Computational Fluid Dynamics and Heat Transfer
Prof. Gautam Biswas
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 21
Solution of N - S equations for Incompressible Flows Using MAC Algorithm

Good morning everybody. Today we will extend the application of MAC Algorithm for solving energy equation.

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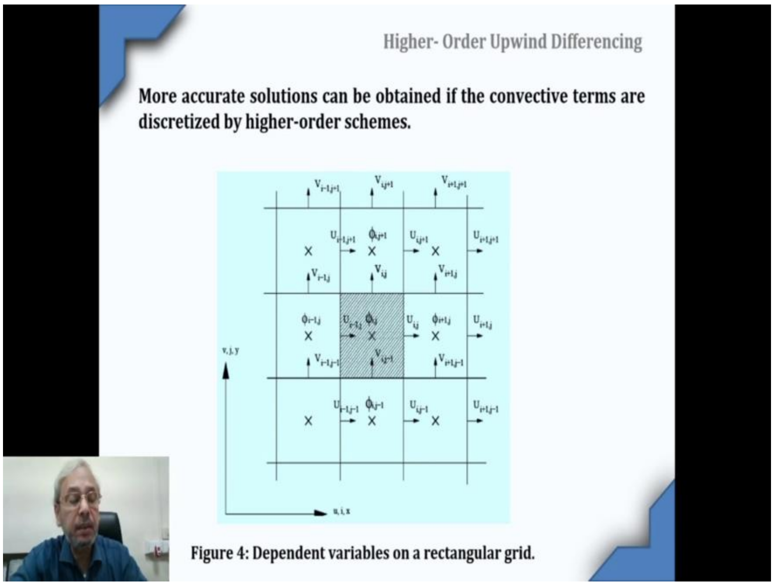

And before we start the solution part, we will just briefly mention about higher order upwind differencing and we have shown a grid template which already you know. Basically, the velocity quantities or the velocities are defined at the centre of the cell faces to which they are normal like we can you can see $u_{i,j}$, $v_{i,j}$ and the pressure and temperature are defined at the centre of the cells.

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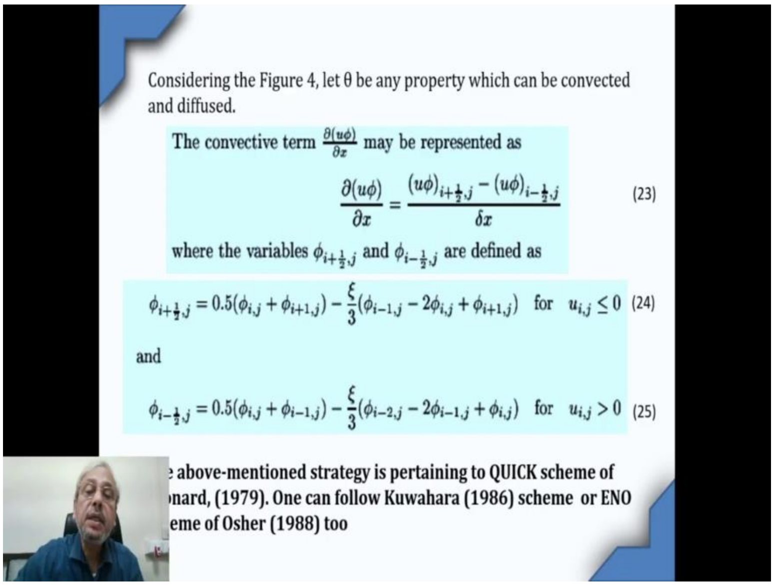

Having said that if we want to discretise a convective term like $\frac{\partial(u\phi)}{\partial x}$, ϕ is a scalar quantity and $\phi_{i,j}$ is the location of phi at a cell identified by i, j. So, or d$\frac{\partial(u\phi)}{\partial x}$ we have written in conservative form will be $\frac{(u\phi)_{i+\frac{1}{2},j} - (u\phi)_{i-\frac{1}{2},j}}{\delta x}$.

And this $\phi_{i+\frac{1}{2},j}$ and $\phi_{i+\frac{1}{2},j}$ can be defined by the expressions 24 and 25 (Refer Slide Time: 01:10), separate expressions are written for upwinding. And you can see if local velocity is negative then it will have the expression which is given by 24 and the local velocity is positive then the expression is given by 25 because depending on the direction of the velocity, we have to engage the upstream points.

And this quantity ξ is basically a parameter involved with upwinding strategy and value of this parameter; for example, if we go by quick scheme as you know QUICK is a very popular scheme and the value of ξ in quick scheme is $3/8$. QUICK stands for quadratic upstream interpolation of convective kinematics. It was developed by Professor Leonard in 1979.

And I have given just an example one can use QUICK scheme one can use several other schemes. Again I have given two more examples like Kuwahara scheme which was developed in 1986 and ENO scheme. ENO scheme, basically ENO stands for Essentially

Non Oscillatory scheme of Stanley Osher. It is a very accurate scheme, ENO scheme used by used for even modelling free surface flows.

