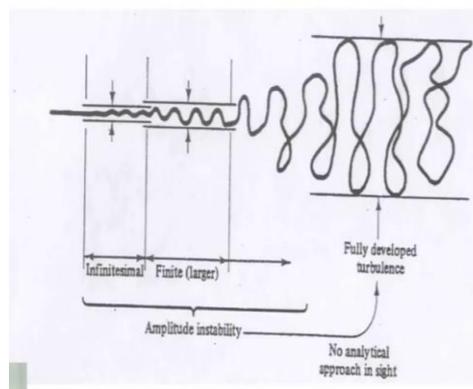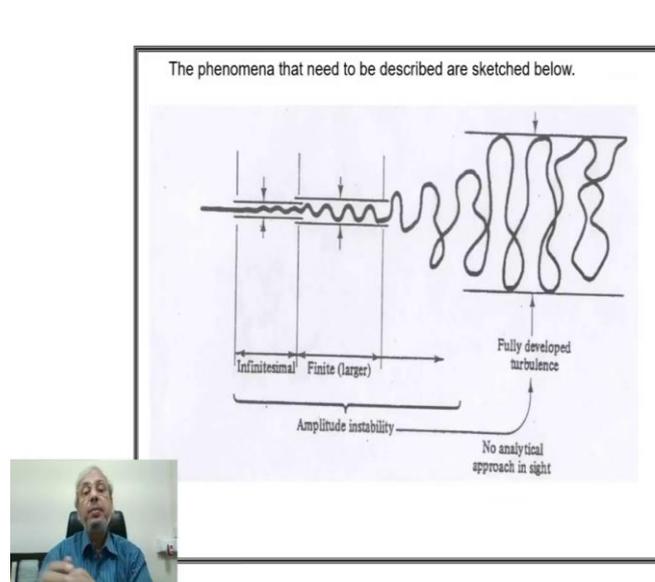
Now, the phenomena that need to be described are sketched below; in a flow field as I am I have already mentioned about it that, in some flows disturbances come from somewhere maybe some external vibration, may be roughness of surfaces and then it grows.

So, one can see infinitesimal small fluctuations, those are growing and now you can see fully developed turbulence. So, this how does it grow? This is very difficult to analyze as I was mentioning that in order to capture this transition very powerful techniques like you know direct numerical stimulation should be used.

But, for the engineering purpose when turbulent flow may be called as fully developed turbulent flow or fully developed turbulence is the state, then we can apply the statistical models that we have started discussing.

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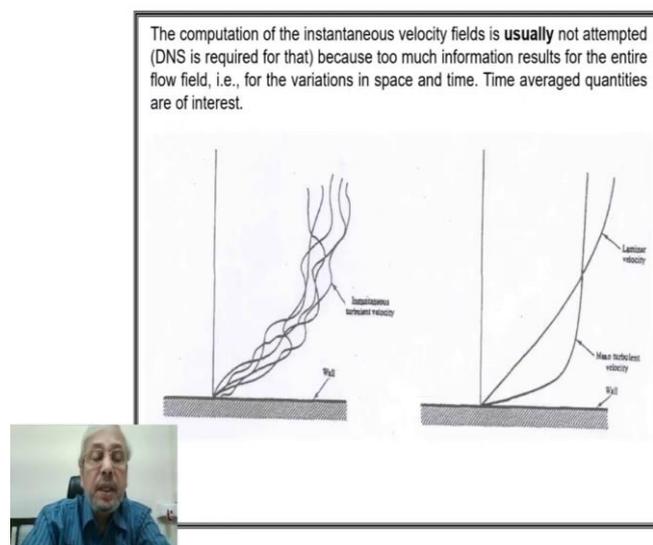
The computation of the instantaneous velocity fields is **usually** not attempted (DNS is required for that) because too much information results for the entire flow field, i.e., for the variations in space and time. Time averaged quantities are of interest.

