Computational Fluid Dynamics and Heat Transfer
Prof. Gautam Biswas
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 18
Solving N-S Equations for Incompressible Flows using SIMPLE Algorithm
(Momentum Equations and the Solution for Pressure)

Good morning, everybody, today we will discuss the remaining part of Simple Algorithm. I may call that we derived till the discretized x momentum equation and also, we discussed discretized continuity equation and some other aspects of the solution algorithm.

(Refer Slide Time: 00:53)

y- momentum equation

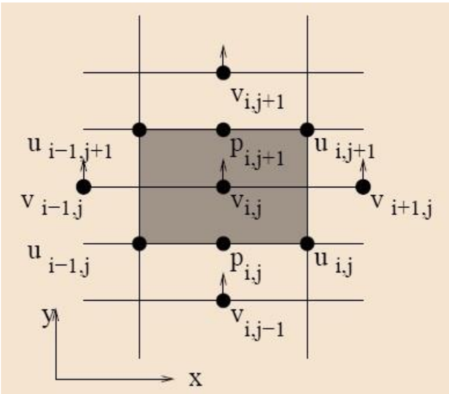


Figure 4: Control volume for y-momentum equation

y- momentum equation

$$\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}(uv)+\frac{\partial (v)^2}{\partial y}+\frac{\partial p}{\partial y}=\frac{1}{Re}\left\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right\}$$

Using the control volume shown in Fig. 4, the y momentum equation can be integrated as

$$\frac{\Delta x\,\Delta y}{\Delta t}\left(v_{i,j}^{n+1}-v_{i,j}^{n}\right)+\iint\left[\frac{\partial}{\partial x}\left\{uv-\frac{1}{Re}\cdot\frac{\partial v}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{v^2-\frac{1}{Re}\frac{\partial v}{\partial y}\right\}\right]dx\,dy+\iint\left(\frac{\partial p}{\partial y}\right)dx\,dy=0 \qquad (8)$$

So, today we will start to with y momentum equation and we can see y momentum equation here at the first place has been written in terms of weak conservative form. But then we change it to strong conservative form; that means,

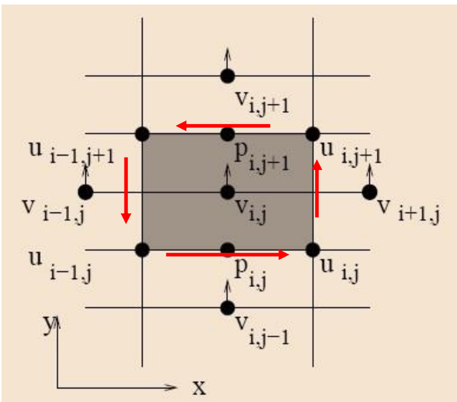
$$\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}(uv)+\frac{\partial (v)^2}{\partial y}+\frac{\partial p}{\partial y}=\frac{1}{Re}\left\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right\}$$

So, we this is something similar to strong conservative form and then we integrate it over v control volume. Now, when we define v control volume $v_{i,j}$ velocity is defined at the center of the cell and the neighbours are adjusted accordingly, means defined they are already defined they are considered accordingly like this point is considered $P_{i,j}$ this point is considered as $P_{i,j+1}$. And this point is $v_{i+1,j}$ this point is $v_{i-1,j}$ etcetera and this control volume is v control volume where $v_{i,j}$ is defined at the center of the cell or center of the control volume. And we will perform the integration of this equation on this control volume. So, we have written double integral dx dy that is equation 8.

$$\frac{\Delta x \Delta y}{\Delta t}\left(v_{i,j}^{n+1} - v_{i,j}^{n}\right) + \iint\left[\frac{\partial}{\partial x}\left\{uv - \frac{1}{Re}\frac{\partial v}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{v^2 - \frac{1}{Re}\frac{\partial v}{\partial y}\right\}\right]dx\,dy + \iint\left(\frac{\partial p}{\partial y}\right)dx\,dy = 0 \qquad (8)$$

(Refer Slide Time: 03:25)

$$\iint\left[\frac{\partial}{\partial x}\left\{uv - \frac{1}{Re}\frac{\partial v}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{v^2 - \frac{1}{Re}\frac{\partial v}{\partial y}\right\}\right]dx\,dy = \iint\left[\frac{\partial}{\partial x}\left\{E^2\right\} + \frac{\partial}{\partial y}\left\{F^2\right\}\right]dx\,dy$$



Performing the integration (Application of Green's Theorem) will produce:

$$= \oint\left(\{-F^2\}dx + \{E^2\}dy\right)$$

The contour integration results in:

$$= \left[\left\{E^2_{i+1/2,\,j} - E^2_{i-1/2,\,j}\right\}\Delta y\right] + \left[\left\{F^2_{i,\,j+1/2} - F^2_{i,\,j-1/2}\right\}\Delta x\right]$$

Now, we will discuss see I will go back again. Now integration of $\partial v/\partial t$ is very easy. So, over the volume Δx Δy by Δt into $v_{i,j}$ at n+1 nth level minus $v_{i,j}$ at nth level , $\partial v/\partial t$. Now, we will consider these two terms of integration. So, double integral $\frac{\partial}{\partial x}\left\{uv - \frac{1}{Re}\frac{\partial v}{\partial x}\right\} +$

$\frac{\partial}{\partial y}\left\{v^2 - \frac{1}{Re}\frac{\partial v}{\partial y}\right\}$. This term we can consider as E^2 and this term we can redefine as F^2 like x momentum equation.

We define these bracketed terms which were derivative of del x and del y E^1 and F^1 respectively here we are doing E^2 and F^2. Now, performing the integration and applying Greens theorem. We can write this surface integral as the contour integral of $-F^2dx + E^2dy$.

So, E^2 first we will integrate then $-F^2$ also we will integrate and this integration will be done this is contour integral and we will follow a cyclic integral. So, we can see like when we calculate E^2 dy we define E^2 here with dy we integrate and then you know here it is F^2 again E^2 is defined again here. And but this time dy is in the negative direction.

