Computational Fluid Dynamics and Heat Transfer
Prof. Gautam Biswas
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture – 02
Finite Difference Methods

Basic Ideas of Discretization – 1

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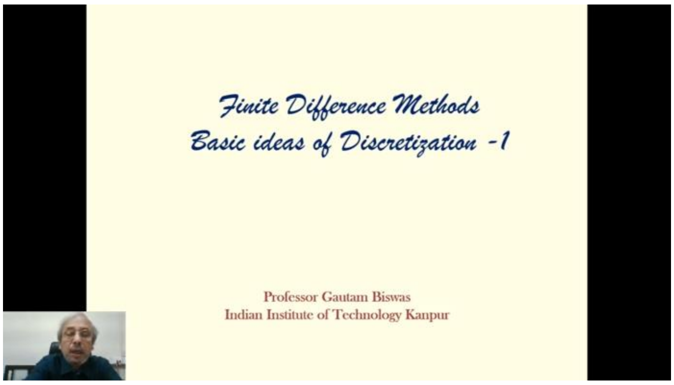

Good afternoon, everybody. Today's lecture we will start, if you recall in the last class, we mentioned about our lesson plan; at the outset you will learn the basic methodology related to finite difference, finite volume and finite element techniques. We will apply these techniques to simplified transport equations and try to understand different characters of the problems and formulation strategies, having done that we will switch over to full-fledged solvers.

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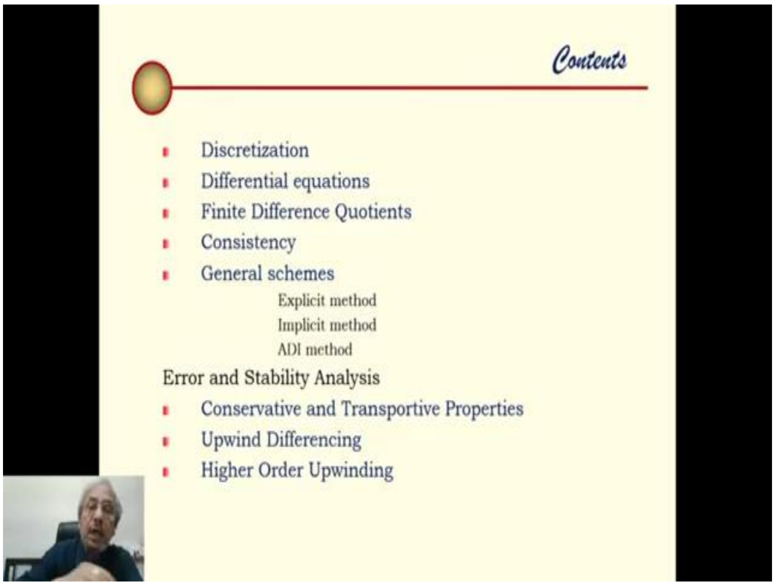

So, today we will start with Finite Difference Methods, in our plan we have you know the following: we will mention about Discretization Techniques; to convert each term of the differential equation into finite difference quotients. And then those quotients are assembly to describe different difference equations into the algebraic equations.

Then we will discuss about consistency, then we will take up some formulation strategies whether to formulate the problems in an explicit method or to follow implicit method or to follow ADI method; this is Alternate Direction Implicit method. While going through it we learn lot more about it, I am just mentioning it now.

Then we will go for error analysis, which we will calumniate into stability analysis. Then we will mention about, conservative and transportive properties of the equations when the differential equations are converted into difference equations. How do they preserve conservative property and transportive property, what can be a better strategy to preserve all these properties?

Then we will mention about upwind differencing and higher order Upwinding, and this you know we will do with the finite volume and finite element also following finite difference strategies. And as I said then we will apply you know all this knowledge for developing full-fledged solvers.

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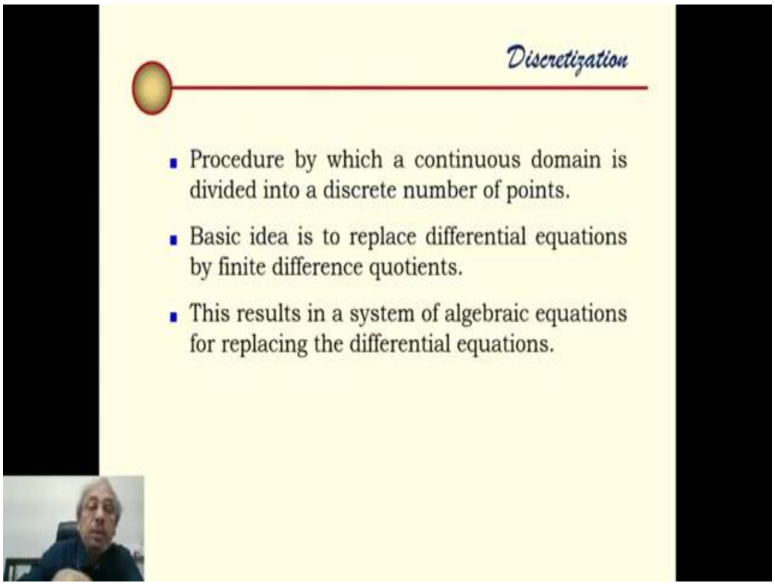

So, discretization procedure is a procedure by which a continuous domain is divided into a discrete number of points. Basic idea is to replace differential equations by finite difference quotients. This results in a system of algebraic equations for replacing the differential equations.