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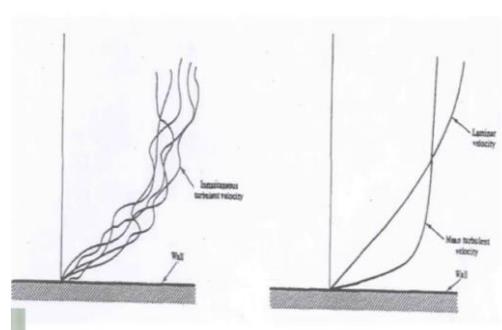
Here I will give you another example. Maybe if you look at a velocity profile close to the wall maybe this is a channel flow close to the bottom wall, maybe flow over a flat plate

again close to the bottom wall. The computation of instantaneous velocity fields usually not attempted again I have repeated DNS is required for that; because too much information results for the entire flow field that is for the variations in space and time.

So, this at every instant it is varying. So, if you take maybe 10000 you know such instantaneous value and average it then only you will get correct average. Usually as again let me repeat for engineering calculation like we know for laminar flow this is the variation similarly for turbulent flow some log law variation we know and we will get some such variation.

So, if we apply DNS we have we will get you know all instantaneous profiles from there we have to get the time average profile. But we would like to go for a technique we are aiming to sort of you know apply some technique. So, that directly we can get this log law profile obeying the basic physics involved in creating such profile.

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The continuity equation for ρ = constant is:

$$\frac{\partial U_i}{\partial x_i} = 0 \rightarrow \frac{\partial}{\partial x_i}(U_i + u_i') = 0 \tag{1}$$

The time average of this equation yields:

$$\lim_{T \to \infty} \int_0^T \frac{\partial}{\partial x_i}(U_i + u_i')dt = 0 \tag{2}$$

Or rewritten:

$$\frac{\partial}{\partial x_i} \lim_{T \to \infty} \frac{1}{T} \int_0^T (U_i + u_i')dt = \frac{\partial \overline{U}_i}{\partial x_i} = 0 \tag{3}$$

Hence, we can also deduce:

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{4}$$

...very point in time the turbulent fluctuations have fulfilled equation (4) ...ctly.

The continuity equation for ρ = constant is:

$$\frac{\partial U_i}{\partial x_i} = 0 \rightarrow \frac{\partial}{\partial x_i}\left(\overline{U}_i + u_i'\right) = 0 \tag{1}$$

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Or rewritten:

$$\frac{\partial}{\partial x_i} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left(\overline{U}_i + u_i^{'} \right) dt = \frac{\partial \overline{U}_i}{\partial x_i} = 0 \tag{3}$$

Hence we can also deduce:

$$\frac{\partial u_i^{'}}{\partial x_i} = 0 \tag{4}$$

Delivery point in time the turbulent fluctuation have fulfilled equation (4) directly.

So, the continuity equation for constant density flows we can write del U i del x i equal to 0 and this U i is instantaneous. So, this is del del x i of U i bar plus u i prime time average plus instantaneous equal to 0.

The time average of this equation yields that you know del del x i of these over a time window is should be equal to 0. And if we write it in this way time T tends to infinity 1 by T this time window the integration over this time U i bar plus u i prime then we will get; obviously, the continuity equation satisfied by time mean quantities.

And if that is true then again substituting in this we will get that the continuity equation is satisfied by the fluctuating components also. So, if we say u v w three directions U del U del x del U bar by del x del v bar by del y plus del w bar by del z equal to 0.

Here del u prime by del x plus del v prime by del y plus del w prime by del z equal to 0. So, continuity equation is satisfied by the time average quantities continuity equation is also satisfied by the instantaneous fluctuating velocity components at every point in time the turbulent fluctuations have fulfilled equation four exactly.

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Instantaneous values:

$$\hat{A} + \overline{A} + a' \quad \& \quad \hat{B} = \overline{B} + b'$$

Averaging yields per definition:

$$\lim_{T \to \infty} \int_0^T \hat{A} dt = \overline{A} \quad \& \quad \lim_{T \to \infty} \int_0^T \hat{B} dt = \overline{B}$$

Or

$$\overline{\overline{A}} = \overline{A} \quad \& \quad \overline{\hat{B}} = \overline{B}$$

This yields:
$$\overline{a'} = 0 \quad \& \quad \overline{b'} = 0$$

In connection with averaging over products:

$$\overline{\hat{A}\hat{B}} = \overline{\left(\overline{A} + a'\right)\left(\overline{B} + b'\right)} = \overline{\overline{A}\,\overline{B}} + \overline{a'\,\overline{B}} + \overline{b'\,\overline{A}} + \overline{a'b'}$$

$$\overline{a'\,\overline{B}} = \overline{a'}\,\overline{B} = 0 \quad \& \quad \overline{b'\,\overline{A}} = \overline{b'}\,\overline{A} = 0$$

$$\overline{\hat{A}\hat{B}} = \overline{\overline{A}\,\overline{B}} + \overline{a'b'} = \overline{A}\,\overline{B} + \overline{a'b'}$$