So, you can see $E^2_{i+1/2,j}$, so this is i, j with respect to v. So, this point will be i+1/2, j. So, $E^2_{i+1/2,j}$ which is defined here into Δy minus $E^2_{i+1/2,j}$ into Δy, since this is contour integral here and this is E^2dy; so on this side dy is positive this side dy is negative we are getting this. Next term is F^2; F^2 is defined here and here. So, this is $F^2_{i,j+1/2}$ now $F^2_{i,j+1/2}$ into Δx, and Δx in the negative direction that makes minus $F^2_{i,j+1/2}\,\Delta x$, but F is having negative sign because of this integration through Green's theorem. So, that then this negative and this negative sign makes it positive. So, $F^2_{i,j+1/2}$ and here $F^2_{i,j-1/2}$ is defined here.

So $F^2_{i,j-1/2}$ into positive Δx, so that makes this quantity positive, but it is having a negative sign, so that makes it negative. So, $F^2_{i,j+1/2}$ - $F^2_{i,j-1/2}$ Δx and that is how we evaluate this integral.

(Refer Slide Time: 08:41)

y- momentum equation

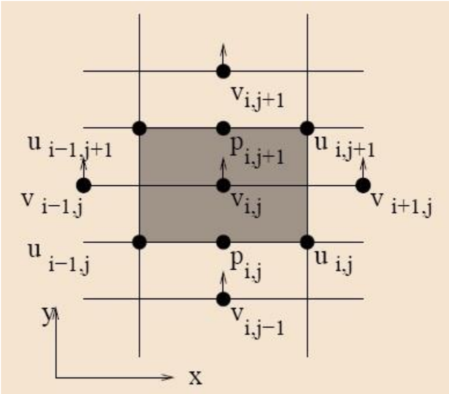


Figure 4: Control volume for y-momentum equation

Using the control volume shown in Fig. 4 the y -momentum equation can be written as

$$\frac{\Delta x \Delta y}{\Delta t}(v_{i,j}^{n+1} - v_{i,j}^{n}) + (E_{i+1/2,j}^{2} - E_{i-1/2,j}^{2})\Delta y + (F_{i,j+1/2}^{2} - F_{i,j-1/2}^{2})\Delta x$$
$$(p_{i,j+1}^{n+1} - p_{i,j}^{n+1})\Delta x = 0 \quad (9)$$

$$E^2 = uv - \frac{1}{Re}\frac{\partial v}{\partial x}, \qquad F^2 = v^2 - \frac{1}{Re}\frac{\partial v}{\partial y}$$

So, finally, we can write

$$\frac{\Delta x \Delta y}{\Delta t}(v_{i,j}^{n+1} - v_{i,j}^{n}) + (E_{i+1/2,j}^{2} - E_{i-1/2,j}^{2})\Delta y + (F_{i,j+1/2}^{2} - F_{i,j-1/2}^{2})\Delta x$$
$$(p_{i,j+1}^{n+1} - p_{i,j}^{n+1})\Delta x = 0$$

$$E^2 = uv - \frac{1}{Re}\frac{\partial v}{\partial x}, \qquad F^2 = v^2 - \frac{1}{Re}\frac{\partial v}{\partial y}$$

So, $P_{i,j+1}$-$P_{i,j}$ into Δx and pressure is at the level of n+1 and we have already seen earlier we did this substitution what is a meaning of E^2 and what is a meaning of F^2. Now, this equation 9 we have to now express in terms of u, v and their derivatives.

(Refer Slide Time: 10:44)

Here E^2 and F^2 are the axial and transverse fluxes of y-momentum

$$E^2_{i+1/2,j} = 0.25(u_{i,j} + u_{i,j+1})(v_{i,j} + v_{i+1,j}) - \frac{1}{Re}\frac{(v_{i+1,j} - v_{i,j})}{\Delta x}$$

$$E^2_{i-1/2,j} = 0.25(u_{i-1,j} + u_{i-1,j+1})(v_{i-1,j} + v_{i,j}) - \frac{1}{Re}\frac{(v_{i,j} - v_{i-1,j})}{\Delta x}$$

$$F^2_{i,j+1/2} = 0.25(v_{i,j} + v_{i,j+1})^2 - \frac{1}{Re}\frac{(v_{i,j+1} - v_{i,j})}{\Delta y}$$

$$F^2_{i,j-1/2} = 0.25(v_{i,j-1} + v_{i,j})^2 - \frac{1}{Re}\frac{(v_{i,j} - v_{i,j-1})}{\Delta y}$$

Having substituted E2 and F2 in equation (9) we require to linerize

So, if we want to do that, we basically we like here it is E^2 at i+1/2 location and this is basically u into v. So, how to find out at this location u into v this is interpolation simply $(v_{i,j}+v_{i+1,j})/2$ will give v here and $(u_{i,j}, u_{i,j+1})/2$ will give u here.

So, we will look at the terms which are basically $u_{i,j}+u_{i,j+1/2}$ that is what we said $u_{i,j}+u_{i,j+1/2}$ multiplied by (vi,j+vi+1,j)/2 this 2 and 2 will become 4 this is 1 by 4 means 0.25 and this quantity, then we have 1 by Re $\frac{\partial v}{\partial x}$.

So, this is 1 by Re $\frac{\partial v}{\partial x}$ here is ($v_{i+1,j}-v_{i,j}$)/Δx; So, similarly this E^2 quantity have to be the way we have done here have to be find out have to be found out here. So, I am not repeating that it is straightforward I am going for calculation of F2 at the point i,j +1/2.

So and what is F^2? F^2 is $v^2 - \frac{1}{Re}\frac{\partial v}{\partial y}$. So, here v square has to be defined that means this is $(v_{i,j}+v_{i,j+1/2})$ square. So, or let us see $(v_{i,j}+v_{i,j+1/2})/2$ square; that means, $(v_{i,j}+v_{i,j+1/2})^2/4$ and at this point 1 by 4 means 0.25.