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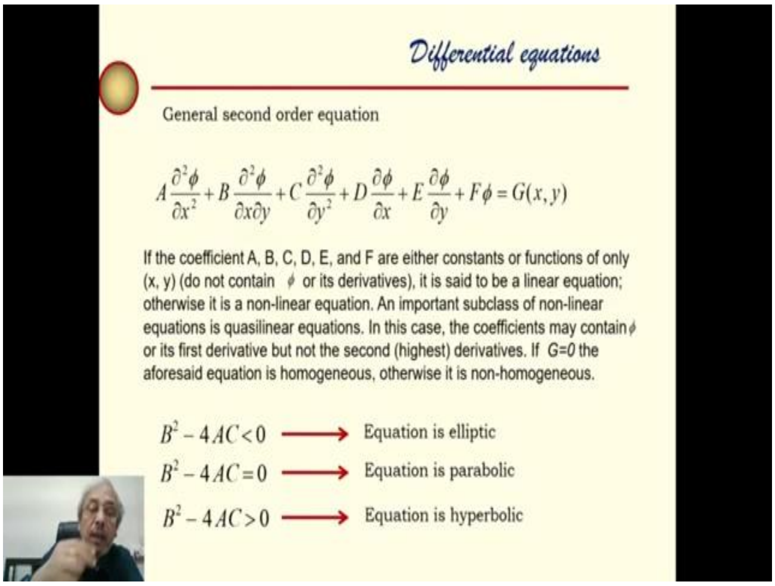

Now, if we have look at a general second order equation given here, as you can see

$$A\left(\frac{\partial^2\phi}{\partial x^2}\right)+B\left(\frac{\partial^2\phi}{\partial x\partial y}\right)+C\left(\frac{\partial^2\phi}{\partial y^2}\right)+D\left(\frac{\partial\phi}{\partial x}\right)+E\left(\frac{\partial\phi}{\partial y}\right)+F\phi=G(x,y)$$

If the coefficients A, B, C, D, E and F are either constants or functions of only x and y; that means, they do not contain ϕ or its derivatives it is said to be a linear equation.

Otherwise, it is a non-linear equation. An important subclass of non-linear equations is quasilinear equations. In this case coefficients may contain phi or its first derivative, but not the second derivatives. If $G = 0$ the aforesaid equation is homogeneous, otherwise it is non-homogeneous.

Now, from this A, B, C, D, E and F we can determine the character of the equation.

If $B^2 - 4AC < 0$ then the equation is elliptic.

If $B^2 - 4AC = 0$ equation is parabolic.

If $B^2 - 4AC > 0$ then the equation is hyperbolic. Depending on these characters of the equations we have to choose different solution strategies.

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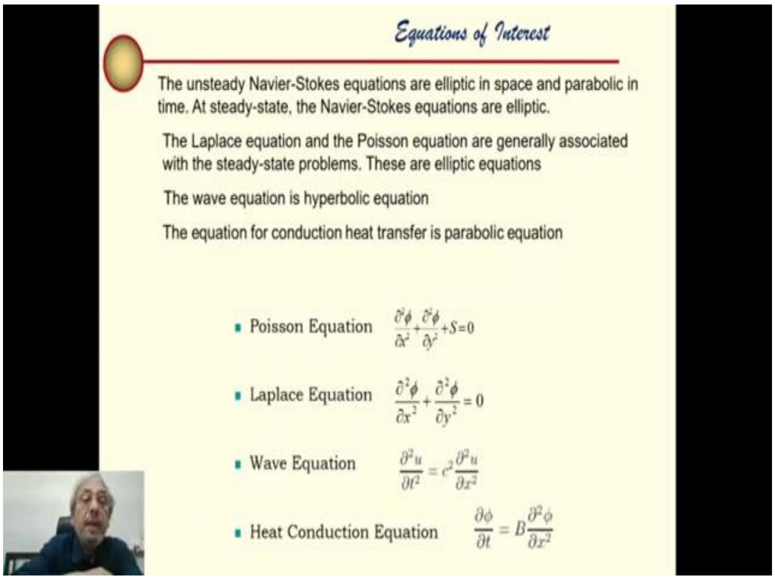

The unsteady Navier Stokes equations as you know these are basically three equations in X, Y and Z directions these are elliptic in space and parabolic in time. At steady state, the Navier Stokes equations are elliptic. The Laplace equation and Poisson equation are generally associated with steady state problem. I have given here example of Laplace equation and Poisson equation.

Now, it is very clear that space variables are involved in it and these are steady state problems and also character wise these are elliptic equations. Next you can see wave equation; wave equation is hyperbolic equation. We will discuss lot more about it in course of our discussions.

And then I have given very simply heat conduction equation, it is you know basically if ϕ will be substituted by temperature T if it becomes its heat conduction equation. And B will be substituted by conductivity of the material, it means conductivity of the material. So, this is basically 1 dimensional heat conduction equation which is parabolic in nature.

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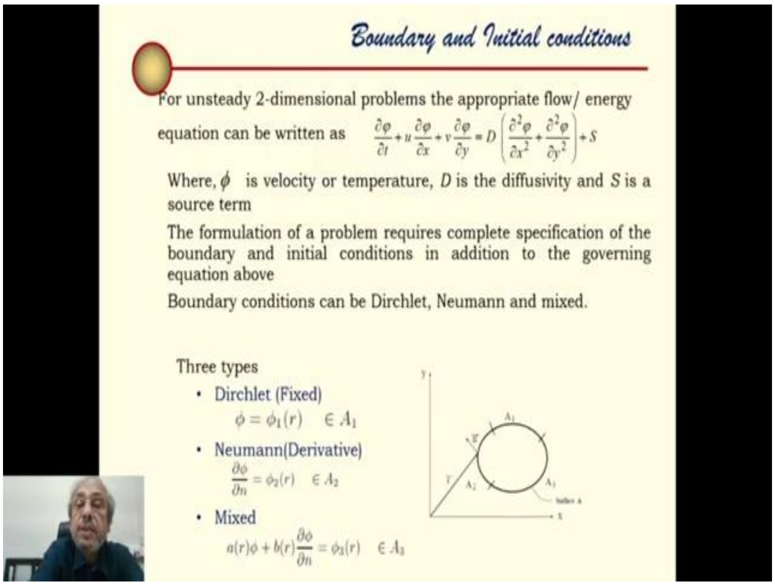

Now, for unsteady two-dimensional problems, which can be described by the equation given here, either flow problem or a heat transfer problem then it will be energy equation. You can see in equation below

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = D\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + S$$

Where, ϕ is a velocity of the field in this direction it is two-dimensional equation. So, basically ϕ will be either u or v or it will be temperature if it is energy equation. D is the diffusivity and S is a source term; if sources are present as a special case. The formulation of a problem requires complete specification of the boundary and initial conditions in addition to the complete equation.

Boundary conditions can be Dirichlet, Neumann and mixed. Dirichlet means, the value of the dependent variable is directly defined as you can see in this example that $\phi = \phi_1(r)$ and in it is defined on A_1. Then Neumann it is basically the derivative in the normal direction as you can see it is defined on A_2 and then also it is possible to have a condition, where Dirichlet and Neumann condition are clubbed together it is called mixed condition, and here you know it is on A_3.