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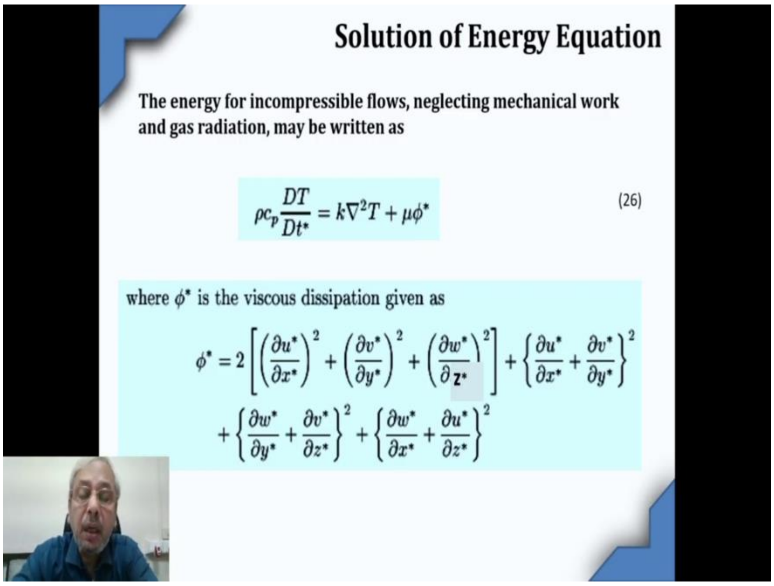

Now, we will come to energy equation. You can see we have written energy equation as a you know general expression

$$\frac{\rho c_p (DT)}{Dt^*} = k\nabla^2 T + \mu\phi^*$$

Now, we have done a little you know sort of change here. Change means, no major change in nomenclature that is starred quantities we are considering as dimensional quantities. Usually, starred quantities are non-dimensional quantities. We have taken starred quantities as dimensional quantities so that after non dimensionalization we get usual terms as non-dimensional terms. So, you can see that u^*, v^*, w^* or $x^*, y^*$and z^*, all these starred terms are here we are using for dimensional quantities. Similarly t^* is dimensional time and uppercase T is dimensional temperature.

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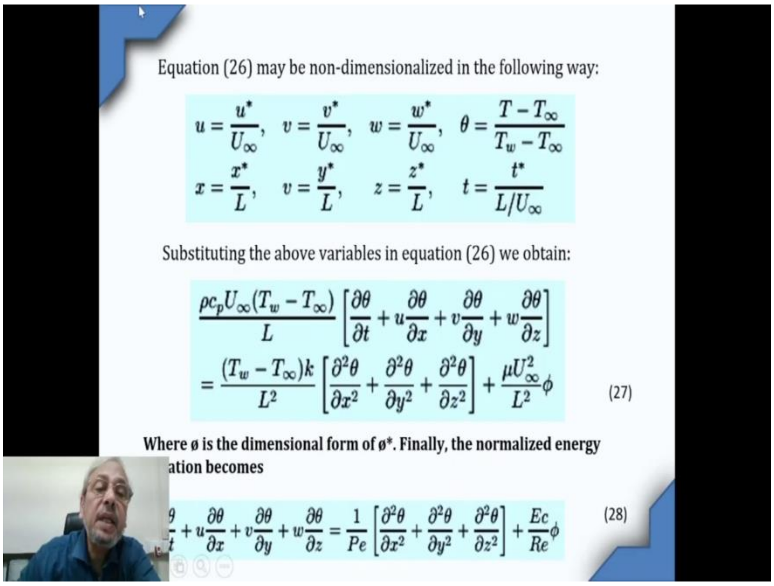

And then we non denationalize in this way that small u is u^*/U_∞, small v is v^*/U_∞, w is w^*/U_∞, θ is $\frac{T-T_\infty}{T_w-T_\infty}$ and x is x^*/L. So, L is a reference dimension of the domain, U_∞ is the reference velocity, T_∞ is the lowest temperature available in the field and T_w is the highest temperature available in the field. So, that non dimensional temperature varies between 0 and 1. And non dimensional time is $\frac{t^*}{L/U_\infty}$. So, if we define the non dimensional variables in this way then we can plug in these variables into the dimensional equation given by equation 26 (Refer Slide Time: 06:17) and we can get you know this non dimensional form of equation given by

$$\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial\theta}{\partial z} = \frac{1}{Pe}\left[\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right] + \frac{Ec}{Re}\phi$$

Peclet number (Pe) means, $Re \times Pr$, ϕ is again non dimensional dissipation function. So, we can then look at whether this viscous dissipation we will retain or not.

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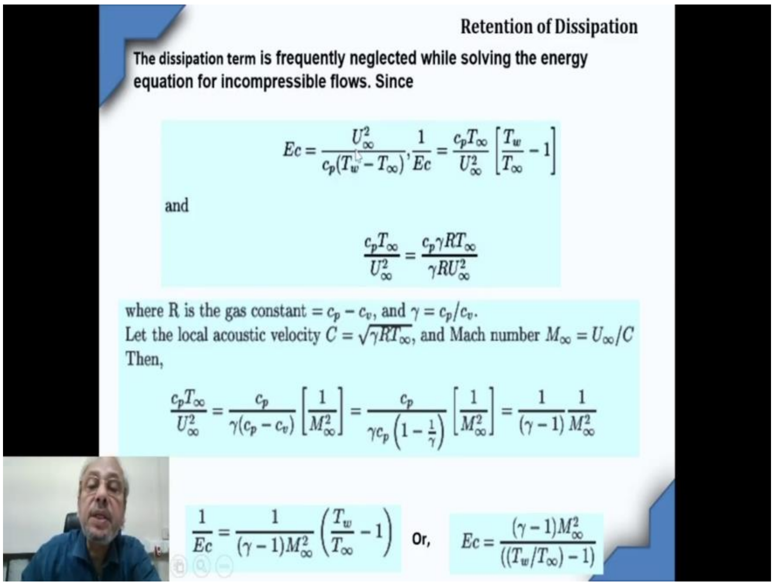

So, if Eckert number Ec is given by $\frac{U_\infty^2}{c_p(T_w - T_\infty)}$ then $1/Ec$ number we can write this way (right hand side of first Eq. of Slide Time: 08:23). From here we can see that

$$\frac{c_p T_\infty}{U_\infty^2} = \frac{c_p \gamma R T_\infty}{\gamma R U_\infty^2}$$

γR is introduced because γ is basically ratio of specific heats c_p/c_v and R is the gas constant given by this is basically $c_p - c_v$.