And then again, we have to see what is $\frac{1}{Re}\frac{\partial v}{\partial y}$ here $v^2 - \frac{1}{Re}\frac{\partial v}{\partial y}$ will be $v_{i,j+1} - v_{i.j}$ divided by Δy. So, this is how again it is straightforward to find out F^2 at this point; that means, $F^2_{i,j-1/2}$ we have found out.

(Refer Slide Time: 14:29)

The linearized forms of this equation are

$$E^2_{i+\frac{1}{2},j} = 0.25\left(u^n_{i,j} + u^n_{i,j+1}\right)\left(v^{n+1}_{i,j} + v^{n+1}_{i+1,j}\right) - \frac{1}{\mathrm{Re}}\frac{\left(v^{n+1}_{i+1,j} - v^{n+1}_{i,j}\right)}{\Delta x}$$

$$E^2_{i-\frac{1}{2},j} = 0.25\left(u^n_{i-1,j} + u^n_{i-1,j+1}\right)\left(v^{n+1}_{i-1,j} + v^{n+1}_{i,j}\right) - \frac{1}{\mathrm{Re}}\frac{\left(v^{n+1}_{i,j} - v^{n+1}_{i-1,j}\right)}{\Delta x}$$

$$F^2_{i,j+\frac{1}{2}} = 0.25\left(v^n_{i,j} + v^n_{i,j+1}\right)\left(v^{n+1}_{i,j} + v^{n+1}_{i,j+1}\right) - \frac{1}{\mathrm{Re}}\left(\frac{v^{n+1}_{i,j+1} - u^{n+1}_{i,j}}{\Delta y}\right)$$

$$F^2_{i,j-\frac{1}{2}} = 0.25\left(v^n_{i,j-1} + v^n_{i,j}\right)\left(v^{n+1}_{i,j-1} + v^{n+1}_{i,j}\right) - \frac{1}{\mathrm{Re}}\left(\frac{v^{n+1}_{i,j} - u^{n+1}_{i,j-1}}{\Delta y}\right)$$

$$\left(p^{n+1}_{i,j+1} - p^{n+1}_{i,j}\right)\Delta x = 0$$

So, all these quantities we find out the way we did for x momentum equation, then what we have to do? These are non-linear terms quadratic terms and these have to be linearized. So, one part we write at nth level another part we write at n+1th level. So, exactly we did that for x momentum equation. So, we can see here 0.25 $(u_{i,j}+u_{i,j+1})$ we will write at nth level and $v_{i,j}+v_{i+1,j}$ will be at the level n+1.

So, these are the level n+1 these are the level n, similarly here also this 0.25 $(u_{i-1,j}+u_{i-1,j+1})$ this will be at the level n and $(v_{i-1,j}+v_{i,j})$ will be at the level n+1. Exactly we have done that this is basically $(v_{i-1,j}+v_{i,j})$ at n+1 and $(u_{i,j}+u_{i,j-1})$ at the level n.

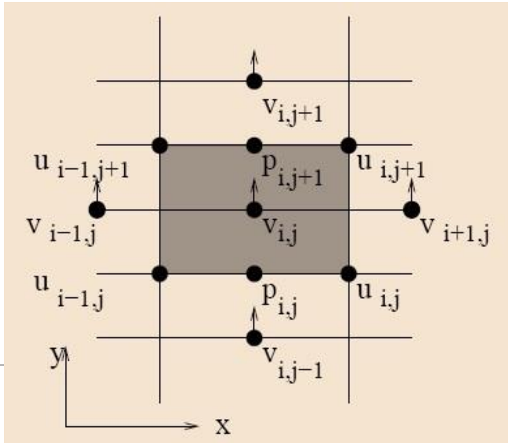
So, this way we do the linearization and then we substitute this in the original equation that means these terms are substituted which are basically discretization of E^2 and F^2.

Pressure gradient term is already discretized and temporal term is already discretized. So, we write all these and nonlinearity is removed through local linearization. And then we collect the terms just we are repeating the same procedure that we followed for x momentum equation.

(Refer Slide Time: 16:55)

Finally, the y-momentum equation becomes:

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^{v}_{i,j}\right) v^{n+1}_{i,j} + \sum a^{v}_{nb} v^{n+1}_{nb} + b^{v} + \Delta x \left(p^{n+1}_{i,j+1} - p^{n+1}_{i,j}\right) = 0 \quad (10)$$



Let us recall the x-momentum equation (derived in the last lecture):

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^{u}_{i,j}\right) u^{n+1}_{i,j} + \sum a^{u}_{nb} u^{n+1}_{nb} + b^{u} + \Delta y \left(p^{n+1}_{i+1,j} - p^{n+1}_{i,j}\right) = 0 \quad (7)$$

All the coefficients of $v_{i,j}$ at the level n+1, so these are the coefficients of $v_{i,j}$. And the neighbouring v_s that means, $v_{i+1,j}$ $v_{i-1,j}$ $v_{i,j+1}$ $v_{i,j-1}$ those neighbours at the level n+1 those terms we have to identify and they are coefficients a and b are respective coefficients of Eastern neighbour, Western neighbour, Northern neighbour and Southern neighbour.

Again, this term will be basically if we follow what we did is x momentum equation ΔxΔy multiplied by $v_{i,j}$ at nth level divided by Δt. We did it earlier Δx into Δy $v_{i,j}$ at nth level all 3 are multiplied divided by Δt and this is the pressure gradient term this is equation 10 which is found by integrating y momentum equation on v control volume.

We found x momentum equation integrating on u control volume and let us recapitulate we found this equation 7. In our last lecture in way term by term explained and much more detail than what we did today because that was the introduction. So, we derived equation 7, now equation 7 and equation 10 are discretized x momentum equation and discretized y momentum equation; equation 7 and equation 10.

