Computational Fluid Dynamics and Heat Transfer
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Lecture - 17
Solution N-S Equations for Incompressible Flows Using SIMPLE Algorithm

Good morning, everyone. Today we will work on solution of Navier-Stokes equations for incompressible flows using SIMPLE algorithm. SIMPLE stands for Semi Implicit Pressure Linked Equations.

Now, I will make few very important comments here. SIMPLE algorithm was developed during the period late 60s and early 70s at the Imperial College London. Prof. D. B Spalding was a very well-known professor and researcher at the Imperial College; also Prof. David Gosman was very well-known.

Now, in this group of Prof. Spalding and Prof. Gosman quite a few very active researchers were working, and among them I can mention about Prof. S. V. Patankar, Prof. S P Banka, Prof. A. K. Ranchal, Prof. Wolfgang Rodi, Prof. Michael Waldstein and few others. All of them are very famous academicians today all over the world, and they are located at different locations, different places. And you know they are very visible in thermo fluids community.

SIMPLE algorithm was primarily developed by Prof. S. V. Patankar and Prof. D P Spalding. And on the same time, another very well-known algorithm was developed; its name is MAC – M A C. It is basically also an acronym; the full name is marker and cell. Now, MAC algorithm was developed at the Los Alamos National Laboratory of the United States. And the scientists who developed are Prof. F. H. Harlow, Prof. J. E Welch, Prof. Antony Amston, Prof. P. D. Nicholls and few others.

Now, today many commercial and open-source scientific software are available to solve extremely complex problems involving fluid flow, heat transfer, mass transfer, combustion, chemical reactions etcetera. Now, in most of such software wherever flow is incompressible, the basic solver flow solver that is used is a variant of either SIMPLE algorithm or MAC algorithm, that is why MAC and SIMPLE both these algorithms are very important for its relevance and usefulness.

And specifically for the learners, these are the pivotal points of learning flows about the flow solvers.

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So, we will discuss about you know in this sequence, staggered grid, semi-implicit method for pressure linked equations, continuity equation, x-momentum equation, y-momentum equation, pressure solver, and line by line TDMA.

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Introduction

Equations for incompressible three dimensional flows are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (1)$$

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \quad (2)$$

$$\frac{\partial v}{\partial t} + \frac{\partial (uv)}{\partial x} + \frac{\partial (v^2)}{\partial y} + \frac{\partial (vw)}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] \quad (3)$$

$$\frac{\partial w}{\partial t} + \frac{\partial (uw)}{\partial x} + \frac{\partial (vw)}{\partial y} + \frac{\partial (w^2)}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] \quad (4)$$

Now, equations for incompressible three-dimensional flows are basically continuity equation and momentum equations. In the momentum equations, you can see we have written $1/Re$ instead of ν coefficient of viscosity, that means, we have non-dimensionalized the equations. So, all the velocities are non-dimensionalized, pressure is non-dimensionalized. And instead of ν we get 1 by Reynolds number together with the viscous terms.

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Staggered Grid

Figure 1 shows a two-dimensional staggered grid where independent variables with the same indices are staggered to one another. Extension to three- dimensions is straight- forward. The computational domain is divided into a number of cells, which are shown as "main control volumes" in Figure1.