So, R is $c_p - c_v$and γ is c_p/c_v and local acoustic velocity is given by uppercase C which is $\sqrt{\gamma R T}_\infty$. And we also define Mach number which is velocity divided by local acoustic velocity. So, this $c_p T_\infty / U_\infty^2$ with little bit of algebraic manipulation we can write it as you can see $\frac{1}{\gamma - 1} . \frac{1}{M_\infty^2}$. So, $1/Ec$ then this is the expression (last equation of of Slide Time: 08:23).

So, $\frac{1}{Ec} = \frac{1}{\gamma - 1} . \frac{1}{M_\infty^2}\left(\frac{T_w}{T_\infty} - 1\right)$

That means, $Ec = \frac{(\gamma - 1)M_\infty^2}{\left(\left(\frac{T_w}{T_\infty}\right) - 1\right)}$, T_w if there is any heated objects it temperature of that heated object or heated wall if it is flowing internal flow may be wall is heated to a temperature T_w.

So, $\frac{T_w}{T_\infty}$ is the temperature ratio available in the flow field. We can say highest by lowest temperature ratio minus 1. So, Eckert number is given by this.

Now, we can go for little order of magnitude analysis, usually gamma for diatomic gases or gamma for air can be taken as 1.3 or 1.4 and then this becomes a decimal quantity $\gamma - 1$. Mach number for incompressible flows it is less than 0.3. Even if we take 0.3, so, it is 0.3 square. It is a small number and finally, divided by temperature ratio which may be 1.8, 1.7, whatever a minus 1 this is also a small quantity or temperature ratio if it is may be 2 then it will be 1.

So, whatever is the denominator we can see that numerator is basically you know if Mach number is 0.3 and if we say gamma is 1.3, so, it is 0.3 cube. So, it is a very small number after I mean this is starts from the decimal point and this multiplier maybe so small and then further it will be divided by Reynolds number and usually Reynolds number is moderate too large for such cases.

So, Eckert number by Reynolds number is a very small quantity. If ϕ is not too large ; that means, viscous dissipation function then this Eckert number by Reynolds number can be ignored. As I said estimation is this is 0.3 cube Eckert number and from the gamma minus 1 into Mach number square and divided by Reynolds number. So, it is a very small number with respect to the contribution of other terms and we can neglect that.

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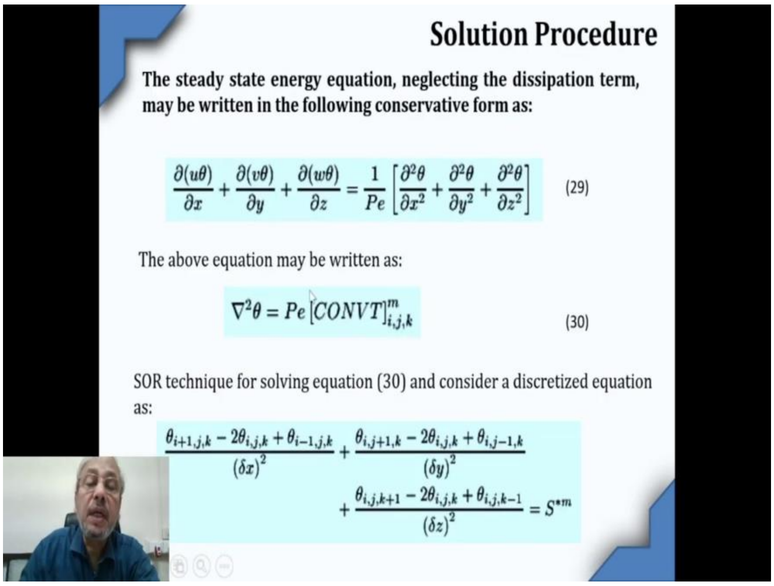

If we neglect that we can retain also then it will be added as a source term, but for the time being we are neglecting it and we are sort of redefining the equation as

$$\frac{\partial u\theta}{\partial x} + \frac{\partial v\theta}{\partial y} + \frac{\partial w\theta}{\partial z} = \frac{1}{Pe}\left[\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right]$$

Here one can also mention that usually this contribution of I have said that it is basically it will behave as source term because when we are coming to solve this equation we have already solved the velocity field. So, even though you know the this multiplier Eckert number by Reynolds number square into viscous dissipation function they will have be having u, v and w, but since u, v, w are known apriority.

So, they can be those can be evaluated and that will be added as a source term. And here we are ignoring it and also you can see that since u, v and w are already known and we are plugging in we will be plugging in those terms from the solution of momentum equation the non-linearity of energy equation is also not there.

So, now we can write this equation as $\nabla^2\theta = Pe[\mathrm{CONVT}]_{i,j,k}^{m}$ and we are solving you can see the unsteady term is dropped you know the steady state energy equation because for a given velocity filed energy equation will finally, converge to a you know steady state temperature field. Velocity field may vary.

So, for any given velocity field we can get a steady temperature field. So, we our solution strategy will be something like pseudo time marching. Here this m index superscript this does not signify time level, but iteration level. And so, convective terms at i, j, k point at m^{th} iteration level.