Now, if we want to evaluate equations 7 and 10 for all the vs; that means and all the I us that means $u_{i,j}$ $u_{i+1,j}$ $u_{i-1,j}$ $u_{i,j+1}$ and $u_{i,j-1}$ at the level n+1; $v_{i,j}$ $v_{i+1,j}$ $v_{i-1,j}$ $v_{i,j+1}$ $v_{i,j-1}$ at the level n+1. We get this equation. But in this equation, we will see that pressure is also at the level n+1.

Now, we have the basic field when we started the calculation for n+1 and that field corresponds to nth level, nth level velocities are there nth level pressures are there. So, all those nth level velocities we have included in this coefficient terms, but nth level pressure is only available pressure we do not have pressure available at the level n+1.

So, these two equations in this form it is difficult to evaluate, because pressure at n+1 this level n+1 at the level n+1 pressures are not known. Basic velocities that mean velocities at nth level pressures at nth level are known and were implicitly calculating velocity at the level n +1 , u velocity and v velocity

(Refer Slide Time: 21:37)

At any intermediate stage, the solution is to be advanced from (n) th time level to $(n+1)$th time level. The velocity is advanced in two steps. First, the momentum equations (7) and (10) are solved to obtain the provisional values of u^*, and v^*. It is not possible to obtain u^{n+1} and v^{n+1} directly since the nth level pressures are available instead of (n+1) th level pressures.

Making use of the approximate velocity solution $\mathbf{u}^*$, a pressure correction δp will evolve which will give $p^{n+1} = p^n + \delta p$ and also a velocity correction $\mathbf{u^c}$ will be obtainable. As such, $\mathbf{u^c}$ will correct $\mathbf{u}^*$ in such a manner that $\mathbf{u^{n+1}} = \mathbf{u}^* + \mathbf{u^c}$ and $\mathbf{u^{n+1}}$ will satisfy the continuity

In order to obtain U*, equation (7) and (10) are approximated as:

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^u_{i,j}\right) u^*_{i,j} + \sum a^u_{nb} u^*_{nb} = -b^u - \Delta y \left(p^n_{i+1,j} - p^n_{i,j}\right) \quad (11)$$

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^v_{i,j}\right) v^*_{i,j} + \sum a^v_{nb} v^*_{nb} = -b^v - \Delta x \left(p^n_{i,j+1} - p^n_{i,j}\right) \quad (12)$$

So, that is what we have written here since the pressure at n+1 th level not available. So, we are when we evaluate this equation we approximate and what is approximation we write pressure at nth level. If we write pressure at nth level, then we can do the calculation, but this u and v velocities will not be exactly n+1 th level, they will advance towards level which is n+1.

But since correct pressure is not there at the level n+1 and we are just advancing them with the help of this equation we are unable at this stage to satisfy continuity also. So, these advancements of velocities are provisional advancements. So, making use of the approximate velocity solution u^* a pressure correction δp will evolve which will give finally.

So, we are that is why we are evaluating $u_{i,j}$ $v_{i,j}$ and all neighbouring us and neighbouring vs, but we are not calling these values are n+1 th level value values we are calling them as star values. So, making use of appropriate velocities solution $\mathbf{u}^*$ this is u vector star; that means, u v both are meant both are addressed by this. A pressure correction δp will evolve which will give p^{n+1} equal to $p^n + \delta p$.

So, if we can evolve δp we can know somehow how δp will be added to p to make it pressures at n+1 th level. Then these u's will or u stars will also be corrected as u^{n+1} th level. As such what we are writing is a pressure correction δp will evolve which will give p^{n+1} equal to p^n plus δp and also a velocity correction u^c which will be obtainable.

As such u^c will correct $\mathbf{u}^*$ in such a manner that $\mathbf{u}^{n+1}$ this is vector u means u and v both are covered u star plus $\mathbf{u}^c$ and then we will get $\mathbf{u}^{n+1}$. So, and we will get p^{n+1} also that is why equation 11 and 12 are making a projection for velocities at n+1 th level. And then these 2 equations require to be corrected when the pressure will be corrected u star will also be corrected to u^{n+1}.

(Refer Slide Time: 26:14)

Solution

The equation (11) and (12) can be evaluated in an implicit manner. Substract equation (11) from equation (7), we get:

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^u_{i,j}\right) u^c_{i,j} = \sum a^u_{nb}\, u^c_{nb} - \Delta y\left(\delta p_{i+1,j} - \delta p_{i,j}\right) \quad (13)$$

In a similar manner, equation (12) is substracted from equation (10) to produce a correction equation for v^c.

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^v_{i,j}\right) v^c_{i,j} = \sum a^v_{nb}\, v^c_{nb} - \Delta x\left(\delta p_{i,j+1} - \delta p_{i,j}\right) \quad (14)$$

In order to make the link between u^c and δp explicit, the equation (13) can be reduced to:

$$u^c_{i,j} = \frac{E_1\ \Delta y\left(\delta p_{i,j} - \delta p_{i+1,j}\right)}{(1+E_1)\, a^u_{i,j}} \quad (15)$$

where $E_1 = \Delta t\ a^u_{i,j} / \Delta x\ \Delta y$

Now, that is why equation 11 and 12 if we look at it and if we look carefully equation 7 and 10. Let us compare equation 7 and 11 both are x momentum equation. Equation 7 everywhere we have written $u_{i,j}$ and their neighbours $u_{i+1,j}$, $u_{i-1,j}$, $u_{i,j+1}$ $u_{i,j-1}$ at the level n+1 and pressures are written at n+1.

Whereas, equation 11 this is also x momentum equation, but all the velocities $u_{i,j}$, $u_{i+1,j}$, $u_{i-1,j}$, $u_{i,j+1}$ $u_{i,j-1}$ are starred quantity. That means, these are not exactly at the level n+1 somewhat similar to that but different, that is why those are starred quantities and pressure is at nth level pressure.

And if we subtract equation 11 from equation 7 everywhere with $u_{i,j}$ $u_{i+1,j}$, $u_{i-1,j}$, we will get difference between level n+1 and star. That means, that will be yet another equation with $u_{i,j}$ $u_{i+1,j}$, $u_{i-1,j}$ $u_{i,j+1}$, $u_{i,j-1}$, but c instead of star we will write c on them. Those are correction terms associated with corresponding starred terms to make these starred terms n+1 and pressure will also be corrected by δp at each pressure point.

