Computational Fluid Dynamics and Heat Transfer
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Lecture - 19
Solution of N-S equations for Incompressible Flows Using MAC Algorithm

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Solution of the Unsteady Navier-Stokes Equations

The MAC method of Harlow and Welch is one of the earliest and most useful methods for solving the Navier-Stokes equations. This method necessarily deals with the momentum equations for the computation of velocity and a Poisson equation for the pressure.

The text discusses the modified MAC method (Hirt and Cook, 1972) and highlights the salient features of the solution algorithm so that the reader is able to write a computer program with some confidence. The important ideas on which the MAC algorithm is based are:

1. Unsteady Navier-Stokes equations for incompressible flows in weak conservative form and the continuity equation are the governing equations.
2. Description of the problem is elliptic in space and parabolic in time. Solution is marched in the time direction. At each time step, a converged solution in space is obtained but this converged solution at the a time step may not be the final solution of the physical problem.

Good morning, everyone, today we will start Solution of Navier Stokes equations for Incompressible Flows Using MAC Algorithm. I have already explained MAC stands for marker and cell and this algorithm was developed in the Los Alamos National Laboratory of the United States, primarily by the research group of Professor Harlow and Professor Welch.

And the version which we will study here is called sometimes simplified MAC method and this was developed by Professor Hirt and Professor Cook in 1972. Now, this algorithm is widely used and quite easy to learn and implement it for solving the problems that we usually face in our research parlance or for solving industrial problems.

Now, this uses unsteady Navier Stokes equations for incompressible flows and we use weak conservative form and continuity equation. Description of the problem is elliptic in space and parabolic in time. Solution is marched in the time direction. At each time step, a converged solution in space is obtained, but this converged solution at a time step may not be the final solution of the physical problem.

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Solution of the Unsteady Navier-Stokes Equations

MAC method

3. If the problem is steady, in its physical sense, then after a finite number of steps in time direction, two consecutive time steps will show identical solutions. However, in a machine-computation this is not possible hence a very small upper bound, say "STAT" is predefined. Typically, STAT may be chosen between 10^{-3} and 10^{-5}. If the maximum discrepancy of any of the velocity components for two consecutive time steps at any location over the entire space does not exceed STAT, then it can be said that the steady solution has evolved.

4. If the physical problem is basically unsteady in nature, the aforesaid maximum discrepancy of any dependent variable for two consecutive time steps will never be less than STAT . However, for such a situation, a specified velocity component can be stored over a long duration of time and plot of the velocity component against time (often called as signal) depicts the character of the flow. Such a flow may be labeled simply as "unsteady".

If the problem is steady in its physical sense; then after a finite number of steps in time direction, two consecutive time steps will produce identical solutions. However, in a machine computation, this is not possible; because numbers everywhere, every location between two-time steps exactly same up to maybe 16th, decimal place or 10 decimal place or sometimes 20th decimal place, it is not possible.

So, there will be a small difference even if it is converged. So, that is why we specify a very small number as the upper bound, let us call it a variable STAT. Specifically, STAT may be chosen between 10 to the power minus 3 and 10 to the power minus 5 that is good enough. If the maximum discrepancy of any of the velocity components for two consecutive time steps at any location over the entire space does not exceed STAT, then it can be said that the steady solution has evolved.

That means, at each location whatever values we get, at the same locations the values that we get in the subsequent time step, they are same. And if there is any difference, that difference is less than say predefined value, which may be 10 to the power minus 4 or 10 to the power minus 5.

If the physical problem is basically unsteady in nature, the aforesaid maximum discrepancy of any dependent variable for two consecutive time steps will never be less than STAT. So, it will you know keep on varying the difference. However, for such a situation, a specified velocity component can be stored over a long duration of time and plot of such you know velocity component against time, you know will give us lot of important information.

These are often you know such storage of velocity over time is called signal. So, you know and the signals are very useful in determining the different characteristics of flow; whether it is unsteady periodic or you know unsteady chaotic or you know it is you know transitional flow, all such things can be depicted very nicely from such signals, which are basically collection of velocity components at specified locations over time.

Now, but at this stage let us call it, when this you know maximum discrepancy of velocity at any time location is not falling below a specified predefined small value; then the flow is oscillating, it is basically we can call such flows as unsteady flows.

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Solution of the Unsteady Navier-Stokes Equations

MAC method

5. With the help of the momentum equations, we compute explicitly a provisional value of the velocity components for the next time step.

Let us start with the continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (1)$$

Consider the weak conservative form of the non dimensional momentum equation in the x-direction:

$$\frac{\partial u}{\partial t}+\frac{\partial (u^2)}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial (uw)}{\partial z}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\nabla^2 u$$

It is assumed that at t= n th level, we have a converged solution. Then for the next time step we can write:

$$\tilde{u}^{n+1}_{i,j,k}=u^{n}_{i,j,k}+\Delta t\,[CONDIFU-DPDX]^{n}_{i,j,k} \qquad (2)$$

Then with the help of momentum equations, we compute explicitly a provisional value of the velocity components for each time step. And then from that provisional value, we get the real value or the converged value or final value for that particular time step. Now, when we discuss about the flow equations, obviously the first equation that comes continuity equation, then let us take up x momentum equation.