And $\nabla^2\theta$ we can simply discretize at as the central difference in x, direction central difference in y direction, central difference in z direction and we can again consider Peclet number into this convective terms contribution. This is known and we can call this as S^{*m} . Just we have used this nomenclature, we will call Peclet number into the contribution of convective terms at m^{th} level m^{th}iteration level as S^{*m} and left-hand side $\nabla^2\theta$ we have discretized following central difference scheme in x direction, y direction and z direction.

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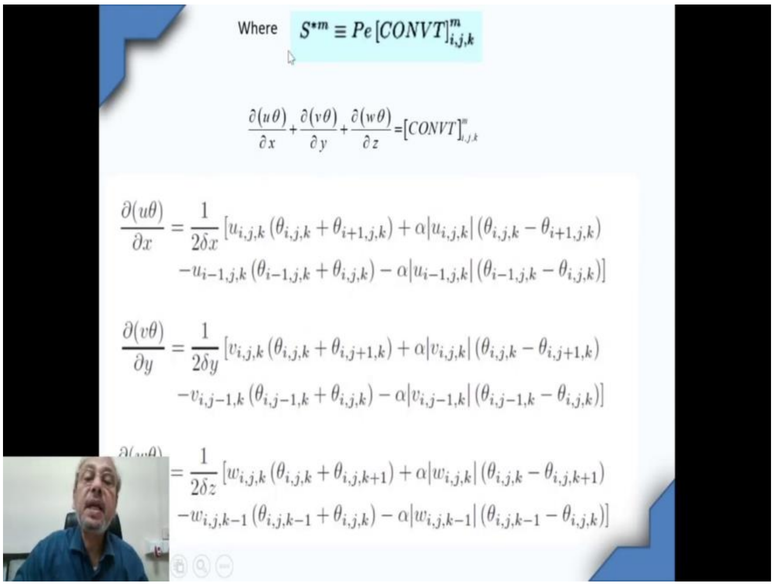

Now, as I said S^{8m} is basically we have defined this. This is Peclet number into the contribution of the convective terms at the m^{th} level at a point i, j, k. So, we have written that $\frac{\partial u\theta}{\partial x}$, $\frac{\partial v\theta}{\partial y}$ and $\frac{\partial w\theta}{\partial z}$ is basically contribution of convective terms at a point i, j, k at a level m and then we have given the expressions for $\frac{\partial u\theta}{\partial x}$, $\frac{\partial v\theta}{\partial y}$ and $\frac{\partial w\theta}{\partial z}$. And we have followed the same second upwind and method for discretization.

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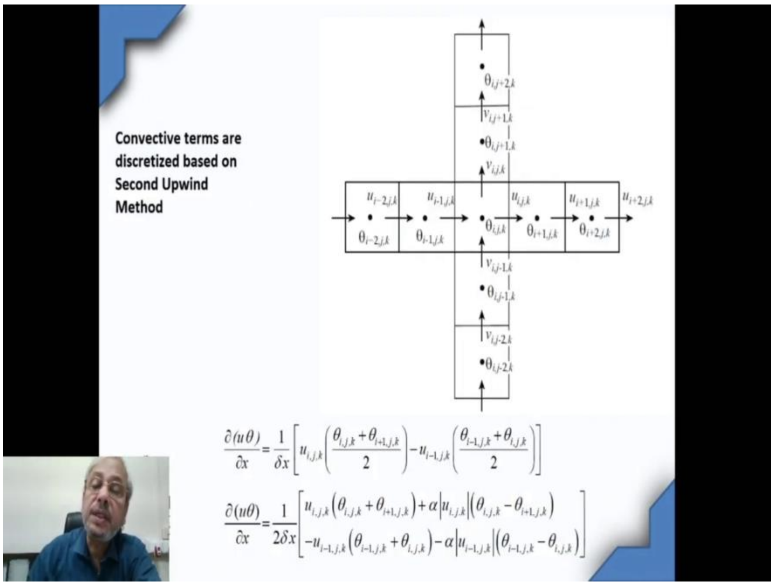

For the convective terms we have followed the second upwind method of discretization and if we recall basically, we have the terms $\frac{\partial u\theta}{\partial x}$, $\frac{\partial v\theta}{\partial y}$ and $\frac{\partial w\theta}{\partial z}$. Now, we have taken up the first term which is $\frac{\partial u\theta}{\partial x}$.

Now $\frac{\partial u\theta}{\partial x}$ we have to discretise where $\theta_{i,j,k}$ has been defined. That means, basically we have to define θ at this point together with $u_{i,j,k}$ because this is, here we are using conservative form. So, u and θ are to be defined here (right hand side of centre cell).

u is defined here as $u_{i,j,k}$ and θ here will be $\frac{\theta_{i,j,k}+\theta_{i+1,j,k}}{2}$. So, u and θ both have been defined here minus $u_{i-1,j,k}\left(\frac{\theta_{i-1,j,k}+\theta_{i,j,k}}{2}\right)$.

So, $u_{i-1,j,k}$ multiplied by $(\theta_{i-1,j,k}+\theta_{i-1,j,k})/2$. So, $u_{i-1,j,k}$ here and $\frac{\theta_{i-1,j,k}+\theta_{i,j,k}}{2}$ that is θ here we have defined. And you know difference between these two divided by δx, sort of central difference why exactly central difference we have done that.

And then we have introduced the upwinding factor and also together with giving upwind based we have introduced modular sign so that the formulation can take care of positive and negative both $u_{i,j,k}$ at this point. And similarly, also we have introduced this modular sign over $u_{i-1,j,k}$.