So, that $p_{i,j}$ $p_{i+1,j}$ $p_{i-1,j}$ all these instead of being defined at the level n after adding Δp they will be at the level of n+1. So, if we subtract equation 11 from equations 7 we will get this equation, instead of $u_{i,j}^{n+1}$ we are writing $u_{i,j}^{c}$ velocity correction and exactly with $u_{i,j}^{n+1}$ the coefficients which were there those coefficients are there.

Similarly, the neighbouring velocities $u_{i+1,j}$, $u_{i-1,j}$, $u_{i,j+1}$ $u_{i,j-1}$ these are not at n plus at the level n+1, but we are writing you know or we are defining them as with a superscript c. So, these are again the velocity corrections, are the velocities at those specified points. And everywhere at every point p at the level n+1 and p at the level n those are now differences are basically δp, because p at nth level if δp is added will be p at the level n+1.

Same philosophy applies for y momentum equation and this we get if we subtract equation 12 from equation 10 we will get equation 14. This is velocity correction equation or v velocity correction equation and the pressure term it is having $\delta p_{i,j+1}$-$\delta P_{i,j}$ into Δx.

Now, from these 2 equations again these 2 equations are basically implicit equations. Here $u_{i,j}$ correction $u_{i+1,j}$ correction $u_{i-1,j}$ correction $u_{i,j+1}$ correction $u_{i,j-1}$ correction all are unknown. But δp if this is also unknown, so we have to break this implicitness through some approximation. And what do we here doing we are saying that this correction terms at the neighbouring cells correction terms v velocity also corrections at the neighbouring points may be ignored.

If we ignore them we definitely mathematically we will not get exact expression, expression will be different. But if we keep on doing iteration we will be able to compensate that. So, from equation 13 neglecting u_{nb} into a_{nb}, that means neighbouring velocity corrections and their coefficients we are getting an explicit expression for $u_{i,j}^c$ velocity correction term.

Simply we are neglecting this and we are expressing $u_{i,j}^c$ equal to this term right hand side divided by it is coefficient on the left hand side. So, and we do some little algebra when algebraic you know sort of rearrangement and we get this expression as $u_{i,j}^c = \frac{E_1 \Delta y(\delta p_{i,j} - \delta p_{i+1,j})}{(1+E_1) a_{i,j}^u}$.

So, and what is this E_1 this E_1 is not to be confused with E^1 , E^2. We used in the momentum equation there E^1 or E^2 whatever we used those were basically superscript, this E_1 is subscript it is a just defined quantity does not have any deeper meaning. The way it was having for x momentum and y momentum equation here it is simply you know to write $u_{i,j}^c$. That means, velocity correction from this expression after neglecting this neighbouring terms.

What we get when you do algebraic rearrangement in this E subscript 1 E_1 is $\Delta t\, a_{i,j}^u$ divided by $\Delta x\, \Delta y$, $a_{i,j}^u$ is this coefficient of the velocity correction and this coefficient is same when we wrote $u_{i,j}^n$ and that when we derived u momentum equation we had term by term expansion of $a_{i,j}^u$ can again look at that.

(Refer Slide Time: 36:24)

An equivalent expression can be obtained for $v^c_{j,k}$

$$v^c_{i,j} = \frac{E_2\ \Delta x\,(\delta p_{i,j} - \delta p_{i,j+1})}{(1+E_2)\,a^v_{i,j}} \qquad (16)$$

where $E_2 = \Delta t\ a^v_{i,j} / \Delta x \Delta y$

Now, substitution of $u^{n+1}_{i,j} = u^*_{i,j} + u^c_{i,j}$ use of $v^{n+1}_{i,j} = v^*_{i,j} + v^c_{i,j}$ (15) and (16) gives

$$a^p_{i,j}\ \delta p_{i,j} = \sum a^p_{nb}\ \delta p_{nb} + b^p \qquad (17)$$

where, $b^p = -\left(u^*_{i,j} - u^*_{i-1,j}\right)\Delta y - \left(v^*_{i,j} - v^*_{i,j-1}\right)\Delta x$

basically a discrete form of Poisson equation equivalent to:

$$\nabla^2(\delta p) = \frac{1}{\Delta t}\nabla.\mathbf{u}^* \qquad (18)$$

The solution converges as $b^p = 0$. Eq. (15) and (16) can be represented as

$$\mathbf{u^c} = -\frac{1}{\Delta t}\nabla(\delta p) \qquad (19)$$

So, everything is known this is E subscript 1 E_1 and from equation 14 then we can write $v_{i,j}$ velocity correction on v velocity correction and here we get this expression. Here also just after getting obtaining the expression from this equation neglecting the neighbouring terms whatever we get.

If we reorganize algebraically everything, we will get this E_2 appearing in numerator and denominator. And $E_2 = \Delta t\ a_{i,j}^v$ is coefficient of $v_{i,j}$ whether at n+1 level or n level that we derive that coefficient of $v_{i,j}$ divided by $\Delta x\ \Delta y$.

So, we have found out v velocity correction and u velocity correction. Now, this u velocity correction and v velocity correction if we add with the u star and v star which we have provisionally calculated, we have provisionally calculated starred quantities using nth level pressure because only that pressure was available n+1 th level pressure was not available. So, you have found out this starred quantity.

Now, in the starred quantity if we add the correction terms, we are likely to get velocities at the level n+1. So, we in the original continuity equation which was equation 5 previous lecture we derived discretized equation 5, they are $u_{i,j}^{n+1}$ and $v_{i,j}^{n+1}$ are substituted through this starred and correction quantities. Similarly, $u_{i-1,j}$ at n+1 will be substituted. Similarly, $v_{i,j-1}$ at the level n+1 will be substituted by corresponding starred and correction terms. Then from there we can get equation 17 which is an equation for pressure correction. You can see $\delta p_{i,j}$ with some coefficient will be δp at the neighbouring points, that means δp at i-1,j δp at i+1,j δp at i,j+1 ; j- 1 and they are coefficient together and b^p.