So, Neumann is on A_2, I showed this side Neumann is on A_2; A_2 is turned on this side, Dirichlet is on A_1 and mixed is on A_3. So, this is how the boundary conditions are defined.

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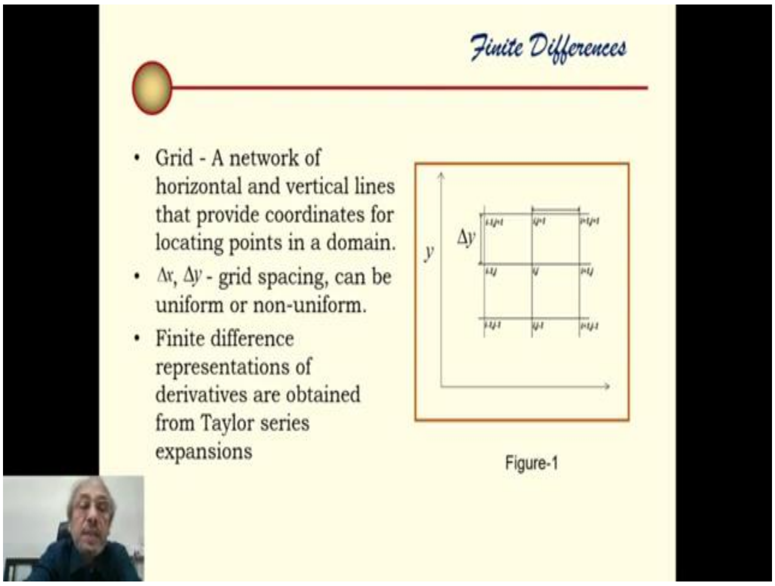

Now, let us imagine a very simple network with horizontal and vertical lines that provide coordinates for locating points in a domain. This is the most important point that the points in a domain should be defined through their coordinates and in order to know that coordinate you may have to draw some such vertical and horizontal lines in cartesian coordinate system.

In cylindrical polar again this will be constant theta and constant r lines. Similarly in spherical polar system also we will have the respective grid system. Now, Δx and Δy are the grid spacing which can be uniform or non-uniform both here we have taken the uniform grid as you can see this is allocation of (i, j) point.

If we move Δx towards east coordinate of this point will be $(i + 1, j)$. If we move Δx towards west then coordinate of this point will be $(i - 1, j)$. If we move Δy in a north, then coordinate of this point will be $(i, j + 1)$. If we move Δy in south then the coordinate of this point will be $(i, j - 1)$. So, finite difference representations of the derivatives obtained on all these points using Taylor series expansion.

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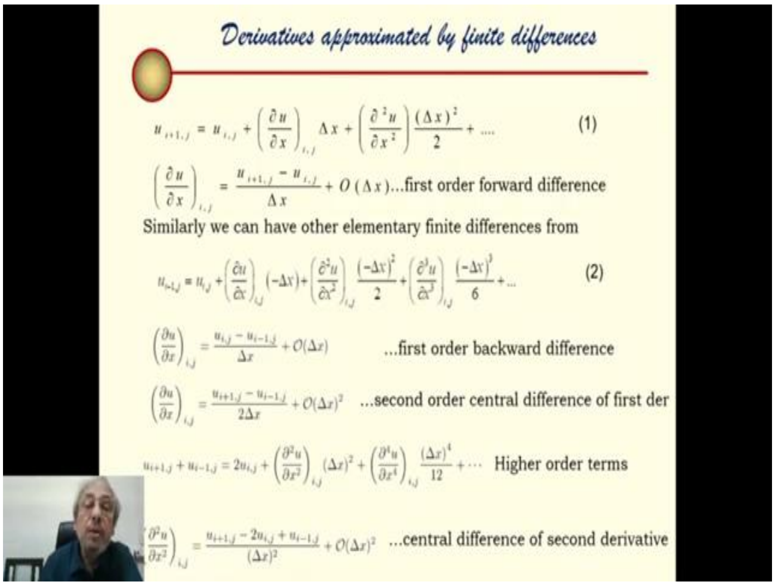

Very simple then again if we say u is a variable dependent variable distributed over this domain and value of u at (i, j) point is known then value of u at $(i, j+1)$ point can be known by using Taylor series expansion. So,

$$u_{i+1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2} + \cdots \tag{1}$$

You know, all of us know that this is Taylor series expansion.

But from here it is possible to find out value of $\left(\frac{\partial u}{\partial x}\right)_{i,j}$, as you can see $\left(\frac{\partial u}{\partial x}\right)_{i,j}$ is $\frac{u_{i+1,j} - u_{i,j}}{\Delta x}$ plus all other terms if we shift to the other side because we are trying to find out $\left(\frac{\partial u}{\partial x}\right)_{i,j}$.

So, if everything is shifted to the other side then after having this expression $\left(\frac{u_{i+1,j} - u_{i,j}}{\Delta x}\right)$ the remaining terms are basically neglected, but you know we can say if you look at the remaining terms, this term $\left(\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{\Delta x}{2}\right)$ will be having highest contribution after going to the other side. It will contain higher order derivative and you know terms of some order.

So, here it is called first order because it is order of Δx, the Δx with $\left(\frac{\partial u}{\partial x}\right)_{i,j}$ will divide then all the terms on the other side. So, the term $\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2}$ will become $\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{\Delta x}{2}$. So, and this $\left(\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{\Delta x}{2}\right)$ will be the highest contribution from the terms that will be neglecting in the Taylor series. So, this is called forward difference because the point of interest is in the forward direction. And the truncated series is having order of Δx. So, this is called first order forward difference.

Similarly, we can have other elementary final finite differences if we take backward or real word difference; that means, a point $u_{i-1,j}$ which is in the westward direction from the point (i, j). So, then it will be:

$$u_{i-1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} (-\Delta x) + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(-\Delta x)^2}{2} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(-\Delta x)^3}{6} + \cdots \quad (2)$$

So, from here we can approximate $\left(\frac{\partial u}{\partial x}\right)_{i,j}$ from $u_{i-1,j}$ and $u_{i,j}$ if we do that since we have moved in the westward direction to get the Taylor series it is called backward difference, and you can just like the forward difference here also order of accuracy you can say it is first order accurate.