So, $u_{i-1,j,k}$ it can be possible locally it is possible somewhere it can be positive somewhere it can be negative. So, in order to take care of that variation of velocity we have introduced modular sign here (last equation of Slide Time: 19:41) and here this $u_{i,j,k}$ this also can be locally positive or negative for introducing that flexibility we have used the modular sign here (last equation of Slide Time: 19:41). And this is basically these are the upwind contributions and upwind contribution is you know identify together with α, α is the upwind factor. And if we recall what we did for the momentum equations recommendation was α may be varied between 0.2 and 0.3, it may be even less lesser is always better, but it will it should not cross 0.3.

Because less α means it is incline more towards central differencing second order accuracy and α is giving the upwind contribution, but it should be kept at a minimum level at a I mean low value of α should be used. So, this is how we have discretized $\frac{\partial u\theta}{\partial x}$ following second upwind method, which we also followed for the momentum equations. And same way we can discretize $\frac{\partial v\theta}{\partial y}$ and $\frac{\partial w\theta}{\partial z}$and then we will have contributions of all the individual terms of the convective part of the energy equation. So, of this is how I have given the example $\frac{\partial u\theta}{\partial x}$, similarly $\frac{\partial v\theta}{\partial y}$, $\frac{\partial w\theta}{\partial z}$ those are to be discretized and these are the expression for discretized. $\frac{\partial u\theta}{\partial x}$, $\frac{\partial v\theta}{\partial y}$ and $\frac{\partial w\theta}{\partial z}$ terms. Once we know all this, we know what is basically discretized convective terms at point i, j, k at a level m and if we multiply it with Peclet number we will get A^{*m}.

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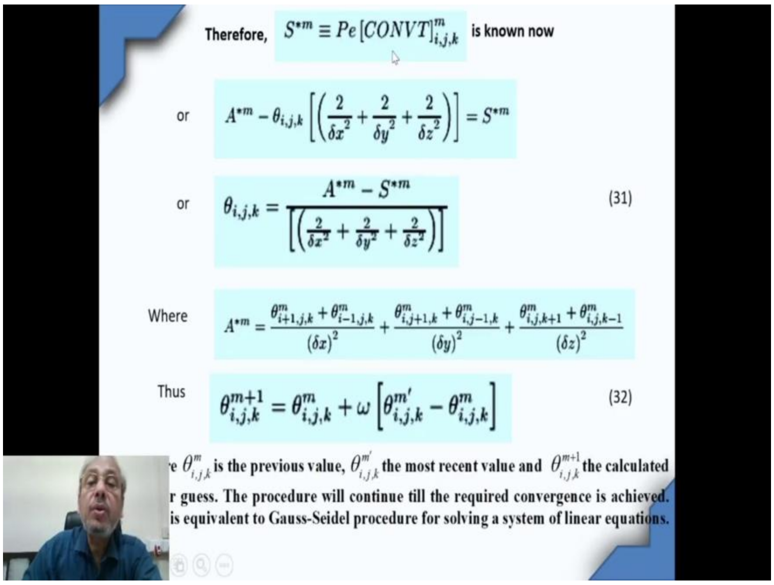

Now, so, S^{*m}.equal to this and this is known. Now, you let us go back again. Do not confuse, go back to equation 30. So, this term (right hand side of Eq. 30 (Slide Time: 14:01)) we have evaluated. Left hand side we had basically central difference use central difference scheme to discretize del square theta in x, y and z direction.

And we can see that it involves $\theta_{i,j,k}$ and its neighbours $i+1, j, k,\ \ i-1, j, k,\ \ \ i, j+1, k,$ $i, j-1, k,\ \ \ i, j, k+1,\ \ \ \ i, j, k-1$. So, $\theta_{i,j,k}$ and the neighbouring θs are involved on the left-hand side.

Now, again let us go back to expression (Slide Time: 26:26). We know now the quantity S^{*m} and left-hand side we had central differencing of θ in x, y and z direction. Now, in that expression for central difference I mean discretized theta, we are pulling out $\theta_{i,j,k}$ terms.

So, $\frac{2\theta_{i,j,k}}{\delta x^2}+\frac{2\theta_{i,j,k}}{\delta y^2}+\frac{2\theta_{i,j,k}}{\delta z^2}$ with negative sign and all the neighbouring terms that means, $(\theta_{i+1,j,k}+\theta_{i-1,j,k})\frac{1}{\delta x^2}$ $(\theta_{i,j+1,k}+\theta_{i,j-1,k})\frac{1}{\delta y^2}$ $and\ (\theta_{i,j,k+1}+\theta_{i,j,k-1})\frac{1}{\delta z^2}$. So, all those neighbouring terms we are calling them as A^{*m}.

We are deliberately doing that we will assume that at a given level θ is known. θ can be when you are starting the computation θ can are initial temperatures or all θs. So, if it is known for the when we are going for the next level, if the previous level is known we can call them a together A^{*m} and we can then from this expression we can get a clearly explicit expression for $\theta_{i,j,k}$.

So, $\theta_{i,j,k}$ then will be $\frac{\{A^{*m}-S^{*m}\}}{\frac{2}{\delta x^2}+\frac{2}{\delta y^2}+\frac{2}{\delta z^2}}$ whereas as I told A^{*m} is at the m^{th} level neighbouring terms of $\theta_{i,j,k}$ like $(\theta^m_{i+1,j,k}+\theta^m_{i-1,j,k})\frac{1}{\delta x^2}$, $(\theta^m_{i,j+1,k}+\theta^m_{i,j-1,k})\frac{1}{\delta y^2}$ $and\ (\theta^m_{i,j,k+1}+\theta^m_{i,j,k-1})\frac{1}{\delta z^2}$.