This b^p if you do this substitution by yourself and try to write down the equation in resulting equation in this form you will be able to write and you will get 1 term b^p, which is given by as you can see $-\left(u_{i,j}^* - u_{i-1,j}^*\right)\Delta y - \left(v_{i,j}^* - v_{i,j-1}^*\right)\Delta x$ As if this is basically the continuity equation written using starred quantities and if it is it produces 0, b^p will be 0.

That means, when it will produce 0 if all starred quantities have been elevated to n+1 level quantities. So, this is a meaning of b^p, so effectively equations 17 is a poisons equation for pressure correction. So, this is $\nabla^2(\delta p) = \frac{1}{\Delta t}\nabla.\boldsymbol{u}^*$, here **u** is small u small v both.

So, basically this divergence of u^* is $\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}$ both are starred quantities, approximated quantity and we can write their differences divided by Δx differences divided by Δy.

Now, this is we keep on changing how? Because you know we will find out solving this Δp these Δp will feed here we will find out velocity correction, we will find out u velocity correction, we will find out v velocity correction we will change this $u_{i,j}^{n+1}$ through velocity correction $v_{i,j}^{n+1}$ through velocity correction.

Again, use that value here that will be new quantities not at the has not reached may be in one go n+1 still we will use star. And then when it will reach really n+1 level, then this divergence quantity will be 0 and there will be no δp correction and whatever velocity correction we got the earlier iteration that is a final velocity correction and that will then that means velocities at the level n+1 have emerged we have got those velocities.

(Refer Slide Time: 43:12)

SIMPLE-Algorithm

The Algorithm may be summarized as

1. **u*** are obtained from Equations (11) and (12).
2. δp is obtained from Equation (18)
3. $\mathbf{u^c}$ are obtained from Equations (15) and (16)
4. p^{new} is obtained from $p^{new} = p^n + \alpha\,\delta p$, where α is a relaxation parameter
5. Calculate updated $u_{i,j}$, $v_{i,j}$ from their starred values using velocity correction formulae $u_{i,j}^{new} = u_{i,j}^* + u_{i,j}^c$ and $v_{i,j}^{new} = v_{i,j}^* + v_{i,j}^c$
6. Treat the corrected pressure p^{new} as the new initial pressure p^n. Use $u_{i,j}^{new}$ and $v_{i,j}^{new}$ as $u_{i,j}^n$ and $v_{i,j}^n$. Calculate new **u*** from Equations (11) and (12). Repeat steps 2-5 until a converged solution is obtained. The convergence will be achieved as $b^p = 0$.

So, here is the systematic way of writing the algorithm u star are obtained from equation 11 and 12, let us go 2 equation 11, 12. That means, if we use pressure which we got which we have gotten through the previous level of calculation nth level of calculation. If we use those we will be only getting u star and v star.

So, this is u star are obtained from equation 11 and 12 δp is obtained equation 18. So, using those u stars if we try to get δp as you can see here it will be a you know Penta diagonal matrix. It will not be known you know at i j, i +1 j, i-1 j, i j+1, i j-1 you have to run in the entire domain i = 2 to imax-1, j=2 to jmax-1 and we will get this matrix.

Solve it δp will be obtained and then pressure corrections are obtained from equation 15 and 16. So, as and when δp is obtained we can go back we can look at equation 15 velocity u velocity correction equation 16 v velocity correction is obtained. Then they are again they will update the velocity quantities and p^{new} is obtained as $p^n + \alpha\delta p$.

Where α is relaxation parameter, usually this is under relaxed we have this discussion later. This so pressure is updated p's, p at nth level with this δp pressure correction decided with p^n and p^{new} emerges. So, similarly we also update u_{ij} v_{ij}, so u_{ij}^{new} is $u_{i,j}^* + u_{i,j}^c$, $v_{i,j}^{new}$ is $v_{i,j}^* + v_{ij}^c$ and p^{new} is $p^n + \alpha\ \delta p$.

So, this way you see you know we are advancing from one level to another level. Treat the corrected pressure p^{new} as the new initial pressure p^n use $u_{i,j}^{new}$ and $v_{i,j}^{new}$ which we have just updated as $u_{i,j}^n$ and $v_{i,j}^n$, again calculate u star from 11 and 12. Repeat the steps 1 to 5 until a converged solution is obtain.

And when you get converged solution? Converged solution you get when b_p equal to 0; that means, effectively b^p is this term this is basically that means divergence of v at every cell will be 0, there will be no longer pressure correction. So, $b^p = 0$ and this is how you get the velocity at n+1 th level or you know it will not change any more if it you reach a steady state.

(Refer Slide Time: 47:32)

Patankar (1980) introduced a revised algorithm, SIMPLER to improve the convergence. The SIMPLER algorithm has the following steps.

1. A velocity field $\hat{u}$ is computed form Equations (11) and (12) with the pressure terms deleted from the right-hand sides.

2. Equation (17) then becomes a Poisson equation for p^{n+1} rather than δp with $\hat{u}$ replacing the **u*** terms in b^p

3. The p^{n+1} (obtained from 2 steps) replaces p^n in equations (11) and (12), which are solved for u* (as it was done in SIMPLE).

4. Equation (17) is solved for δp and it is used to provide $u^{n+1} = u^* + u^c$ but no further adjustment is made to p^{n+1} found from step 2

This is basic simple algorithm a Professor Patankar revised it in 1980 and gave a name SIMPLER. While in know pronouncing it pronounces SIMPLER, but it is actual meaning is simple revised and simple revised what is you know basically little quicker convergence.

What is done you go for equation 11 and 12, you remove the pressure terms try to evaluate instead of u^* through implicit calculation a quantity given by $\hat{u}$.