However, now if you take equation 1 and 2, and you subtract 2 from 1, and then try to find out $\left(\frac{\partial u}{\partial x}\right)_{i,j}$. So, obviously you will get this:

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right) + O(\Delta x^2)$$

So, $u_{i,j}$ from 1 and 2 will cancel out, you will get on the right-hand side $2\left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x$ and then if you want to find out the quotient $\left(\frac{\partial u}{\partial x}\right)_{i,j}$ you will be able to write $\left(\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}\right)$.

And again, after subtracting 2 from 1 the equation you get from there if you find out the expression for $\left(\frac{\partial u}{\partial x}\right)_{i,j}$ as $\left(\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}\right)$, you will have remaining terms which you can neglect. These remaining terms are having higher order derivatives and higher order terms and since you are neglecting them and the highest order of the neglected series is of order of Δx^2. So, it is called second order central difference, and as you can see it is accuracy is second order.

So, this is basically second order central difference of first order derivatives. Now, if you add up 1 and 2 you will see that you will get as a result on left hand side $u_{i+1,j} + u_{i-1,j}$, and on the right-hand side you added up, you will get $2u_{i,j} + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \Delta x^2 + \left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j} \frac{(\Delta x)^4}{12} + \cdots$, all even terms will get cancelled out and all odd terms will be there and then you will be able to write an expression $u_{i+1,j} + u_{i-1,j} = (2u_{i,j} + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \Delta x^2 + \left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j} \frac{(\Delta x)^4}{12} + \cdots)$, and from here you can easily write down $\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}$ as $\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}$,and all remaining term will be dropped. So, this quotient can be called as central difference of the second derivative.

So, this way whether first derivative or second derivative or third derivative whatever is the term in the finite difference equation. So, differential equation you can get finite difference equivalent of that equation through these quotients.

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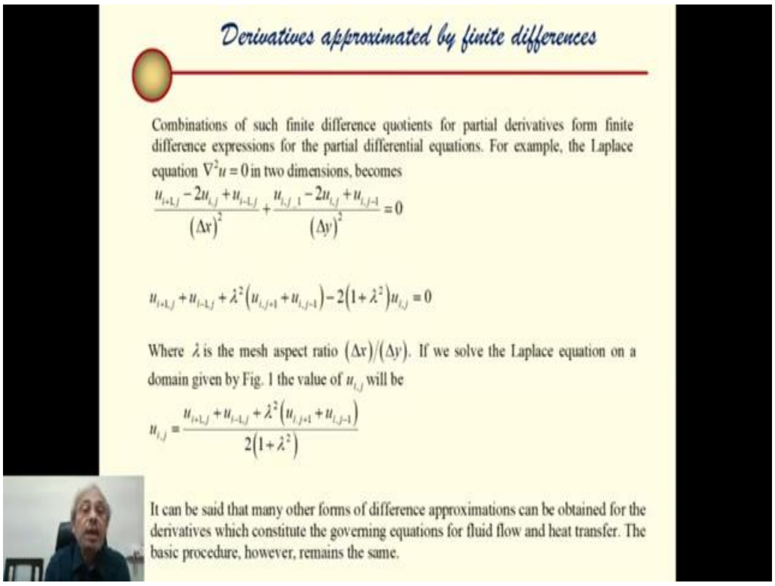

Now, combination of such finite difference quotients for partial derivatives forms finite difference expressions for the partial differential equations. For example, if we consider Laplace equation; that means, $\nabla^2 u = 0$. So, double derivative of u in X direction plus double derivative of u in Y direction equal to 0. And then $\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}$ we have substituted by this $\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2}\right)$ quotient $\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j}$ we have substituted by this $\left(\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}\right)$ quotient.

Now, we can perform simple algebra from here we can write down

$$u_{i+1,j} + u_{i-1,j} + \lambda^2(u_{1,j+1} + u_{i,j-1}) - 2(1+\lambda^2)u_{i,j} = 0$$

Where, λ is aspect ratio of the grids. If Δx and Δy are same then λ is 1, but if they are different then λ will be having some value depending on the ratio of Δx and Δy.

Now, from here we can again simplify and get an expression for $u_{i,j}$. If you go back the figure 1 (Refer Slide Time: 11:31) where, (i, j) template is there $(i+1, j)$, $(i-1, j)$, $(i, j+1)$, $(i, j-1)$ etc. There you can see that $u_{i,j}$ is now expressed in terms of neighboring nodes and then divided by $2(1+\lambda^2)$ if λ is 1 then it will be 4 and it will be simple average of all the neighboring values of neighboring nodes.

So, that means, if you have some initial distribution of u known in the domain, and then maybe the domain is subjected to some change maybe you have started heating from one end, you have started cooling from one end and you want to find out and domain is may be conducting material one to find out final temperature of that domain. So, that will be equivalent to $u_{i,j}$ at each point.

So, at each point it will be the basically function of 4 neighboring nodes divided by $2(1+\lambda^2)$ and this way if you can vary i location from maybe $i = 1$ to maximum number of points you have in the domain $j = 1$ from maximum number of points you have in the domain maybe 1 to 100 for i and 1 to 100 for j.

Then every $u_{i,j}$ you will be able to calculate in terms of its full neighbors, and if the values are changing you will be able to update $u_{i,j}$ and this way iteratively if you keep on calculating at one point of time you will see that these values of $u_{i,j}$ are not changing anymore it has reached a steady state; that will be the solution of this $\nabla^2 u = 0$.

This equation, the example I gave there, u is main temperature. So, you will get the final temperature distribution on that domain. So, it can be said that many other forms of difference approximations can be obtained for the derivatives which constitute the governing equations for fluid flow and heat transfer the basic procedure; however, remains the same.