And as you can see, we have taken x momentum equation in weak conservative form;

$$\frac{\partial u}{\partial t}+\frac{\partial (u^2)}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial (uw)}{\partial z}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\nabla^2 u$$

It is assumed that at time t equal to nth level, we have a converge solution. This can be initial solution also or initial values can be you know used in such a way that, you know that can be considered as the nth level solution. Then the next time step. So, basically, we progress from n to n+1, n+1 to n+2 so on and so forth.

So, when we have converged value of the velocity field and a pressure field, we have the converged values everywhere at n^{th} level. From there for the next time step, we can write for u component of velocity; we have started with x momentum equation, so obviously u component of velocity $u_{i,j,k}$ at a point i, j, k at the next time level that is n+1. And we have used another sign tilda; because there is a reason, this is not really n+1 as I said, a provisional value.

Therefore, to identify its provisional state, we have used this tilda value. So, this is equal to $u_{i,j,k}$ at n^{th} level plus Δt into CONDIFU, which is convection diffusion a part of U velocity minus DPDX. Just the way it is written ($\tilde{u}_{i,j,k}^{n+1} = \tilde{u}_{i,j,k}^{n+1} + \Delta t[CONDIFU - DPDX]_{i,j,k}^{n}$) there, you can see we have just you know numerically transferred those values; $\partial u/\partial t$ means, u at n+1 minus u at n, by Δt.

So, u at n plus 1, there will be u at n, Δt; if we transfer all these components on the other side, Δt multiplied by all these terms. Now, among these terms, this term these first three terms are convective terms and terms associated with 1/Re are diffusive terms. So, delta t into convective and diffusive terms of u momentum equation, that is why you have written CONDIFU minus DPDX, all these values are evaluated at i, j, k point at a time level n at n^{th} level.

Those multiplied by t plus $u_{i,j,k}$ at nth level is $u_{i,j,k}$ at n plus 1th level provisional; basically $\partial u/\partial t$ is $u_{i,j,k}$ at n plus 1 minus $u_{i,j,k}$ at n by Δt equal to all these terms which are we are calling as CONDIFU and DPDX. So, $u_{i,j,k}$ at n plus 1 is basically, then Δt into all these terms plus $u_{i,j,k}$ at n. And as I say that, when we write this way or when we calculate this way; this velocity is projected velocity at $n+1^{th}$ level, but not really the final velocity at the level n+1.

Why, I will explain shortly. And since this is provisionally calculated, not converged as yet; we will call this u as provisional u and that is why together with n+1, we will use this tilde. So, this is equation 2.

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$[CONDIFU - DPDX]^n_{i,j,k}$ consists of convective and diffusive terms, and the pressure gradient. Similarly, the provisional values for $\tilde{v}^{n+1}_{i,j,k}$ and $\tilde{w}^{n+1}_{i,j,k}$ can be explicitly computed. These explicitly advanced velocity components may not constitute a realistic flow field. A divergence-free velocity field has to exist in order to describe a plausible incompressible flow situation. Now, with these provisional $\tilde{u}^{n+1}_{i,j,k}$, $\tilde{v}^{n+1}_{i,j,k}$ and $\tilde{w}^{n+1}_{i,j,k}$ values, continuity equation is evaluated in each cell. If $(\nabla \cdot V)$ produce a nonzero value, there must be some amount of mass accumulation or annihilation in each cell which is not physically possible.

Here we have explained CONDIFU minus DPDX at you know point i, j, k at the level n consists of convective and diffusive terms and the pressure gradient. Similarly, the provisional values of $v_{i,j,k}$ at the level n plus 1 tilde provisional and $w_{i,j,k}$ at the level n+1 again provisional tilde can be explicitly computed. I will go back by one slide.

So, those u momentum equations can be casted in this mold, and v momentum, w momentum equation can be also formatted in this way to calculate the provisionally advanced v velocity and w velocity, the way we have calculated provisionally advanced u velocity v velocity and w velocity. These explicitly advanced velocity components may not constitute a realistic flow field. A divergence free velocity field has to exist in order to describe a plausible incompressible flow situation.

Now, with this provisional value of u, v and w, continuity equation is evaluated in each cell. If, so continuity equation divergence of V, this v is vector V signifying u, v and w all three components. So, if divergence of V produces a non-zero value; there must be some amount of mass accumulation or annihilation in each cell which is not physically possible.

So, this, why we are calling these velocities as provisionally advanced u or provisionally advanced v or provisionally advanced w? Because we do not know, whether they satisfy continuity equation and we will see indeed; they do not after this provisional advancement, they do not really satisfy continuity equation. So, non-zero continuity means, mass annihilation or mass accumulation which is not possible.

So, we have to reach a situation when divergence of V=0; then only these provisional velocities will be the velocity, at velocities at the level n+1.

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Solution of the Unsteady Navier-Stokes Equations

Therefore, the pressure at any cell is directly linked with the value of the $(\nabla \cdot V)$ of that cell. Now, on one hand the pressure has to be calculated with the help of the nonzero divergence value and on the other, the velocity components have to be adjusted. The correction procedure continue through an iterative cycle until the divergence-free velocity field is ensured. Details of the procedure will be discussed in the subsequent section.