So, $\theta_{i,j,k}$ when at m^{th} level this S^{*m} which is comprising of Peclet number in two convective terms at m^{th}level these are known. θ values at the neighbouring cell at m^{th} level those are known divided by this you know $\frac{2}{\delta x^2}+\frac{2}{\delta y^2}+\frac{2}{\delta z^2}$.

We can get new $\theta_{i,j,k}$ distribution on all cells if we run from $i=1$ to iim, $j=1$ to jim and $k=1$ to kim, but here point by point we will be able to get that. So, if that is improved or (Refer Time: 31:36) new $\theta_{i,j,k}$ we can call it m prime. So, $\theta^{m'}_{i,j,k}$ prime let us assume we are calculating through equation 31.

Then in equation 32

$$\theta^{m+1}_{i,j,k}=\theta^m_{i,j,k}+\omega[\theta^{m'}_{i,j,k}-\theta^m_{i,j,k}]$$

So, we are accelerating this calculation by introducing an over relax over relaxation factor which is may be 1.3, 1.4, 1.5. We have to do little bit of numerical apriori study to find out a proper ω for a good level of acceleration. So, we have written in order to describe equation 32 that $\theta_{i,j,k}^{m}$ is the previous value, $\theta_{i,j,k}^{m'}$ is the most recent value calculated from equation 31 and $\theta_{i,j,k}^{m+1}$ the calculated better guess.

So, it is even more improved case when we are trying to converge going for the steady state temperature. The procedure will continue till the required convergence is achieved. This is equivalent to Gauss-Seidel procedure for solving a system of linear equations. So, that is how just following and this Gauss-Seidel procedure Gauss-Seidel for procedure for solving linear equations.

We can solve then θ and conjugate the you know iterative loop till the steady state temperature is reached. So, this is the overall strategy for solving the energy equation and finding out temperature at each cell.

But you know we must remember that in this case specifically the what we have discussed we have solved the Navier-Stokes equations; that means, momentum equations in advance and found out the velocity quantities u, v and w at each cell then we are solving the energy equation.

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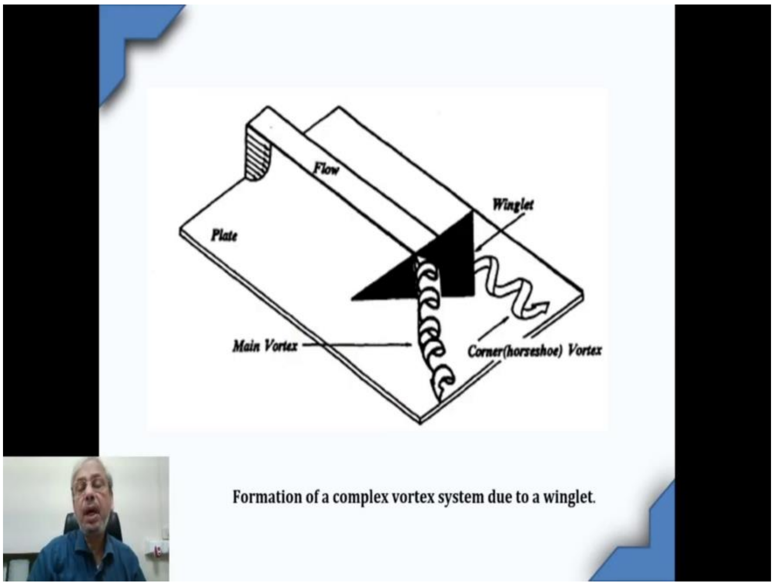

So, now I will give one or two very successful computations. There are many, but I am giving just one or two examples because you know these are very a complex flow field.

First example I am giving that flow passed a delta winglet in a channel. See this is the delta winglet and approach flow coming from you can well understand from the other side.

Obviously, the side of the winglet which we cannot see that is facing the flow and there will be you know higher pressure on that side and downstream side we will have lower pressure. And because of creation of this pressure surface and suction surface aerodynamically this side is called suction surface, the vortices will be created.

So, this is if we call main vortex we will get and this is a longitudinal vortex. We will get another such longitudinal vortex which will be created on the opposite side which we cannot see which will be created on that side due to the difference of stagnation pressure from the top of the wing to the bottom of the wing.

Because you can see the velocity head will be converted into pressure head and; obviously, velocity is having a gradient and that is why this stagnation pressure will also have gradient. And depending on the velocity like on the top side, it will be having higher pressure and the bottom side it will be having lower pressure. Due to that another you know vortex will be created which is called coroner vortex.

The creation of these vortex follows the basic philosophy of horseshoe vortex. So, this will be another vortex and the fluid between these two vortices will also undergo a rotation.

So, it will be a very complex filed of this combine vortices. If we go in the downstream on any plane, we have to get footprint of all these 3 vortices; that means, main vortex, horseshoe vortex or corner vortex and through it in between will be induced we will create induced vortex.