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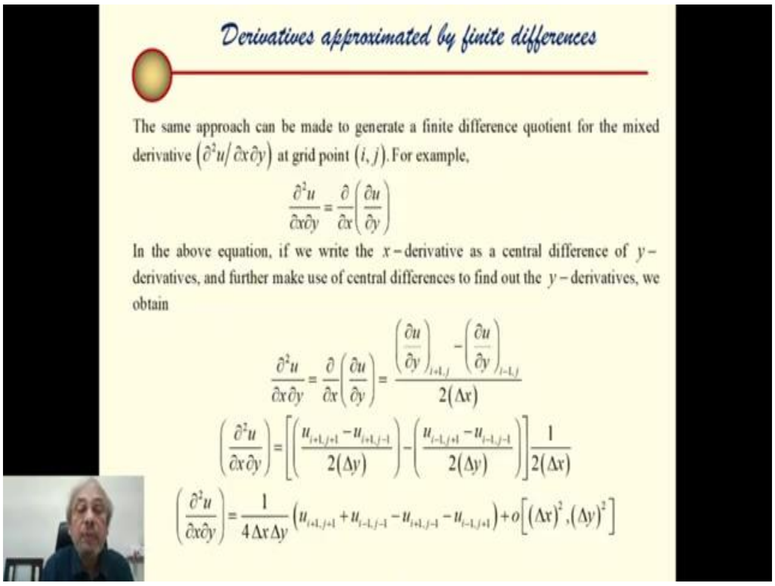

So, here I will give an example of a mixed derivative. The same approach can be made to generate the finite difference quotient for the mixed derivative say $\left(\frac{\partial^2 u}{\partial x\partial y}\right)$. So, we can write $\left(\frac{\partial^2 u}{\partial x\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$. In this equation if we write x derivative as a central difference of y derivatives and further make use of central difference of x derivatives then we will be able to obtain the expression.

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{\left(\frac{\partial u}{\partial y}\right)_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)_{i-1,j}}{2(\Delta x)}$$

retaining second order accuracy. Now, these $\left(\left(\frac{\partial u}{\partial y}\right)_{i+1,j} \; and \; \left(\frac{\partial u}{\partial y}\right)_{i-1,j}\right)$ terms we will expand in the next step. So, these $\left(\left(\frac{\partial u}{\partial y}\right)_{i+1,j} \; and \; \left(\frac{\partial u}{\partial y}\right)_{i-1,j}\right)$ will be expressed as the $(i+1, j)$, so $i+1$ point has been fixed at this point we will find out y derivative.

So, we will go $j+1$; that means 1 point in the northward direction, $j-1$, one point in the southward direction and divide by $2(\Delta y)$. Same will do at $(i-1, j)$ point we will find out quotient of $\left(\frac{\partial u}{\partial y}\right)_{i-1,j}$ by writing down difference between the northern point, southern point divided by the distance between the northern point and southern point.

So, we can summarize this as

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{1}{4\Delta x \Delta y}\left(u_{i+1,j+1} + u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1}\right) + \mathcal{O}[(\Delta x)^2, (\Delta y)^2]$$

All neighboring points at the point of interest; that means, (i, j) we will be able to find out the quotient this way, and you can see this is second order accurate; second order accurate in X direction second order accurate in Y direction.

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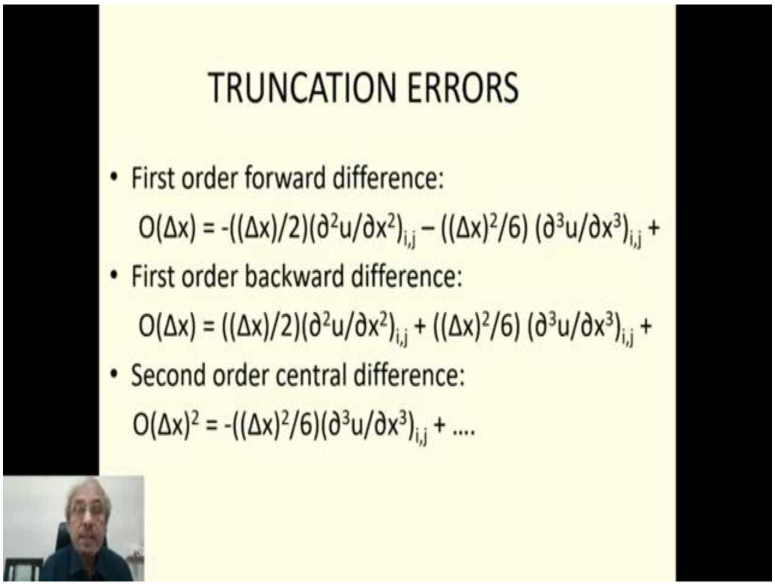

So, now let us look at the truncation errors. So, when we found out forward difference, we said that we will find out quotient and neglect the remaining terms of the series. So, if we neglect the remaining terms of the series in the first order forward difference, we try to recall we wrote $\frac{u_{i+1,j} - u_{i,j}}{\Delta x}$.

And there $\left(-\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{\Delta x}{2} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^2}{6} + \cdots\right)$ was the truncation series and highest order term in the truncated series is $\left(-\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{\Delta x}{2}\right)$ because next terms are even higher Δx you know $(\Delta x)^2$, $(\Delta x)^3$. So, obviously, these will be having less and less contributions.

So, $\left(-\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}\frac{\Delta x}{2}\right)$ will be having highest contribution. Similarly, first order backward difference we will get you know the truncated series and if you write the truncated series $\left(\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}\frac{\Delta x}{2}\right)$ term will have the highest contribution. So, this is the case of first order backward difference and in the case of central difference if we look at the truncated series then highest contribution is made by $\left(\left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}\frac{(\Delta x)^6}{6}\right)$ term which contains third order derivative and second order term (second order (Δx^2)).

So, these are the you know expressions for truncated series and the highest contribution in the truncated series and since these are being neglected. So, they contribute in errors and we will call them truncation errors.

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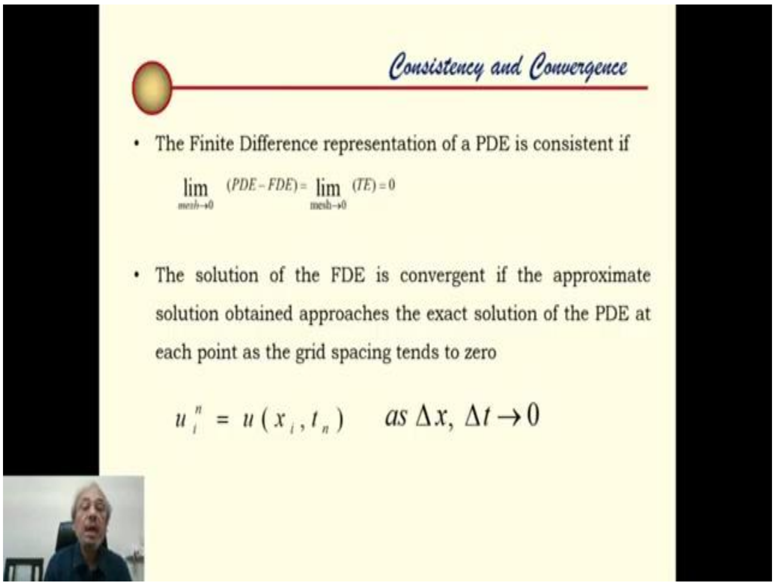

So, a Finite Difference representation of a partial differential equation will be consistent. If we certainly meet that partial differential equation minus finite difference equation, finite difference equivalent as mesh → 0 that means truncation error as mesh → 0 what it will be?