6. Boundary conditions are to be applied after each explicit evaluation for the time step is accomplished. Since the governing equations are elliptic in space, boundary conditions on all confining surfaces are required. Moreover, the boundary conditions are also to be applied after every pressure-velocity iteration. The five types of boundary conditions to be considered are rigid no-slip walls, free-slip walls inflow and outflow boundaries, and periodic (repeating) boundaries.

Therefore, the pressure at any cell is directly linked with the value of divergence V of that cell. Since divergence is not satisfied, you cannot accept pressure of that cell as converge pressure or the final pressure. Now, on one hand, pressure has to be calculated with the help of non-zero divergence value and on the other, the velocity components have to be adjusted.

The correction procedure continue through an iterative cycle until the divergence free velocity field is ensured. Detailed calculation will be acquainted with slightly later how to go about that. Now, I have to mention about boundary conditions too. So, boundary conditions are to be applied after each explicit evaluation of the time step.

So, we go from, when you go from one time step to another; obviously you have to apply boundary conditions on all confining surfaces. Since the governing equations are elliptic in space, boundary conditions on all confining surfaces are required. Moreover, the boundary conditions are to be applied after every pressure velocity iteration.

Not only after advancing from one time step to the next time step; but when we calculate I have mentioned, we have to correct velocity and we have to correct pressure and this is an iterative process. And each every and for every iteration, we have to apply the boundary conditions; because boundary conditions have to be satisfied for a converged flow field.

So, we will discuss five types of boundary conditions: rigid no slip walls, free slip walls, inflow plane, outflow boundaries and the periodic boundaries; we will specifically discuss all this.

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MAC Formulation

The region in which computations are to be performed is divided into a set of small cells having edge lengths δx, δy and δz as shown in Figure 1:

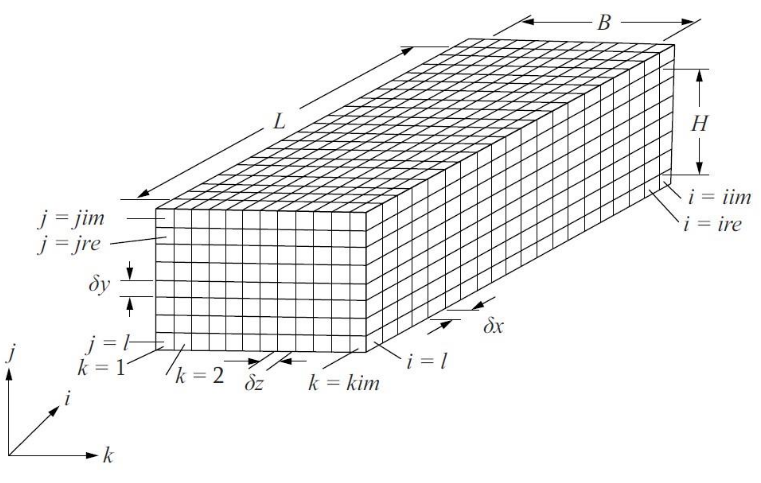

Figure 1: Discretization of a three -dimensional domain

The region in which computations are to be performed is divided into a set of small cells having edge length δx, δy, and δz shown in this figure; you can see that this δx's form the length. So, it extends from i = 1 to i = maximum value iim.

The last, but one cell is called ire; here also the second cell is equal to 2. Why this ire has been specifically used, we will discuss little later. So, in the x direction, which covers l dimension; we will go from i equal to 1 to iim. Similarly, in the y direction we go from j = 1 to jim maximum number of cells. And in the z direction, that is varies with k, k direction will go from k = 1 to kim.

And each small segment in z direction is δz, each small segment or you know the width cell width in y direction is δy and x direction is δx. So, this is how we discretize a domain.

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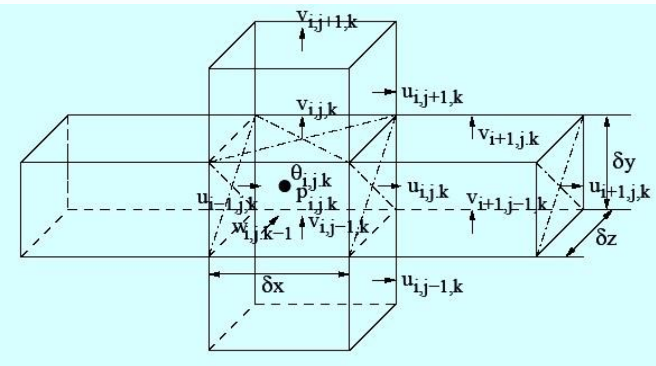
The region in which computations are to be performed is divided into a set of small cells having edge lengths $\delta x, \delta y$ and δz (Fig. 1). With respect to this set of computational cells, velocity components are located at the centre of the cell faces to which they are normal and pressure and temperature are defined at the centre of the cells. Cells are labeled with an index (i, j, k) which denotes the cell number as counted from the origin in the x, y and z directions respectively. Also $p_{i,j,k}$ is the pressure at the centre of the cell (i, j, k), while $u_{i,j,k}$ is the x-direction velocity at the centre of the face between cells (i, j, k) and $(i+1, j, k)$ and so on (Fig. 2).