Now, let us discuss some laws of averaging. Let A be some instant A hat be some instantaneous quantity which can be expressed as A bar plus small a prime. And B hat is

B bar plus small b prime this is average this is fluctuation, this is average this is fluctuation. Averaging yields per definition 1 by T 0 to capital T A hat d t equal to A bar.

Similarly 1 by T integration 0 to T B hat d t is B bar so; that means, A hat bar is A bar B hat bar is B bar; which means a prime bar is 0 and b prime bar is 0. So, time average of fluctuations is 0 time average of b fluctuations is also 0; then if we take two quantities A hat and B hat.

And perform the averaging of that, we can write A bar plus a prime B bar plus b prime over bar equal to A bar B bar over bar plus a prime B bar over bar plus b prime A bar over bar plus a prime b prime over bar.

Now a prime B bar over bar and b prime A bar over bar these quantities are 0. And so, because you know we have already mentioned about a bar over bar is 0 b bar over bar is 0. So, that will make these two quantities 0. So, A hat B hat over bar we can write as A bar B bar over bar plus a prime b prime over bar which will give us A bar into B bar plus a prime b prime over bar. So, a prime over bar is 0 b prime over bar is 0, but a prime into b prime over bar is non zero ok.

So, and here we have a prime B bar over bar is a prime over bar into B bar and a prime over bar 0 so, this quantity is 0. Similarly b prime A bar over bar this operation will give us b prime bar into A bar b prime bar is 0 so, this is 0. So, that is how this quantity will be 0; this quantity will be 0, but this quantity is non zero and this quantity will produce A bar into B bar. So, this is what is usually called also Reynolds decomposition.

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The momentum equation reads:

$$\rho\left[\frac{\partial \hat{U}_j}{\partial t} + \hat{U}_i \frac{\partial \hat{U}_j}{\partial x_i}\right] = -\frac{\partial \hat{P}}{\partial x_j} - \frac{\partial}{\partial x_i}\hat{\tau}_{ij} + \rho g_j$$

If one adds the continuity equation multiplied with $\hat{U}_j$

$$\hat{U}_j\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho \hat{U}_i\right)\right] = 0$$

One obtains:

$$\frac{\partial\left(\rho \hat{U}_j\right)}{\partial t} + \frac{\partial\left(\rho \hat{U}_i \hat{U}_j\right)}{\partial x_i} = -\frac{\partial \hat{P}}{\partial x_i} - \frac{\partial}{\partial x_i}\hat{\tau}_{ij} + \rho g_j$$

Again $\hat{U}_j = \overline{U}_j + u'_j$ and $\hat{U}_i = \overline{U}_i + u'_i$ introduced into equation yields:

$$\overbrace{\frac{\partial}{\partial x_i}\left(\rho \hat{U}_i \hat{U}_j\right)}^{\text{total momentum transport term}} + \overline{\tau}_{ij} + \rho \overline{u'_i u'_j} = -\frac{\partial \overline{P}}{\partial x_j} + \rho g_j$$

And we can apply this to momentum equation. So, the intention notation this is usual momentum equation rather momentum equations because we have we are writing U j hat which we represent three velocity quantities in three directions. And U i hat so, basically

this is temporal term convection term minus d P hat d del P hat del x j plus del del x i of tau i j hat plus rho g j.

If one adds continuity equation multiplied with U j hat we can get this and this is basically allows we allows us to write the momentum equation in conservative form. So, this we have done earlier, but here we are using the attention notation and doing that. So, this is plugging in continuity equation; means we are basically writing momentum equation in conservative form.

Again U instant U j instant u j hat is U j bar plus u j prime and U i hat is U i bar plus u i prime. If we introduce that and perform the over bar operation, we will get del del x i into rho U i bar U j bar plus tau i j bar plus u i prime u j prime over bar equal to minus del P bar del x j plus rho g j.

So, basically we write conservative form. Then we substitute basically instantaneous quantities as mean and fluctuations instantaneous quantities as mean and fluctuations, then we do this substitution and apply over bar and this equation will result in.