Now, this truncation error as mesh → 0 if it becomes 0, then it is basically consistent. The solution of partial different equation in addition to consistency this is another condition.

The solution of finite difference equation is convergent if the approximate solution that is obtained approaches the exact solution of the partial difference partial differential equation at each point as the grid spacing $\rightarrow 0$. These are almost equivalent statements this is qualified little further to give the convergence and this is called consistence or consistency or formulation is consistent.

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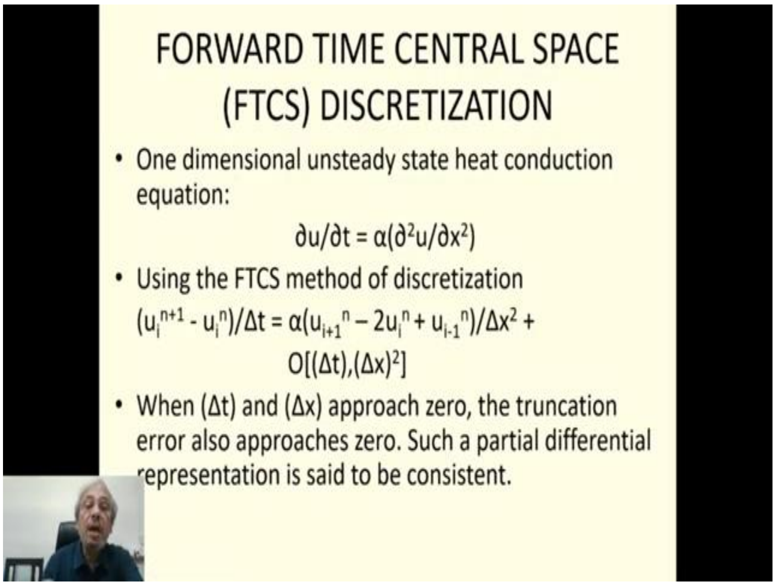

Now, let us look at different formulation strategies. One dimensional unsteady state heat conduction equation again we can write $\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2}\right)$. We have written it in a generic way 'u' usually used for flow variables, but you know we can here use it as temperature. Then alpha will be the basically thermal diffusivity or in this case basically you know thermal conductivity.

So, using FTCS method the discretization; that means,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha\left(\frac{u_{i+1}^n - 2u_i + u_{i-1}^n}{\Delta x^2}\right) + \mathcal{O}[(\Delta t), (\Delta y)^2]$$

you must have noticed that we are using n for the time direction for the time domain and i and j for the spatial direction. So, u_i^{n+1} in the time direction you know from the present time level to forward direction $n+1$. So, in X direction one point eastward, one point westward and the point of interest at n^{th} level; that means, to start with you know if the values are known, then you will be able to easily find out what is this.

And from there you can do the time marching of the solution in $n+1$ direction in the time direction. When Δt and Δx, these two approaches zero, the truncation error also approaches zero. Such a partial differential representation is said to be consistent that is what I mentioned earlier.

That you know if truncation error becomes zero then the formulation is consistent. Here we can collect the truncation errors and we can see if we set Δt and Δx to zero whether they are tending to zero or they are becoming zero. If they become zero then there is no problem it is a consistent formulation. If they become non zero then there is a problem it may not be a consistent formulation.

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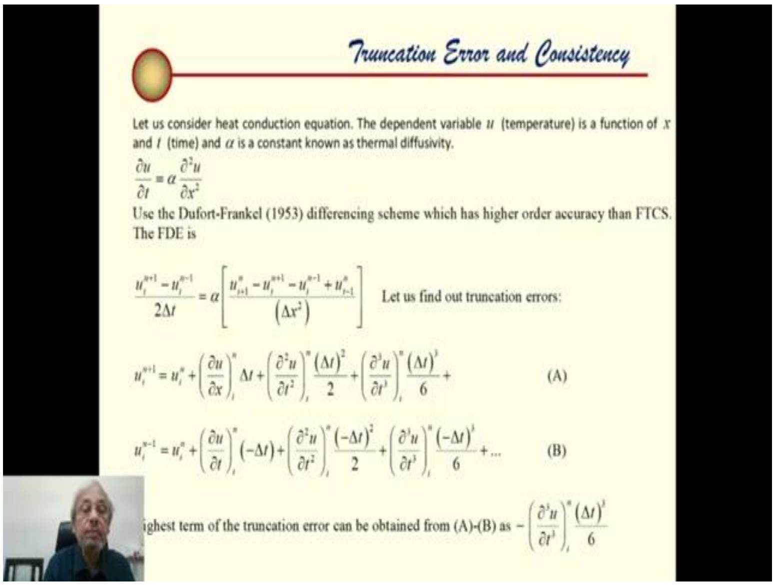

So, can it really happen? that we will discretize, assemble and the equation, that means, finite difference equivalent of the partial differential equation will be such that when the limits are tending towards zero the truncation error is not diminishing or not becoming zero.

Let us take up an example earlier example was FTCS scheme, which is called Forward Time and Central in Space. As we have mentioned that we can really feel the truncation error goes to zero, but let us look at another formulation strategy. Again, let us consider heat conduction equation here again u should be imagined as temperature and α is the thermal conductivity or thermal diffusivity. And basically our $\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2}\right)$ equation we discretized using FTCS method. Here we are going for a very well-known scheme although we have not introduced all these schemes to you, we will do it in course of time maybe through some assignments and you know through discussions as well.

Now, Dufort Frankel scheme this scheme is very much used for compressible flow problems and it is a higher order scheme than FTCS, formally it is second order in time and second order in space.