The region in which computations are to be performed divided into a set of small cells having edge lengths δx, δy and δz we have already seen that. With respect to this set of computational cells, velocity components are located at the center of the cell faces to which they are normal; I have already mentioned about staggered grid and the location of the variables.

So, let me repeat it again; velocity components are located at the center of the cell faces to which they are normal, and pressure and temperature are identified at the center of the cells. Cells are labeled with an index i, j, k which denotes the cell number as counted from the origin in the x, y and z directions. So, from the origin in the x, y and z direction, we make progress by δx, δy, δz and cover the full domain.

So, basically which cells are labeled with an index i, j, k, which denotes the cell number as counted from the origin in the x, y and z directions respectively. Also, $p_{i,j,k}$ is the pressure as at this center of the cell i, j, k, while $u_{i,j,k}$ is the x direction velocity at the center of the cell face between the cells i, j, k and i+1, j, k, which is explained in this figure.

See this is the central cell one, we can call this as i, j, k cell. And at the center of the cell, we can see that $p_{i,j,k}$ has been defined; we will also calculate later temperature, that is why $\theta_{i, j, k}$ has also been defined at the center of the cell. So, this is i, j, k cell and this eastern neighbor is i+1, j, k cell three-dimensional cell. And what we have said here that, $u_{i,j,k}$ in the x direction velocity at the center of the cell face between the cells i, j, k and i+1, j, k.

So, this is i, j, k cell, this is i+1, j, k cell; between these two cells, this is a cell wall and at the center $u_{i,j,k}$ has been defined. Similarly, look at this cell i, j, k and this is i, j+1, k cell. So, between these two cells at the center on the cell face of i, j, k at the center $v_{i,j,k}$ has been defined. So, this is how the velocities and the pressures are located on the grid mesh.

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MAC Formulation

Because of the staggered grid arrangements, the velocities are not defined at the nodal points, but whenever required, they are to be found by interpolation. For example, with uniform grids, we can write $u_{i-1/2,j,k} = \left[u_{i-1,j,k} + u_{i,j,k}\right]/2$.

Where a product or square of such a quantity appears, it is to be averaged first and then the product is to be formed. Convective terms are discretized using a weighted averaged of second upwind and space centered scheme (Hirt et al., 1975). Diffusive terms are discretized by a central differencing scheme.

Because of the staggered grid arrangements, the velocities are not defined at the nodal points; but whenever required, they are to be found by interpolation. So, velocity pressure they are located in specific location; but if somewhere it is needed, but it is not defined, then those have to be interpolated. Like you know for example, here center of the cell $v_{i,j,k}$ is defined; here also center of the cell $v_{i+1,j,k}$ has been defined.

Now, on this line at center of this line, if we want to find out $v_{i,j,k}$, or if we want to identify v value here; then that v value will be basically $v_{i,j,k}$ + $v_{i+1,j,k}$ by 2, these two will be you know interpolated at this point. Similarly, $u_{i,j,k}$ is defined here, $u_{i-1,j,k}$ is defined here; but at the center of the cell u has u is not defined, p is defined, theta is defined, but not u.

So, if we need at the center of this cell u, then that will be $u_{i,j,k}$ + $u_{i-1,j,k}$ by 2; here it is linear interpolation. And as I said that appropriate interpolation has to be done at the point, where the velocity is not directly defined; velocities are defined at specified locations. So, that is what we have said here, where a product or square of such a quantity appears, it is to be averaged first and then product is to be formed.

Since we have used uniform grid, we have written averaged; otherwise, we can also use the word interpolated. So, basically why it is not defined, there it is to be defined either through interpolation or you know averaging and then product is to be formed. Convective terms are discretized using a weighted average of second upwind and space centered scheme.

This is a central differencing scheme and second upwind scheme, average of that which is available as I have mentioned, this work was done by (Refer Time: 27:32) Hirt. So, in his work, I we will discuss this. And diffusive terms are discretized by a central differencing scheme.

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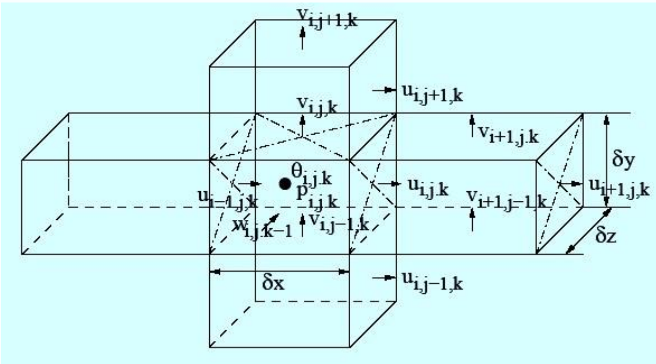

Figure 2: Three-dimensional staggered grid showing the locations of the discretized variables.