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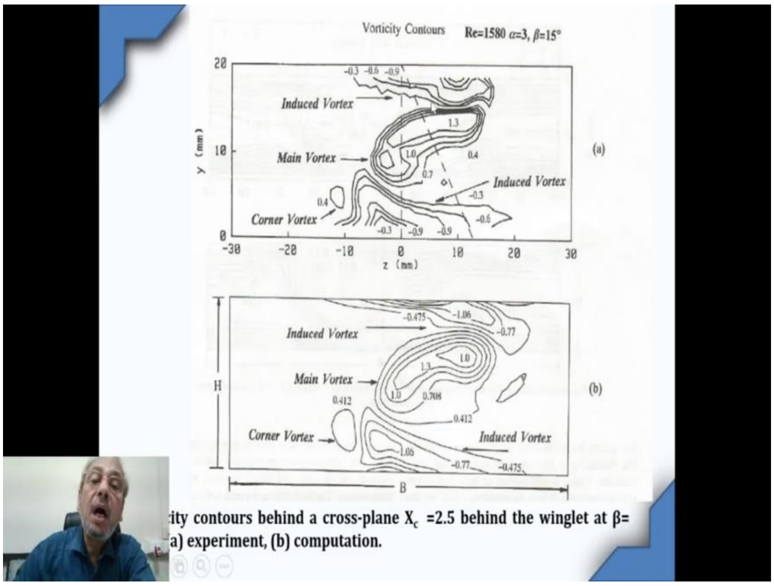

So, now this is the you can see computational result and we can see main vortex, corner vortex, induced vortex I mean because after these vortices are created, they will be disseminated. They will be in the downstream and you know they will spread also and you can see you know here this induced vortex, corner vortex, main vortex.

And this is these are all experimentally determined field and hot wire anemometer was used and on a cross plane about 2000 points of were taken to determine u, v and w velocity on 2000 points using hot wire anemometer. And this is the you know after data reduction you know it was a through a DTA card it was fed to a computer and a computer, I mean calculation was done in vorticity field in the post processing program and this was plotted.

And this is computational vorticity field where we also can see corner vortex, induced vortex, main vortex etcetera and you know the closer comparison even gives a better satisfaction. So, vorticity contours behind a cross plane $X_c = 2.5$, this is a distance behind the winglet.

Winglet is placed at β equal to this angle you know β angle at 150 degree and (a) are experimental results, (b) are computational results. Why I shown this? This is a very complex field because here whatever I have shown schematically, this scheme does not stay static there.

Because these vortices are keep interacting and they spread all over the domain, all over the cross plane and you know depending on distance from the winglet trailing edge the structure keeps changing and we have shown the flow structure at $x = 2.5$. And as I have said that we got amazing match with the experimentally determined flow field.

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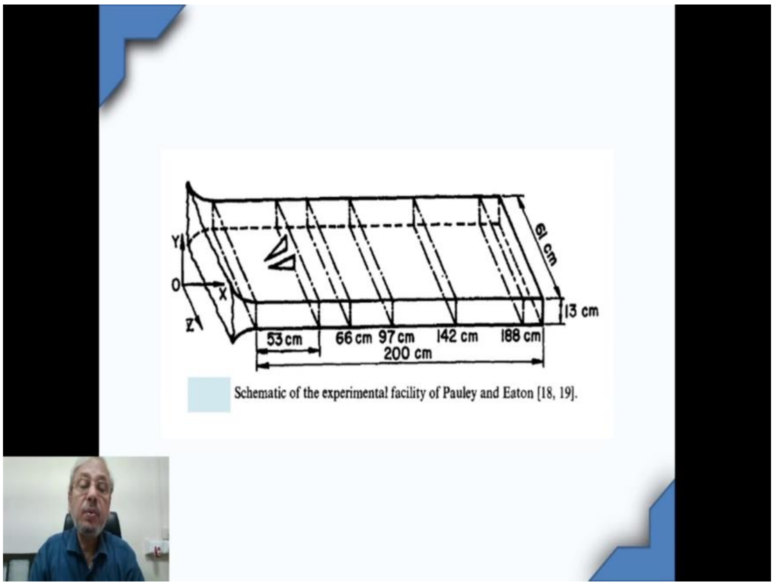

Then we will take up yet another example, here instead of delta winglet we have a I mean used delta winglet pair and this is a very well-known experiment. This experiment was done in Stanford University, Professor Pauley and Professor Eeaton where involved.

This was basically Professor John Eaton's experimental group and this is a channel. You can see channel dimensions and these are relatively you know smaller dimensions and delta winglet pair is sitting here. And again, behind the winglet the flow field at different cross sections flow fields where mat experimentally by Professor Eaton and Professor Pauley.

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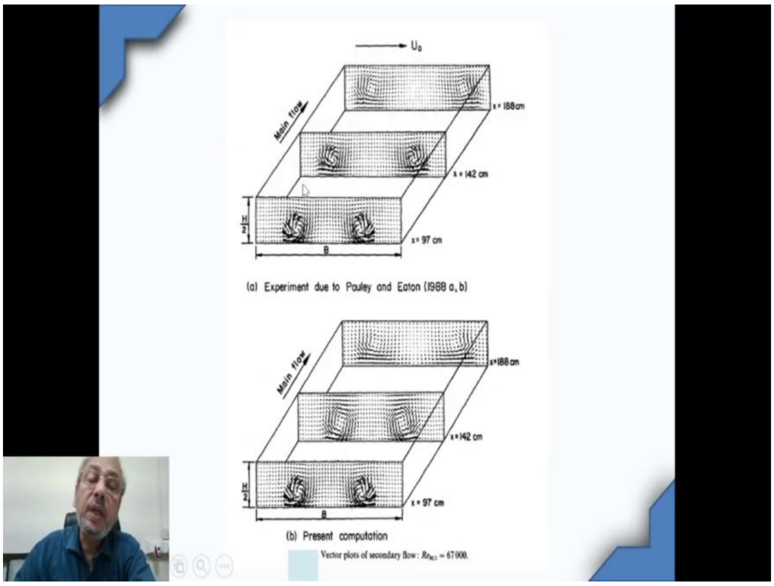

And we calculated those I mean the velocity fields at the same axial location. I mean we calculated inter-velocity field and then plotted at the same axial locations where the results of Professor Eaton and Professor Pauley where available. So, you can see at a distance $x = 97\ cm$ behind the winglet pair, at a distance 142 cm behind the winglet pair, at a distance 188 cm behind the winglet pair.