So, Dufort Frankel differencing scheme which has higher order accuracy than FTCS; we can write it and this is

$$\frac{(u_i^{n+1} - u_i^{n-1})}{2\Delta t} = \alpha \left[\frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{(\Delta x^2)}\right]$$

You can clearly see that in the time domain it is second order accurate. Let us not get confused what has been done here? Usually if we imagine these two $(-u_i^{n+1} - u_i^{n-1})$ terms, we will put it in a bracket take minus sign common then this is $(u_i^{n+1} + u_i^{n-1})$. So, $2u_i^n$ has been substituted by $(u_i^{n+1} + u_i^{n-1})$, originally $\left[\frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{(\Delta x^2)}\right]$ was central difference and supposed to have $u_{i+1}^n - 2u_i^n + u_{i-1}^n$ in this $2u_i^n$ has been substituted by u_i^{n+1} and u_i^{n-1}. That means, $2u_i^n = u_i^{n+1} + u_i^{n-1}$, and that substitute $2u_i^n$ then expression becomes $\left[\frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{(\Delta x^2)}\right]$ and this is yet increasing the order of accuracy. So, let us look into the truncation errors. So, simply first looked at the look at the left-hand side u_i^{n+1} you expand it in Taylor series as you can see,

$$u_i^{n+1} = u_i^n + \left(\frac{\partial u}{\partial t}\right)_i^n \Delta t + \left(\frac{\partial^2 u}{\partial t^2}\right)_i^n \frac{(\Delta t)^2}{2} + \left(\frac{\partial^3 u}{\partial t^3}\right)_i^n \frac{(\Delta t)^3}{6} + \cdots$$

this is there is a mistake $\frac{\partial u}{\partial x}\Delta t$ will be $\frac{\partial u}{\partial t}\Delta t$ remaining terms are correct $\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n \frac{(\Delta t)^2}{2} + \left(\frac{\partial^3 u}{\partial t^3}\right)_i^n \frac{(\Delta t)^3}{6}$. So, u_i^{n-1} is correctly written, again Taylor series $\left(\frac{\partial u}{\partial t}\right)_i^n$ please read this as $\frac{\partial u}{\partial t}$. So, the equation A. So, it is again usual Taylor series equation B. Now, if we subtract B from A to formulate $\frac{(u_i^{n+1} - u_i^{n-1})}{2\Delta t}$ (left-hand side quotient) will be able to do that, but they will be truncating the terms of higher order derivatives and higher order terms of the Taylor series and then the highest order term $-\left(\frac{\partial^3 u}{\partial t^3}\right)_i^n \frac{(\Delta t)^3}{6}$.

So; obviously, you can see that here basically u_i^n terms will be cancelled out and $\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n \frac{(\Delta t)^2}{2}$ term will be cancelled out because we are doing A - B. So, $\left(\frac{\partial^3 u}{\partial t^3}\right)_i^n \frac{(\Delta t)^3}{6}$ term will be there; that means, third order derivative term will contribute to the truncation error.

So, $-\left(\frac{\partial^3 u}{\partial t^3}\right)_i^n \frac{(\Delta t)^3}{6}$ will create maximum contribution in the truncation error. So, we have retained that term you can see $\left(\frac{\partial^3 u}{\partial t^3}\right)_i^n \frac{(\Delta t)^3}{6}$ this is the truncation error for the left-hand side temporal derivative.

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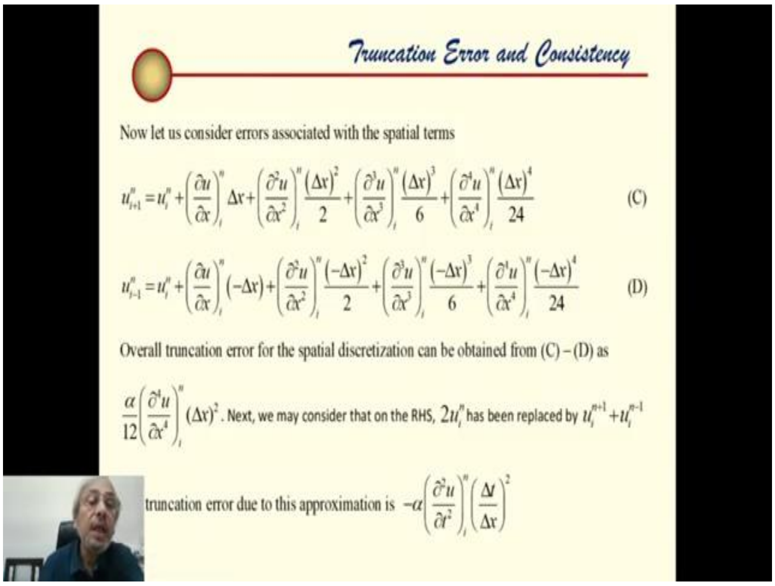

Then we have again written the spatial derivatives u_{i+1}^n at n^{th} level u_{i+1}^n at n^{th} level. In the spatial direction again, we have written the Taylor series expansion and we have tried to find out because here $\frac{(u_i^{n+1} - u_i^{n-1})}{2\Delta t} = \alpha \left[\frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{(\Delta x^2)}\right]$ truncation error if you consider this $(u_i^{n+1} + u_i^{n-1})$ as $2u_i^n$ then first truncation error you will be able to evaluate if you subtract expression of u_{i-1}^n from expression for u_{i-1}^n and that is how you know we get $\frac{\alpha}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n (\Delta x)^2$ as a truncation error. Then we have expressed $2u_i^n$ as $(u_i^{n+1} + u_i^{n-1})$ here; due to that we have introduced some error and that error basically easily can be found out by adding A and B and then you know since $\left(\frac{(u_i^{n+1} - u_i^{n-1})}{2\Delta t} = \alpha \left[\frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{(\Delta x^2)}\right]\right)$ in the right-hand side will be divided by $(\Delta x)^2$. So, this $\left(-\alpha \left(\frac{\partial^2 u}{\partial t^2}\right)_i^n \left(\frac{\Delta t}{\Delta x}\right)^2\right)$ will be the truncation error arising out of that.