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The individuals steps to derive the averaged momentum equations:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{\partial}{\partial t} \left(\rho \hat{U}_j \right) dt = 0$$

Furthermore:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{\partial}{\partial x_i} \underbrace{\left[\rho \left(\overline{U}_i + u_i' \right) \left(\overline{U}_j + u_j' \right) \right]}_{\rho \left(\overline{U}_i \overline{U}_j + u_i' \overline{U}_j + u_j' \overline{U}_i + u_i' u_j' \right)} dt = \ldots$$

Yielding the averaging laws of slide 17 we obtain:

$$\frac{\partial}{\partial x_i} \left[\rho \overline{U}_i \overline{U}_j + \rho \left(\overline{u_i' u_j'} \right) \right] = -\frac{\partial \overline{P}}{\partial x_j} - \frac{\partial \overline{\tau}_{ij}}{\partial x_i} + \rho g_j$$

$\rho \overline{U}_i \overline{U}_j$ = convective momentum transport

$\rho \overline{u_i' u_j'}$ = turbulence caused momentum transport

$\overline{\tau}_{ij}$ = molecular momentum transport

The individual steps to derive the averaged momentum equation:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{\partial}{\partial t} \left(\rho \hat{U}_j \right) dt = 0$$

Furthermore

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\partial}{\partial x_i} \left[\underbrace{\rho\left(\overline{U_i}+u_i^{'}\right)\left(\overline{U_j}+u_j^{'}\right)}_{\rho\left(\overline{U_i}\,\overline{U_j}+u_i^{'}\overline{U_j}+u_j^{'}\overline{U_i}+u_i^{'}u_j^{'}\right)} \right] dt = ...$$

Yielding the averaging laws of slide 17 we obtain:

$$\frac{\partial}{\partial x_i}\left[\rho\overline{U_i}\,\overline{U_j} + \rho\left(\overline{u_i^{'}u_j^{'}}\right) \right] = -\frac{\partial \overline{P}}{\partial x_j} - \frac{\partial \overline{\tau_{ij}}}{\partial x_i} + \rho g_j$$

$\rho\overline{U_i}\,\overline{U_j} =$ Connective momentum transport

$\rho\overline{u_i^{'}u_j^{'}} =$ Turbulence caused momentum transport

$\overline{\tau_{ij}} =$ Molecular momentum transport

The individual steps to derive average momentum equations the as I said that, what we did is 1 by T 0 to T del del x i rho U i bar plus u i prime u j bar plus u j prime d t. And then we have written you know expanded this U i bar into a U j bar plus u i prime into U j bar plus u j prime into U i bar plus u i prime into u j prime. And this after over bar operation this term will be there this term will be there these two terms will not be there.

So, we will get del del x i rho U i bar into U j bar plus rho u i prime into u j prime over bar equal to minus del del x j of P bar minus del del x i of tau i j bar plus rho g j; where rho U i bar U j bar is the convective momentum transport. Rho u i prime u j prime over bar is the turbulence caused momentum transport and tau i j bar is molecular momentum transport. This is basically shear stress arising out of molecular viscosity.

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Introducing simplification arising out of continuity equation we shall obtain:

$$\rho\frac{\partial \overline{u}}{\partial t}+\rho\left[\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}+\overline{w}\frac{\partial \overline{u}}{\partial z}\right]=-\frac{\partial \overline{P}}{\partial x}+\mu\nabla^2\overline{u}-\rho\left[\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right]$$

The above equation looks like laminar flow equation with the exception that the velocities are averaged velocities and there are additional stresses arising out of the fluctuating components!

So, if we want to write this equation as especially u or x direction momentum equation this term is 0 here time average term for del del t we can if u bar is 0. We can call I mean this rho del del t of u bar you know for a steady flows in a time average sense steady flow.

It can be made 0 which has been done here, but while writing this I have I mean separately x direction momentum equation, I have written that then rho u del u bar del x v bar del u bar del y w bar del u bar del z equal to minus del del x of p bar plus mu del square u bar.

And here we have written minus rho; that means, we transfer this entire quantity from left to right. And then we can write minus rho del del x of u prime square bar del del y of u bar v bar u prime v prime over bar sorry, del del x of u prime square over bar del del y of u prime v prime over bar and del del z of u prime w prime over bar.