So, you can see that $\hat{u}$ is computed from equation 11 and 12 with the pressure terms deleted from the right-hand sides. Equations 17 then becomes a Poisson equation; that means, equation 17 becomes a Poisson equation for pressure not pressure correction equation for p^{n+1}. Because now you bring in p^{n+1} you did not use $\hat{p}$ when you evaluated equation 11 and 12. This is p^{n+1} rather than δp with $\hat{u}$ replacing u^* terms in b_p.

The p^{n+1} obtained from these two steps replaces p^n in equation 11. So, this way equation 11 you drop p terms you evaluate u^*. Now this u^* there is no p you are calling it $\hat{u}$, this $\hat{u}$ you are feeding these us in equation 17. And you are solving not for or 18 you can say really not 17.

And when you get the final solution that is basically not you are not solving for δp, but p at n+1 th level. So, you replace p^n in equation 11 and 12 which are solved for sorry equations 17 then becomes Poisson equation for p^{n+1} rather than δp with $\hat{u}$ replacing the u star terms in b_p. The p^{n+1} obtained from these two steps replaces p^n in equation 11 and 12.

So, then this replaces p^n now we rewrite equation 11 and 12 whatever p we get from the Poisson's equation for pressure not for pressure correction, now it becomes Poisson's equation for pressure. And then we use 11 and 12 and find out u star this u^* , with this u star we again come back to equation 17 that and we keep on calculating δp 17 or 18.

But we do not change the pressure because already we have finalized the pressure as p^{n+1}. But whatever δp we get we make use of that to correct to get velocity correction and we keep on updating velocity till we get the convergence. That means, divergence of velocity in each cell is 0 and there is no longer δp from equation 17 this is called simple revised simpler.

There is yet another modification which is called simple c, it was developed by Van Doormal and Raithby. I am not discussing this in this lecture, because this is again you know modification of the procedure to have little better convergence or little bit little more accelerated calculation. If we have understood basic simple algorithm then these modifications like simple revised or simple c or easy absolutely no problem.

(Refer Slide Time: 53:28)

Two-dimensional System of equations and Line-by- Line TDMA

Whether the discretized momentum Eqns. (11) and (12) or the pressure correction Eq. (17) the final outcome is a system of algebraic equation given by $a_p\phi_p = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + b.$ The current values of the dependent variables $(\phi's)$ are to be evaluated from the known values of the coefficients (a's). The evaluation algorithm will be same for the momentum equations and the pressure correction equation. On a rectangular grid, the dependent variables at one point (i,j) may be expressed in terms of its neighbors as

$$a_{i,j}u_{i,j} = b_{i,j}u_{i+1,j} + c_{i,j}u_{i-1,j} + d_{i,j}u_{i,j+1} + e_{i,j}u_{i,j-1} + f_{i,j}$$

where $b_{i,j}$ is equivalent to $a_E,$ $c_{i,j}$ to a_W, $d_{i,j}$ to a_N, $e_{i,j}$ to a_S and $f_{i,j}$ to $b.$ The evaluation of u can be accomplished in the following ways:

Now, we will since in this calculation everywhere we are we will be getting equations in this form, whether it is 11 or 12 or 17. In this form ϕ_p, ϕ maybe u, ϕ maybe v, ϕ maybe pressure our variable of interest and p is point $u_{i,j}$. So, ϕ_p that means $u_{i,j}$ or $v_{i,j}$ or $p_{i,j}$ with it is coefficient this is Eastern neighbor; that means, $u_{i+1,j}$ and it is coefficient Western neighbour, $u_{i-1,j}$ and it is coefficient northern neighbour $u_{i,j+1}$ and it is coefficient southern neighbour $u_{i,j-1}$, then it is coefficient plus b

Same is a style of express expressing you know v momentum equation pressure equation, always we get in terms of you know neighbouring values, neighbouring variables and their coefficients. The current value of in the dependent variables that is what is this we are saying phi is it may be u, it may be v, it may be p are to be evaluated from the known values of coefficients.

The evaluation algorithm will be same for the momentum equations and the pressure correction equation. On a rectangular grid the dependent variables at 1 point i, j point which is point p may be expressed in terms of his neighbours. Again, whatever I have narrated I have written it again instead of ϕ_P if we write $u_{i,j}$ coefficient is $a_{i,j}$ instead of ϕ_E if I write Eastern neighbour $u_{i+1,j}$ coefficient is $b_{i,j}$. Western neighbour ϕ_W $u_{i-1,j}$, it is coefficient $u_{i,j+1}$ it is coefficient $u_{i,j-1}$ it is coefficient plus $f_{i,j}$ all the remaining terms. Where $b_{i,j}$ is equivalent to a_E, $b_{i,j}$ is equivalent to coefficient of Eastern neighbour a_E $c_{i,j}$ is coefficient of Western neighbour $d_{i,j}$ is coefficient Northern neighbour $e_{i,j}$ coefficient of Southern neighbour and $f_{i,j}$ to the free term b.

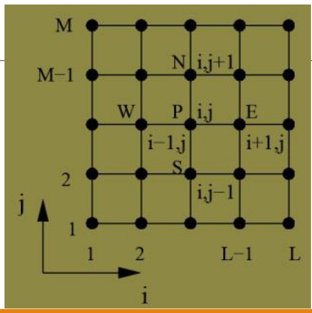
(Refer Slide Time: 56:41)

(a) Vertical Sweep Upwards may be written in a pseudo FORTRAN code as

DO 10 J = 2, M – 1
DO 10 I = 2, L – 1

$$10\ a_{i,j}u_{i,j} = b_{i,j}u_{i+1,j} + c_{i,j}u_{i-1,j} + \left(d_{i,j}\bar{u}_{i,j+1} + e_{i,j}\bar{u}_{i,j-1} + f_{i,j}\right)$$

Here $\bar{u}$ is the currently available value in storage and all the coefficients including $f_{i,j}$ are known. For each j, we shall get a system of equations if we substitute $i = 2......L-1$. In order words, for each j, a tridiagonal matrix is available which can be solved for all i row of points at that j. Once one complete row is evaluated for any particular j, the next j will be taken up, and so on.