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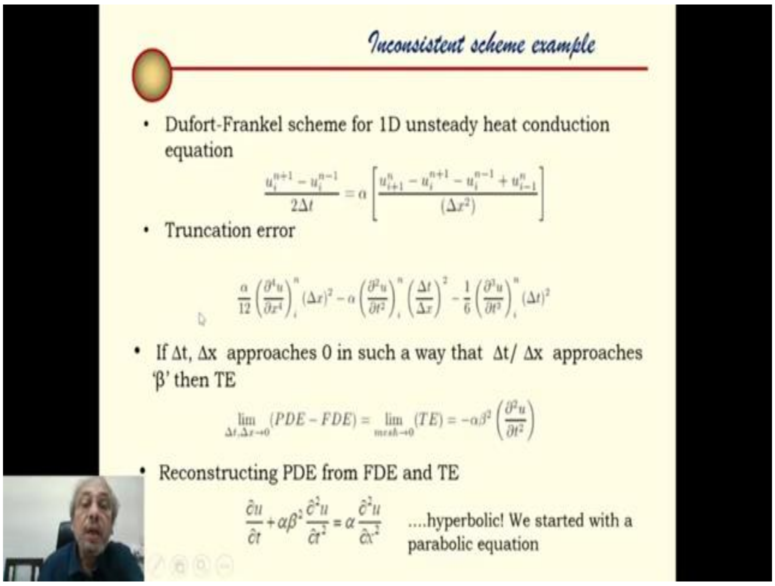

Now, if we combine them, we can see that this $\left(\frac{(u_i^{n+1}-u_i^{n-1})}{2\Delta t}=\alpha\left[\frac{u_{i+1}^n-u_i^{n+1}-u_i^{n-1}+u_{i-1}^n}{(\Delta x^2)}\right]\right)$ is the representation of finite difference equivalent of the partial differential equation using Dufort Frankel scheme. And the truncation error we have already calculated individual truncation error of the finite difference quotients and if we add them up you will get $\frac{\alpha}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n(\Delta x)^2-\alpha\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n\left(\frac{\Delta t}{\Delta x}\right)^2-\left(\frac{\partial^3 u}{\partial t^3}\right)_i^n\frac{(\Delta t)^2}{6}$.

Now, we are lucky if when we shrink the grid size and when we; that means, we make the grid refinement and in the time domain also we refine the time step and both approach 0; in such a way that this entire truncation error approaches 0 then we are lucky life is simple, but it may so, happen because this Δt and Δx are different independent variables.

So, if we approach them in such a way that Δx approaches 0 faster than Δt. Then we will see that instead of this term $(\frac{\alpha}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n(\Delta x)^2-\alpha\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n\left(\frac{\Delta t}{\Delta x}\right)^2-\left(\frac{\partial^3 u}{\partial t^3}\right)_i^n\frac{(\Delta t)^2}{6})$ becoming 0 it will be a nonzero term. Since, we are getting $\frac{0}{0}$ form so, there is a possibility for some conditions, that this entire truncation error will not converge to 0 instead of it will be something which is nonzero. And although we are unable to see it mathematically or you know computationally because we have already written finite difference quotients and from the you know finite difference equations of the partial differential equation, we have you know set the initial condition, we have given the boundary condition, we have solved it, but what we are solving we do not know if the formulation is not consistent, because this truncation error supposed to be 0.

If this remains nonzero the way Δt and Δx are changing that makes a situation where truncation error is nonzero. Then effectively we are solving this $\left(\frac{\partial u}{\partial t} + \alpha\beta^2 \frac{\partial^2 u}{\partial t^2} = \alpha\left(\frac{\partial^2 u}{\partial x^2}\right)\right)$ equation because original equation was $\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2}\right)$, but we have got $\alpha\beta^2 \frac{\partial^2 u}{\partial t^2}$ term here as truncation error.

So, if we add them up together, we will get you know some such expression our basic idea is that in order to have a consistent formulation the truncation error will be 0 and the finite difference equation will become partial differential equation exactly here it is not becoming that leaving behind some nonzero terms.

So, and if you try to characterize this equation you will see this is hyperbolic equation reconstructed equation is hyperbolic. So, original equation was parabolic. So, this is not a consistent formulation.

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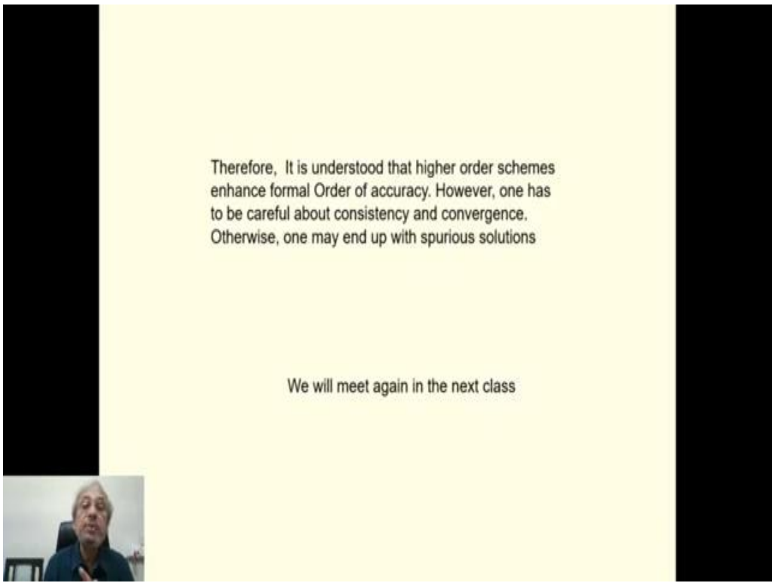

So, therefore, it is understood that higher order schemes enhance formal order of accuracy. It is very well appreciated and we will do that we will try to increase formal order of accuracy, but one is to be careful the formulation should be done in such a way that it should be consistent.

So, one has to be careful about consistency and convergence if it is not consistent, we will have trouble in getting convergence in computer and even if we get convergence, it will be a spurious solution. So, as I said written here one may end up with spurious solution.

So, in conclusion it can be said that finite difference formulation is apparently easy you have the governing equations, you substitute term by term of the governing equations by finite difference quotients, you get the finite difference equations. You apply initial condition apply boundary conditions and then solve it you get the solution after the convergence is achieved.

In order to increase the accuracy, you go for higher discretization of course, with simply 1 dimensional heat conduction problem it is very simple you do not have many terms and you do not have you know many equations, but if it is full Navier Stokes equations or if it is you know basically even energy equation in 3 dimensions then you will see you will have a truncated series and in that truncated series combined truncated series you know all the individual quotients will contribute the error.

And after combining them when you are refining the grid; that means, in principle you are trying to give a solution of the finite difference equation which is equivalent to the exact solution of the partial differential equation. You will see you are not you know consistent either problem is not converging or even the problem converges the result is spurious. So, one has to keep in mind this aspect of finite difference formulation. I will stop here today, we will meet again in the next class.