Let us consider the discretized terms, the first term of x-momentum eqn.

$$\frac{\partial(u^2)}{\partial x} = \frac{1}{\delta x}\left[\left(\frac{(u_{i,j,k} + u_{i+1,j,k})}{2}\right)^2 - \left(\frac{(u_{i-1,j,k} + u_{i,j,k})}{2}\right)^2\right]$$

$$= \frac{1}{4\delta x}\left[(u_{i,j,k} + u_{i+1,j,k})(u_{i,j,k} + u_{i+1,j,k}) - (u_{i-1,j,k} + u_{i,j,k})(u_{i-1,j,k} + u_{i,j,k})\right]$$

$$= \frac{1}{4\delta x}\begin{bmatrix} (u_{i,j,k} + u_{i+1,j,k})(u_{i,j,k} + u_{i+1,j,k}) + \alpha\left|(u_{i,j,k} + u_{i+1,j,k})\right|(u_{i,j,k} - u_{i+1,j,k}) \\ -(u_{i-1,j,k} + u_{i,j,k})(u_{i-1,j,k} + u_{i,j,k}) - \alpha\left|(u_{i-1,j,k} + u_{i,j,k})\right|(u_{i-1,j,k} - u_{i,j,k}) \end{bmatrix}$$

So, this is now we will give an example how u momentum equation is discretized or x momentum equation is discretized. So, this is three-dimensional grid has been drawn. And if we see first term of x momentum equation convective term, first term is temporal term $\partial u/\partial t$; but in the among the convective terms first term is ∂/∂x which is conservative form, first term is ∂/∂x of u square.

So, simply it has to be defined, where $u_{i,j,k}$ is defined at this point. So, at this point if we have to find out $\partial/\partial x$ of u square; we can write central difference and that will be u somewhere here, which is centered between these two u's minus u somewhere here, which is at centrally located between $u_{i,j,k}$ and $u_{i-1,j,k}$ somewhere here.

So, this u which is not shown here; but, average of $u_{i,j,k}$ + $u_{i-1,j,k}$ square. So, basically u square we have write $\partial/\partial x$ of central difference; that means u at this location square u at central of the center of the cell square divided by δx. So, 1 by δx; now this u means $u_{i,j,k}$ + $u_{i+1,j,k}$ by 2 square and this u which is between $u_{i,j,k}$ and $u_{i-1,j,k}$ is $u_{i-1,j,k}$ + $u_{i,j,k}$ by 2 square.

So, this you know just simple algebra 4 will come out, 1 by 4 δx and then we can write $u_{i,j,k}$ + $u_{i+1,j,k}$ again into $u_{i,j,k}$ plus $u_{i+1,j,k}$; just we have done you know little in detail this calculation $u_{i-1,j,k}$ plus $u_{i,j,k}$, so, $u_{i-1,j,k}$ + $u_{i,j,k}$ into $u_{i-1,j,k}$ plus $u_{i,j,k}$.

So, we have just expanded these two terms and then what we have done; we have applied up winding. And for applying up winding, otherwise this is central difference; we have expanded this up to this it is central difference, up winding has been done here. As you can see we have plot this $u_{i,j,k}$ + $u_{i+1,j,k}$ under modulus sign and then multiplied with $u_{i,j,k}$ minus $u_{i+1,j,k}$. And we have also multiplied this quantity which is under modulus sign with alpha.

This represents basically these two terms and the terms which are subtracted, they are represented this way $u_{i-1,j,k}$ plus $u_{i,j,k}$ into $u_{i-1,j,k}$ plus $u_{i,j,k}$ minus alpha. Again $u_{i-1,j,k}$ plus $u_{i,j,k}$ these two terms together are brought under modulus sign and then $u_{i-1,j,k}$ - $u_{i,j,k}$. Now, if you look at this discretization, if you set alpha equal to 0, it will remain central difference; if you set alpha equal to non-zero value, then up winding effect will start coming.

But if you set alpha equal to 1, then it will be full up winded, first order up winding. And this sign is under modulus sign, this terms $u_{i,j,k}$ + $u_{i+1,j,k}$, this quantity is under modulus sign in order to give the flexibility; because locally in a low field, it is you know quite natural that there will be some obstructions, there will be some undulations and there may be flow separation.

If there is any flow separation, then there will be locally negative velocity. So, this we have given that flexibility that this $u_{i,j,k}$ + $u_{i+1,j,k}$ we have put it under modulus sign; that means this will remain valid whether this that velocity is positive or negative. Similarly, we have also used modulus sign for $u_{i-1,j,k}$ and $u_{i,j,k}$; that means u at the center of the cell.

So, that velocity locally may be negative or positive depending on its location; if it is located you know after the point of separation, u velocity can be negative there and otherwise it is positive. So, positive or negative to play with both, we have applied modulus sign here.

So, that is what and if you recall, we have already discussed this way of discretizing the finite difference questions in our first; probably if first lecture or second lecture, where we discussed about you know basically the forward difference, backward difference. And then we wrote the full form probably in 7th lecture, where we introduced formally several up-winding methods and higher order up winding.

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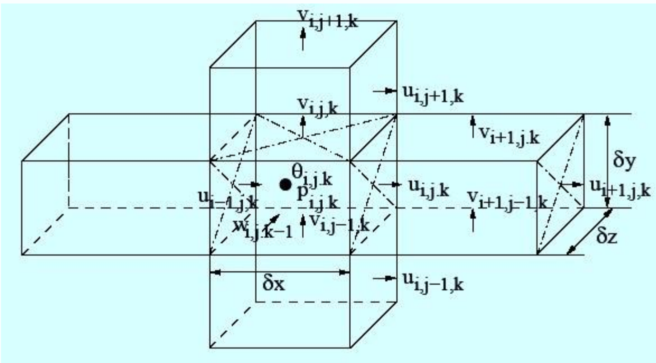

Figure 2: Three-dimensional staggered grid showing the locations of the discretized variables.