And we have already said that u prime bar is 0, v prime bar is 0, w prime bar is 0, but quadratic quantities u prime square bar u prime v prime bar and u prime w prime bar those are non zero.

So, above equation, here this equation looks like laminar flow equation with the exception that the velocities are average velocities and there are additional stresses arising out of the fluctuating components. So, if you consider our x direction momentum equation for laminar flows this term is there only thing we are writing u as u bar all these terms are there only thing here u and v we are writing as average quantities.

This term is also there difference is p we are writing as p bar and this term is also there mu grad square u here u is u bar. So, this part is just like laminar flow equation the velocity and pressure quantities are replaced by time average quantities this is additional term. And this additional term is generated because of fluctuating components activities of the fluctuating components and this behaves as additional stress in the flow field.

So, when we will get corresponding terms, for y momentum equation and z momentum equation then these three again three again three those nine terms will behave as a stress tensor.

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$$\sigma_T = \begin{bmatrix} \sigma'_{xx} & \tau'_{xy} & \tau'_{xz} \\ \tau'_{xy} & \sigma'_{yy} & \tau'_{yz} \\ \tau'_{xz} & \tau'_{yz} & \sigma'_{zz} \end{bmatrix} = -\rho \begin{bmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2} \end{bmatrix}$$

σ_T is the Reynolds stress tensor and written in compact form as $-\rho \overline{u_i u_j}$

$$\sigma_{xx} = -p + 2\mu \frac{\partial \overline{u}}{\partial x} - \rho \overline{u'^2}$$

$$\tau_{xy} = \mu \left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x} \right) - \rho \overline{u'v'}$$

$$\sigma_T = \begin{bmatrix} \sigma'_{xx} & \tau'_{xy} & \tau'_{xz} \\ \tau'_{xy} & \sigma'_{yy} & \tau'_{yz} \\ \tau'_{xz} & \tau'_{yz} & \sigma'_{zz} \end{bmatrix} = -\rho \begin{bmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2} \end{bmatrix}$$

σ_T is the Reynold stress tensor and written in compact form as $-\rho \overline{u'_i u'_j}$

$$\sigma_{xx} = -p + 2\mu \frac{\partial \overline{u}}{\partial x} - \rho \overline{u'^2}$$

$$\tau_{xy} = \mu \left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x} \right) - \rho \overline{u'v'}$$

Exactly that is what we have written sigma T is then here we have written del del x of u prime square bar del del y of u prime v prime over bar del del z of u prime w prime over bar.

So, basically gradients of them will be appearing in the equation, but we can write them as you know this fluctuating components u prime square bar u prime v prime bar u prime w prime bar. Similarly you know contribution for from y momentum equation will be u prime v prime bar v prime square bar v prime w prime bar.

And z direction momentum equation will be u prime w prime bar v prime w prime bar and w prime square bar. So, sigma T is the Reynolds stress tensor written in compact form as minus rho u i prime u j prime over bar. And obviously, the when we get them in the momentum equation gradients of them appear just like you know molecular stress; I mean the usual shear stress and the normal stress here also we will get the contribution of this stress tensor in the momentum equation.

So, now sigma xx we can write as you know nu sigma xx which includes both you know this molecular viscosity and that velocity gradient will include this; similarly shear stress will include contribution of fluctuating components.

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The quantities $\rho \overline{u_i' u_j'}$ are the turbulence caused momentum transport terms

$$\rho \overline{u_i' u_j'} = \begin{bmatrix} \rho \overline{u_1' u_1'} & \rho \overline{u_1' u_2'} & \rho \overline{u_1' u_3'} \\ \rho \overline{u_2' u_1'} & \rho \overline{u_2' u_2'} & \rho \overline{u_2' u_3'} \\ \rho \overline{u_3' u_1'} & \rho \overline{u_3' u_2'} & \rho \overline{u_3' u_3'} \end{bmatrix} \tag{5}$$

This tensor is symmetric and has 6 unknowns.