The evaluation of u can be accomplished in the following ways to use, because it is you know again if we can it is not a tridiagonal matrix, but if we can solve in the format of tridiagonal matrix, we get some advantages. Vertical sweep upwards may be written in a pseudo FORTAN code or pseudo c code also you can say DO 10 J varies from 2 to $M-1$, M is a maximum number of points 1 to M. So 2 to $M-1$ will run the loop.

Similarly in the other direction it is 1 to L, so we will run 2 to $L-1$. So, this do loop J = 2 to M – 1, I = 2 to L - 1 and this is the algorithm $a_{i,j}$ $u_{i,j}$=$b_{i,j}$ $u_{i+1,j}$ + $c_{i,j}$ $u_{i-1,j}$….

So, this is you know these are implicitly related these will be evaluated by varying I = 2 to L-1, J = 2 to M-1. We will get system of algebraic equation, whereas when are you know setting this loop that these barred quantities $\bar{u}$ is a currently available value in storage and all coefficients including $f_{i,j}$ are known.

So, we will assume j+1, j-1 f_{ij} this are known we are varying in i. So, for each j we shall get a system of equation if we substitute i = 2 to L−1. In other words, for each j a tridiagonal matrix is available which can be solved for all i row of point at that j. Once one complete row is evaluated then we go for next j and at every j we will see that j+1 and j−1 will be needed.

And those values are which are available latest value we will club them together as known values. And they will be contributing as from their known quantities.

(Refer Slide Time: 59:43)

(b) Horizontal Sweep Forward may be written in pseudo FORTRAN code as

DO 20 I = 2, L – 1
DO 20 J = 2, M – 1

$$20 \quad a_{i,j}u_{i,j} = b_{i,j}u_{i,j+1} + e_{i,j}u_{i,j-1} + \left(b_{i,j}\bar{u}_{i+1,j} + e_{i,j}\bar{u}_{i-1,j} + f_{i,j}\right)$$

Again $\bar{u}$ is the currently available value in storage from previous calculations. for each i we get a system of equation if we substitute $j = 2......M-1$. A tridiagonal matrix is available for each i. Once one complete column of points are evaluated for any particular i, the next i will be taken up, and so on. The vertical sweep upward and downward are repeated. Similarly, the horizontal sweep forward and rearward are also repeated until convergence is achieved. For solving tridiagonal system, the tridiagonal matrix algorithm (TDMA) due to Thomas (1949) is deployed. The above-mentioned evaluation procedure is known as **Line-by-Line TDMA.**

So, this is a Vertical sweep and the earlier 1 this is a Horizontal sweep, here you can see we have said that do loop DO 20 I=2 to L-1 DO 20 J=2 to M-1. And now j+1 see earlier we bracketed this j+1 j-1, terms and they were all given bar this barred quantities are value in store.

So, we are through algebraic equation we are updating these quantities i, i+1, i-1 at a given j. Now, next what at this stage, what we will do? We will this barred quantities are now i+1, i-1 and equation is implicit in j direction i, j, j+1, j-1 these are not known. Again $\bar{u}$ is currently available value in stage from previous calculations, for each i we get a system of equation if we substitute j equal to 2 to M-1.

So, this is basically you know j, j+1, j-1 you know for every j value 2, 3, 4, 5 up to M-1, we get system of equations. A tridiagonal matrix is available for each i location, once one complete column of points is evaluated for any particular elements all j's then next i will be taken up and so on.

The vertical sweep upward and downward are repeated, we will not only go from 2 to $L-1$. We will go over come from you know the top most value to the bottom most value also. We will just you know reverse, the sweep vertical sweep upward and downward are repeated similarly horizontal sweep forward and rearward this can also be repeated until convergences achieved. For solving tridiagonal system, the tridiagonal matrix algorithm TDMA due to Thomas which we have already used it Thomas algorithm is deployed.

So, these this is called line by line TDMA this is quite similar to alternating direction implicit scheme. But here we are just from the field variables we are updating the values keeping basically one variable direction implicit. Once we are keeping j variables implicit and for every i we are running basically tridiagonal matrix in j direction.

Another sweep we are keeping j direction fixed; that means, at given j we are forming TDMA in i direction i equal to beginning to end point and evaluating. So, this is even if the number of unknowns is not 3 i j, i+1 j, i-1 j, i j+1, i j-1. These are unknown implicitly related.

But through line-by-line TDMA we can again evaluate such you know matrix without going into a different type of matrix solver using similar to what we did similar process. What we did for solving tridiagonal matrix TDMA. So, this is called Line by Line TDMA.

(Refer Slide Time: 64:28)

Useful References

S. V. Patankar, A Calculation Procedure for Two-Dimensional Elliptic Situations, **Numerical Heat Transfer**, Vol. 4. pp. 409-425, 1981.

S. V. Patankar, ***Numerical Heat Transfer and Fluid Flow***, Hemisphere Publishing Co., 1980.

S. V. Patankar, and D. B. Spalding, A Calculation Procedure for the Heat, Mass and Momentum Transfer in Three-Dimensional Parabolic Flows, **Int. J. Heat and Mass Transfer**, Vol. 15, pp. 1787-1805, 1972.

C.A.J. Fletcher, ***Computational Techniques for Fluid Dynamics***, Vol. 2, Springer, 2003

R. Peyret and T. D. Taylor, ***Computational Methods for Fluid Flow***, Springer Verlag, 1983.

Now, these are some useful references as you can see 1st one is a published paper by professor Patankar, 2nd one is professor Patankar book, 3rd one is the first paper on solving Full Navier stokes equations on solution of Full Navier stokes equations as I mentioned by Patankar and Spalding.

And the 4th one is text book volume 2 of Computational Techniques and for Fluid dynamics by professor Fletcher and 5th one is also another book by Professor Roger Peyret and T. D. Taylor this is a Springer book very good book. So, whatever I have explained you will get you know the same material in different way of writing through different explanations in these papers and books.

Thank you very much. Thank you.