Let us consider the discretized terms of the x-momentum equation as shown in Figure 2:

$$\frac{\partial(u^2)}{\partial x} = \frac{1}{4\delta x}[(u_{i,j,k} + u_{i+1,j,k})(u_{i,j,k} + u_{i+1,j,k}) + \alpha|(u_{i,j,k} + u_{i+1,j,k})|(u_{i,j,k} - u_{i+1,j,k}) - (u_{i-1,j,k} + u_{i,j,k})(u_{i-1,j,k} + u_{i,j,k}) - \alpha|(u_{i-1,j,k} + u_{i,j,k})|(u_{i-1,j,k} - u_{i,j,k})] \equiv DUUDX$$

Now, so this is what we have and done and we are writing this discretize form again and giving it a variable name; we are calling it, we are defining this as DUUDX, simply you know d dx of u square. So, we are as if you know this variable is a variable in a program.

So, we have given a name following that sequence in a code; this is being defined as DUUDX and this is exactly the same term that we have written here, del del x of u square.

So, we can look at it alpha is there and $u_{i,j,k}$ + $u_{i+1,j,k}$; that means between these two u by 2 is obviously hidden has come here and that velocity a under modulus sign. And also $u_{i-1,j,k}$ and $u_{i+1,j,k}$ average of that velocity; that means at the cell center whatever is the velocity, u velocity is under modulus sign.

And we are using α which we are calling as basically the upwind up-winding contribution or contribution upwind factor, right. Usually this should be a small value; we will come to that again.

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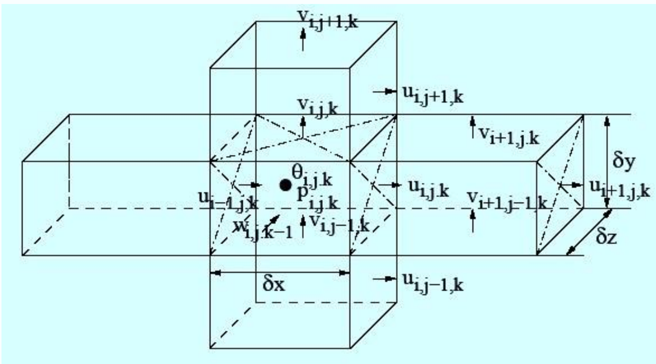

Figure 2: Three-dimensional staggered grid showing the locations of the discretized variables.

$$\frac{\partial(uv)}{\partial y} = \frac{1}{\delta y}\left[\left(\frac{(v_{i,j,k}+v_{i+1,j,k})}{2}\right)\left(\frac{(u_{i,j,k}+u_{i,j+1,k})}{2}\right)-\left(\frac{(v_{i,j-1,k}+v_{i+1,j-1,k})}{2}\right)\left(\frac{(u_{i,j-1,k}+u_{i,j,k})}{2}\right)\right]$$

$$= \frac{1}{4\delta y}\left[\begin{array}{c}(v_{i,j,k}+v_{i+1,j,k})(u_{i,j,k}+u_{i,j+1,k})+\alpha\left|(v_{i,j,k}+v_{i+1,j,k})\right|(u_{i,j,k}-u_{i,j+1,k})\\ -(v_{i,j-1,k}+v_{i+1,j-1,k})(u_{i,j-1,k}+u_{i,j,k})-\alpha\left|(v_{i,j-1,k}+v_{i+1,j-1,k})\right|(u_{i,j-1,k}-u_{i,j,k})\end{array}\right]$$

Then we have gone for the second term $\partial(uv)/\partial y$. Now, see when you find out $\partial(uv)/\partial y$ we have already mentioned how to do that; here also we will apply central difference, but this $\partial(uv)/\partial y$ that has to be defined, where $u_{i,j,k}$ has been defined. So, if we have to define at the point of $u_{i,j,k}$; then we have to define u into v at this point and also u into v at this point.

Then u into v at this point minus u into v at this point divided by δy, we will give $\partial(uv)/\partial y$ here. So, terms associated with x momentum equation, we have to determine all the quantities at a point, where $u_{i,j,k}$ has been defined. So, the second term of the momentum equation $\partial(uv)/\partial y$; we have to calculate at this point, where $u_{i,j,k}$ has been defined.

So, we have to find out u into v at this location and uv at this location. We have exactly done that 1 by delta y. And this first term; that means del into u is at this location will be $v_{i,j,k} + v_{i+1,j,k}$ by 2 multiplied by $u_{i,j,k}$ plus $u_{i,j+1,k}$ by 2.

So, that makes at I mean available u into v at this point that makes available this term. Now, again u into v at this point; that means $v_{i,j-1,k}$ is defined here, $v_{i+1,j-1,k}$ is defined here. So, average of these two terms $v_{i,j-1,k}$ this term, $v_{i+1,j-1,k}$ by 2 and u will be $u_{i,j,k}$ plus $u_{i,j-1,k}$ by 2, so $u_{i,j-1,k}$ plus $u_{i,j,k}$ by 2.