Analogous to $\overline{\tau_{ij}}$ (for $\rho = $ const):

$$\overline{\tau}_{ij} = -\mu \left(\frac{\partial \overline{U}_j}{\partial x_i} + \frac{\partial \overline{U}_i}{\partial x_j} \right) \tag{6}$$

We introduce:

$$\rho \overline{u_i' u_j'} = -\mu_t \left(\frac{\partial \overline{U}_j}{\partial x_i} + \frac{\partial \overline{U}_i}{\partial x_j} \right) + \frac{2}{3} k \delta_{ij} \tag{7}$$

='eddy viscosity' simplest way to evaluate as

So, the quantities rho u i prime u j prime are the turbulence caused momentum transport terms and if we use the tensor notation we can write rho u i prime u j prime over bar equal to all these quantities.

We have already mentioned it earlier that rho u 1 prime into u 1 prime bar rho u 1 prime into u 2 prime bar rho u 1 prime into u 3 prime over bar like that we get other 6 components. This tensor is symmetric and has 6 unknowns basically you this if we really look into this is unknown, this is unknown, this is unknown, this is unknown, this is unknown, this is unknown.

So, since it is symmetric we need not count these three, but 1, 2, 3, 4, 5, 6; 6 unknowns and this is analogous to T i j bar which is coming out of molecular viscosity. So, we can say that this stress additional stress which is coming out of fluctuating components just following the usual stress components we can model it as mu t instead of molecular viscosity this is turbulent viscosity.

Into again this average quantities the which is del del x i of U j bar plus del del x j of U i bar plus two third k delta i j this is Kronecker delta. I will specially discuss why this has been as to why this has been added, but here if we accept adding this term as I said we will specifically discuss about it.

But keeping that aside there is striking similarity between these two only the molecular viscosity is substituted by eddy viscosity. And again this is a basic property and this is not a basic property this is created because of the interaction of the fluctuating components of turbulence.

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$$-\overline{u_i' u_j'} = v_t \left(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_j} \right) - \frac{2}{3} k \delta_{ij}$$

The term involving the Kronecker delta δ_{ij} is perhaps a somewhat unfamiliar addition to the eddy-viscosity expression. It is necessary to make the expression applicable also to normal stresses (when I=j). The first part involving velocity gradients would yields normal stresses.

$$\overline{u_1'^2} = -2v_t \frac{\partial \overline{U}_1}{\partial x_1}, \overline{u_2'^2} = -2v_t \frac{\partial \overline{U}_2}{\partial x_2}, \overline{u_1'^2} = -2v_t \frac{\partial \overline{U}_3}{\partial x_3}$$

Whose sum is zero because of the continuity equation. However all normal stresses are by definition positive quantities and their sum is twice the kinetic energy k of the fluctuating motion:

$$k = \frac{1}{2}\left(\overline{u_1'^2} + \overline{u_2'^{2}} + \overline{u_3'^{2}} \right)$$

Inclusion of second part of eddy viscosity expression assures that sum of stresses is equal to 2k. The normal stresses act like pressure forces endicular to the faces of a control

volume and because like the pressure energy k is a scalar quantity the second part constitute a pressure.

And now I will discuss that part that you know why we wrote two third k delta i j. So, minus u i bar u minus u I prime u j prime over bar sorry minus u i prime u j prime over bar is written as nu t.

So, this is same as what we wrote here only we are transferring rho on right hand side. So, having done that we can write that nu t into del del x j of U i bar plus del del x i of U j bar plus two third k delta i j. The term involving Kronecker delta ij is perhaps a somewhat unfamiliar addition to the eddy viscosity expression.

It is necessary to make the expression applicable also to normal stresses when i equal to j. So, why did we add this in order to explain that let us say you know when let us substitute i equal to j. Then we can write u 1 prime square bar equal to twice nu t del U 1 over bar del x 1 then u 2 i equal to j. So, another will be u 2 prime square bar then this will be minus 2 nu t into del del x 2 of U 2 bar and this is sorry this should be u 3 square bar.

And this 1 should be substituted by 3. I am sorry this is a mistake this u 3 prime square bar like u 2 prime square bar u 1 prime square bar this will be u 3 prime square bar equal to minus twice nu t del del x 3 of U bar or U 3 bar. So, now, because you know if we do not have this only the first part we are writing, let us imagine this was not there this is not there then we get this.

And then if we add it up, it is basically del U del x plus del v del y plus del w del z continuity equation. So, sum is 0 because of the continuity equation. However, all normal stresses are by definition positive quantities and their sum is twice the kinetic energy k of the fluctuating motion. Where k is given by half u 1 prime square bar u 2 prime square bar u 3 prime square bar this is the turbulent kinetic energy.