And at the same we computed the inter-field and at the same location cross plants we plot it the velocity victors you can see excellent match. But, since this is turbulent flow, I will not say this is excellent match because such quantity is in matching such quantities in turbulent flow are I mean is possible, but that is not the complete confirmation.

When it is turbulent flow there are quantities like second movement which are root mean square velocities or Skewness factor flatness factor. These are you know third moment and high order moments these are to be evaluated critically, but from the engineering point of view from the matching of velocity vectors this produced reliable result.

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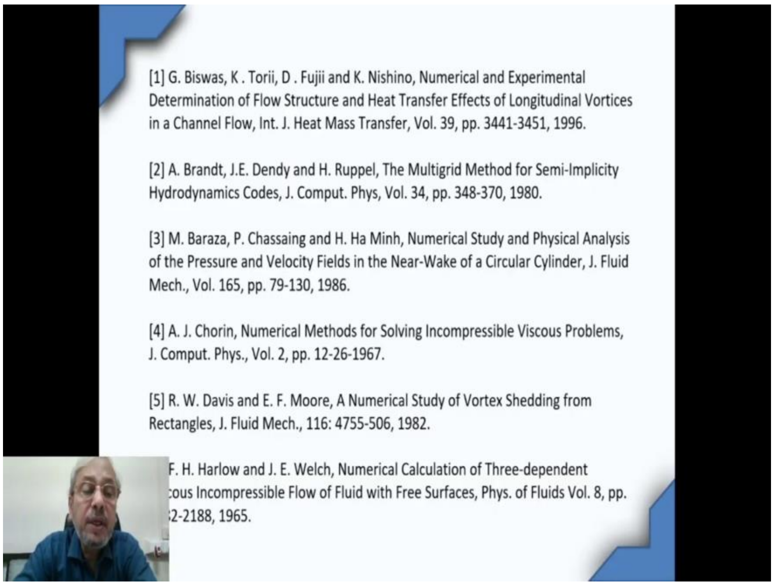

So, I will just give you some references which are very useful references with respect to the MAC algorithm and its application for solving flow and temperature fields.

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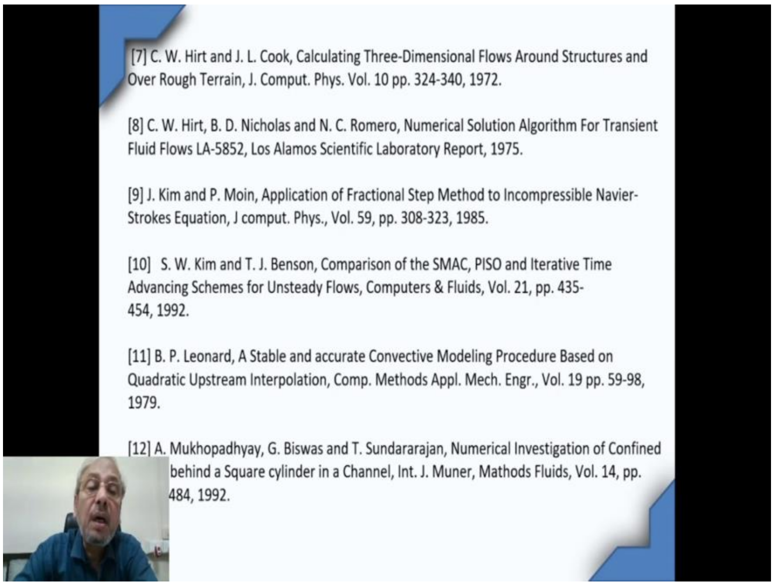

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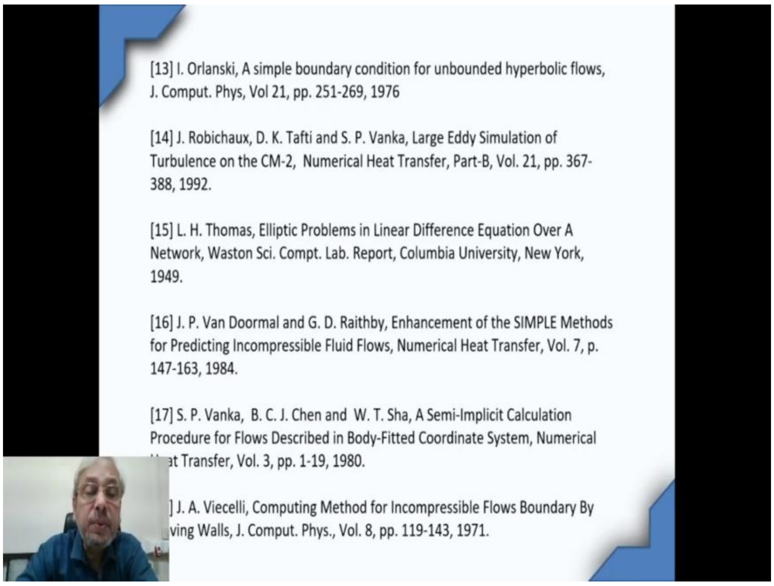

Whatever references I used in my lecture I have discussed are given those references. If you like you can look at these papers and you know gather more knowledge and you know enhance your understanding.

Thank you very much for you know listening to the lecture with attention.

Thank you.