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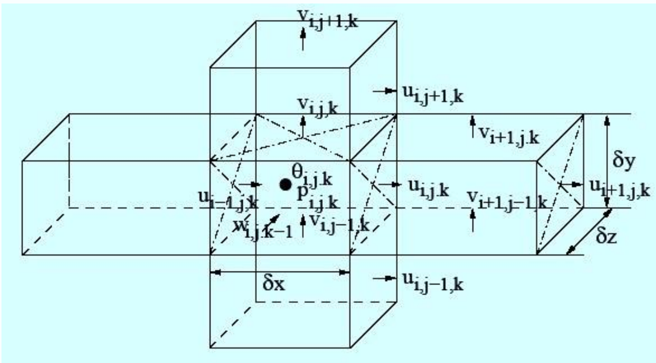

Figure 2: Three-dimensional staggered grid showing the locations of the discretized variables.

$$\begin{aligned}\frac{\partial(uv)}{\partial y} = \frac{1}{4\delta y}[&(v_{i,j,k}+v_{i+1,j,k})(u_{i,j,k}+u_{i,j+1,k})\\ &+\alpha|(v_{i,j,k}+v_{i+1,j,k})|(u_{i,j,k}-u_{i,j+1,k})\\ &-(v_{i,j-1,k}+v_{i+1,j-1,k})(u_{i,j-1,k}+u_{i,j,k})\\ &-\alpha|(v_{i,j-1,k}+v_{i+1,j-1,k})|(u_{i,j-1,k}-u_{i,j,k})]\\ \equiv DUVDY&\end{aligned}$$

So, simply we have found out into by interpolating u at this point, between these two points we have interpolated; between these two points, we have to interpolate u. So, we have done that; between these two points again we have to interpolate v here and using these two points, you have to interpolate u here.

So, u into v at this point is this quantity and u into v at this point is basically this quantity. So, 4 comes out and then we get these terms; again, we go for up winding and by the factor is a contribution of up wind factor is alpha. And we make this locally this v plus this v; that means v here and this v plus this v by 2.

So, these 2 v's locally we have added flexibility on these two views, so that expression can be independent of v these two v is sorry, these two v's, whether they are positive or negative will not affect the discretization. The discretization will assume the correct form of up winding. So, u at i, j, u at this point; that means $u_{i,j,k}$ plus $u_{i+1,j,k}$ by 2.

That means $u_{i,j,k} + u_{i+1,j,k}$ this quantity under mod and also $v_{i,j-1,k}$ and $v_{i+1,j-1,k}$ and $v_{i,j-1,k}$; average of these two v's here also under modulus sign to create the flexibility that it can be positive or negative both.

So, we have again you know just written those term and we are calling it as DUVDY, this is $\partial(uv)/\partial y$; we have rewritten basically the previous slide. And the final result of this slide is same, basically we are evaluating or discretizing $\partial(uv)/\partial y$ and assigning it a variably name DUVDY.

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$$\frac{\partial(uv)}{\partial y} = \frac{1}{4\delta y}[(v_{i,j,k} + v_{i+1,j,k})(u_{i,j,k} + u_{i,j+1,k}) + \alpha|(v_{i,j,k} + v_{i+1,j,k})|(u_{i,j,k} - u_{i,j+1,k}) - (v_{i,j-1,k} + v_{i+1,j-1,k})(u_{i,j-1,k} + u_{i,j,k}) - \alpha|(v_{i,j-1,k} + v_{i+1,j-1,k})|(u_{i,j-1,k} - u_{i,j,k})] \equiv DUVDY$$

$$\frac{\partial(uw)}{\partial z} = \frac{1}{4\delta z}[(w_{i,j,k} + w_{i+1,j,k})(u_{i,j,k} + u_{i,j,k+1}) + \alpha|(w_{i,j,k} + w_{i+1,j,k})|(u_{i,j,k} - u_{i,j,k+1}) - (w_{i,j,k-1} + w_{i+1,j,k-1})(u_{i,j,k-1} + u_{i,j,k}) - \alpha|(w_{i,j,k-1} + w_{i+1,j,k-1})|(u_{i,j,k-1} - u_{i,j,k})] \equiv DUWDZ$$

So, $\partial(uv)/\partial z$ that is how we have done; similarly, the third term of x momentum equation is $\partial(uv)/\partial z$. I am not explaining it, we have explained first two terms; if you follow the way first two terms have been handled, third term will also be handled in a similar way. We have to write the expression first in central differencing, then we have to create provision for up winding and then we have to make the local velocity.

We have to consider local velocity can be positive or negative and make the expression or the discretize form independent of that by using modulus sign. And we will call these as DUWDZ.

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$$\frac{\partial p}{\partial x} = \frac{p_{i+1,j,k} - p_{i,j,k}}{\delta x} \equiv DPDX$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{(\delta x)^2} \equiv D2UDX2$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{(\delta y)^2} \equiv D2UDY2$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}}{(\delta z)^2} \equiv D2UDZ2$$

with

$\alpha \longrightarrow 1$ Scheme $\longrightarrow$ Second Upwind

$\alpha \longrightarrow 0$ Scheme $\longrightarrow$ Space centred

Then $\partial p/\partial x$, we have written $p_{i+1,j,k}$ minus $p_{i,j,k}$ by δx; simply $\partial p/\partial x$ we are calling it, we are assigning this variable name DPDX. Now, the diffusive terms CONDIFU we called the total term; so diffusion terms are $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$, and $\partial^2 u/\partial z^2$ at the point where $u_{i,j,k}$ has been defined.