So, kinetic energy of the fluctuating motion, we are supposed to get when we add all normal stresses. But if we add all normal stresses here we will get continuity equation equal to 0. And this incoherence is coming and in order to remove that we are adding minus two third k Kronecker delta inclusion of the second part of the eddy viscosity expression assures that the sum of the normal stresses is twice k.

Now, it is assured this will be this is twice k the normal stresses act like pressure forces that is perpendicular to the faces of a control volume and because like the pressure itself the energy k is a scalar quantity the second part contributes to modified pressure. So, now, it is consistent it produces of course, with all normal stresses two third k two third k two third k that two third k can be absorbed in the pressure and then everything is consistent.

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*The derivations showed:

*That time averaging the continuity equation and the momentum equation yield the following equation:

$$\frac{\partial \overline{U}_i}{\partial x_i} = 0 \tag{8}$$

$$\frac{\partial \overline{U}_j}{\partial t} + \overline{U}_i \frac{\partial \overline{U}_j}{\partial x_i} = -\frac{1}{\rho} \frac{\partial \overline{P}}{\partial x_j} + v \frac{\partial^2}{\partial x_i \partial x_i} \overline{U}_j - \frac{\partial \overline{u_i' u_j'}}{\partial x_i} \tag{9}$$

*Equation 8 and 9 referred to as Reynold equation. One sees that the correlation of velocity fluctuation yield a new term in this equation. This term is usually referred to as Reynold shear stresses it presents the momentum transport caused by the turbulence

fluctuation. Tensor contains 6 independent components. Additional equations are to solved flow problems.

So, now what we are going to do we have basically continuity equation and this is del del x i of U i bar in all time averaged velocity components. And we have basically the new momentum equations in three directions. And in the momentum equations we have this additional stress term which is u i prime u j prime over bar and as I said gradient of that will come in the momentum equation del del x i.

So, this if we look into it I have already mentioned that there are 6 unknowns u 1 prime bar square over bar u 1 prime bar u 2 prime bar over bar u 1 prime bar u 3 prime bar over bar, then u 2 prime bar u 3 prime bar over bar and u 2 prime u 2 prime; that means, u 2 prime square bar and u 3 prime square bar.

So, this 6 components means u one prime square bar u 1 prime into u 2 prime over bar u 1 prime into u 3 prime over bar u 2 prime into u 3 prime over bar u 2 prime square over bar and u 3 prime square over bar.

These quantities will again just go back for readily remembering it like this quantity, this quantity, this quantity, this quantity, this quantity, this quantity. Or, as I have already mentioned this quantity, this quantity, this quantity, this, this, this 1, 2, 3, 4, 5, 6.

So, these 6 unknowns are there plus u 3 I mean u components in three directions u average components and p average. So, till unknowns, but available equations are 4. So, there is a problem it is called closure problem equation 8 and 9 are referred to as Reynolds equations.

One sees that correlation of the velocity fluctuations u i prime u j prime over bar yield a new term in this equation. This term is usually referred to as Reynolds shear stresses, it represents the momentum transport caused by the turbulence fluctuation.

This tensor contains 6 independent components additional equations are needed to solve these problems. But just I am proposing something that what we have done here if we can write this u i prime u j prime over bar as mu t which is turbulent viscosity into del del x i U j over bar plus del del x j U i over bar.

Then in this equation again this u i prime into u j prime over bar will be expressed in terms of u i bar u j bar and their gradients. So, basically then again we will get 4 equations and 4 unknowns in terms of velocity and pressure.

But when we do that we will get one term which is basically mu t or nu t this mu t or nu t is not known. This nu which is molecular viscosity is known fluid property, but when we will express this in terms of the velocity gradients; that means, shear stresses we will get nu t.

So, in terms of number of equations and number of unknowns again we will have a balance, but nu t is not known. So, entire challenge will be to find out nu t it is not a fluid property. So, it is not known a priori it is a function of the interaction of fluctuating components.

So, depending on turbulent fluctuations and in the field whatever is the turbulent kinetic energy and their dissipation rate this will be controlled or the nu t will be determined. So, in the next lecture we will discuss that aspect thoroughly. Today we will stop here thank you very much; thank you for your interest.

Thank you.