So, again if we go back by one or two slides; so, we have two diffusions, it is associated with u momentum equation or x momentum equation, where $u_{i,j,k}$ has been defined. Here we have to find out final difference quotient of $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$, and $\partial^2 u/\partial z^2$; $\partial^2 u/\partial x^2$ will be $u_{i+1,j,k}$ minus twice $u_{i,j,k}$ + $u_{i-1,j,k}$ divided by $(\delta x)^2$.

So, we can see $u_{i+1,j,k}$ minus twice $u_{i,j,k}$ plus $u_{i-1,j,k}$ by $(\delta x)^2$. Similarly, $\partial^2 u/\partial y^2$ if I am again going back by few slides. So, we have to make use of this $u_{i,j+1,k}$ minus; then twice $u_{i,j,k}$ + $u_{i,j-1,k}$ divided by $(\delta y)^2$. So, $u_{i,j+1,k}$ minus twice $u_{i,j,k}$ plus $u_{i,j-1,k}$ by $(\delta x)^2$.

Similarly, in z direction, if we find out the second derivative of u at the point $u_{i,j,k}$; we will get this quotient. And we will define this as a variable, which will be used in computing; this is simply we have written D2UDX2, D2UDY2, the variable name is taken from directly in the terminology that $\partial^2 u/\partial z^2$ is D2UDZ2. And as we have said that, all these discretized equations are containing α, α is the upwind contribution.

And α = 1, it is second upwind; $\alpha = 0$ it is central differencing scheme. We will not use be only central difference scheme; because we have seen this may give or produce numerical instability. So, we will for the stable calculation, we will introduce up winding; but this α value will be a very small value.

(Refer Slide Time: 48:15)

A typical value of α is between 0.2 and 0.3, the quantity i $\tilde{u}^{n+1}_{i,j,k}$ evaluated explicitly from the discretized form of equation (2) as:

$$\tilde{u}^{n+1}_{i,j,k} = u^n_{i,j,k} + \delta t\,[CONDIFU - DPDX]^n_{i,j,k}$$

where

$$[CONDIFU - DPDX]^n_{i,j,k} = [(-DUUDX - DUVDY - DUWDZ) - DPDX + (1/Re)(D2UDX2 + D2UDY2 + D2UDZ2)]$$

Similarly, we evaluate:

$$\tilde{v}^{n+1}_{i,j,k} = v^n_{i,j,k} + \delta t\,[CONDIFV - DPDY]^n_{i,j,k} \qquad (3)$$

$$\tilde{w}^{n+1}_{i,j,k} = w^n_{i,j,k} + \delta t\,[CONDIFW - DPDZ]^n_{i,j,k} \qquad (4)$$

A typical α is between 0.2 and 0.3, the quantity $\tilde{u}^{n+1}_{i,j,k}$ tilde is evaluated explicitly from the discretized equation 2; we mentioned about it earlier, that is $\tilde{u}^{n+1}_{i,j,k}$ is $\tilde{u}^{n}_{i,j,k}$ plus δt into CONDIFU. We know all the components of CONDIFU now minus DPDX at i, j, k point at nth level.

And components of CONDIFU are we have evaluated them one by one DUUDX, then DUVDY, DUVDY, DUWDZ, DPDX, D2UDX2 etcetera. But when we transfer the governing equation, when we transfer first three terms on the right-hand side, all of them will have negative sign.

So, in CONDIFU when we, so in CONDIFU when we are writing DUUDX; that means, $\frac{\partial u^2}{dx}$, that is with negative sign $\frac{\partial (uv)}{dx}$ DUVDY with negative sign DUWDZ with negative sign minus DPDX. And diffusive terms will be without any change in side as a change in sign; we are not changing side also, no change in sign, so plus 1/Re into all the diffusive terms.

So, this way if we can calculate CONDIFU minus DPDX at nth level at the point i, j, k and if we add it with u velocity at nth level, after multiplying by the time step; we will get provisionally advanced value of $u_{i,j,k}$ at the level of n+1. And since it is provisionally advanced value, we are also using tilde sign.

So, this is the way provisionally at every point u velocity is advanced. Similarly, provisionally at every point v velocity can be advanced and w velocity can be advanced. And they will follow same convective kinematics and diffusive laws and the effect of pressure gradient; instead of CONDIFU, this will be CONDIFV convective and diffusive terms of v momentum equation. DPDY instead of DPDX; in v momentum equation, it will be DPDY.

And in w momentum equation or z momentum equation, it will be CONDIFW minus DPDZ. And we will get provisionally advanced u at every point provisionally advanced v at every point provisionally advanced w at every point. And this is the discretized form; discretization has been done u for the terms associated with x momentum, where $u_{i,j,k}$ has been defined.

Similarly, terms which are associated with y momentum equation, all those terms will be determined, where $v_{i,j,k}$ is located. And the w momentum equation or z momentum equation all the terms will be evaluated at the point, where $w_{i,j,k}$ has been defined.

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We will stop here today; remaining parts we will take up in subsequent lectures.

Thank you very much. Thank you.