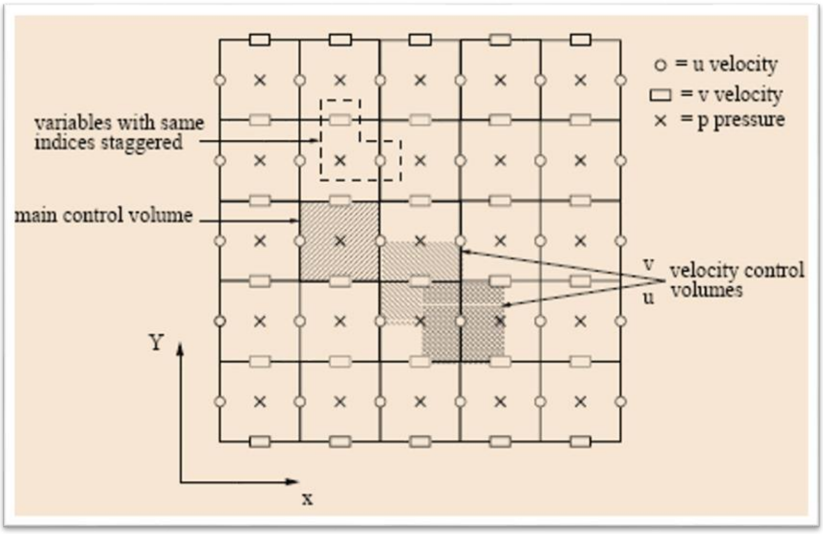

Figure 1: Staggered grid

I will mention one feature that is common with both MAC and SIMPLE algorithms. Both the algorithms use staggered grid. Now, what is staggered grid? Let me start with the basic definition of you know grid topology a domain is divided by number of cells; we can call it number of control volumes. And in these control volumes, you can see location of u velocity is identified by open circle, locations of v velocities are identified by this open rectangle, and locations of pressures are identified by this cross mark.

So, if you take a cell like this, so these are locations of u velocities, these are locations for v velocities, this is the location for pressure. Now, u velocity here is given by u_{ij}, u velocity here is given by $u_{i-1,j}$, v velocity here is given by v_{ij}, and v velocity here is given by $v_{i,j-1}$, and this is p_{ij}. So, p_{ij}, u_{ij} , v_{ij} , you can see they are with the same indices u_{ij}, v_{ij}, p_{ij} variables with same indices are staggered with each other.

Velocities are located at the center of the cell faces to which they are normal. Velocities are located at the center of the cell faces to which they are normal. And pressure, temperature etcetera are located at the center of the cell. Now, this is two-dimensional grid topology. Extension in three dimension is straight-forward.

Now, there is there is a reason for choosing staggered grid. If you look at the governing equations, I am going back by one slide you will see that momentum equations have dp/dx, dp/dy, dp/dz, but we do not get any obvious expression for pressure. Pressure is given in terms of pressure gradients – that is for incompressible flows we know very well.

In the case of compressible flow, we get an obvious expression for pressure that comes from the equation of state. But here that obvious expression for pressure is absent instead we have pressure gradients. So, we require to establish some relationship between the pressure gradient and the velocity.

As I said this location if this is u_{ij}, this is p_{ij}, then this point is $p_{i+1,j}$. Now, there is a pressure gradient between these two points $p_{i+1,j}$, $p_{i,j}$ this pressure gradient is directly responsible for this velocity u_{ij}. Similarly, in the y-direction, this pressure if this is p_{ij}, this pressure is $p_{i,j+1}$.

So, there is a gradient between $p_{i,j+1}$ and $p_{i,j}$. And this gradient dp/dy is responsible or related or linked to $p_{i,j}$. So, pressure gradients in the respective directions are linked to the interfacial velocities. This gives a weaker convergence.

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SIMPLE

SIMPLE algorithm is based on finite-volume discretization of the Navier-Stokes equations. The discretization indicated below corresponds to a uniform grid. Consider the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

For the control volume shown in Figure 2. The application of the finite-volume method (Green's Theorem) to the continuity equation produces

$$\iint \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx\, dy = \oint (-v\, dx + u\, dy) = 0$$

The contour integral on the control volume of interst results in:

$$\left(u_{i,j}^{n+1} - u_{i-1,j}^{n+1}\right) \Delta y + \left(v_{i,j}^{n+1} - v_{i,j-1}^{n+1}\right) \Delta x = 0 \qquad (5)$$

So, let us now look at the continuity equation. We are considering two dimensions. So,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

So, we will integrate this in the over the main control volume, that means, you know the control volume given by this arrangement this is u_{ij}, this is $u_{i-1,j}$, this is v_{ij}, this is $v_{i,j-1}$, and this is p_{ij}. So, we are integrating over a cell volume or cell area $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\ dy$ area integral.

And we apply Greens theorem for evaluating such area integrals which becomes line integral of $-vdx + udy$. And then if we perform the line integral, we will get $u_{ij} - u_{i-1,j}$ both at the level of n+1 into Δy plus $v_{ij} - v_{i,j-1}$ both at the level of n + 1 into Δx equal to 0. We can give a little deeper insight into it.

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$$\oint(-v\,dx + u\,dy) = 0 \quad : \quad \left(u_{i,j}^{n+1} - u_{i-1,j}^{n+1}\right)\Delta y + \left(v_{i,j}^{n+1} - v_{i,j-1}^{n+1}\right)\Delta x = 0$$

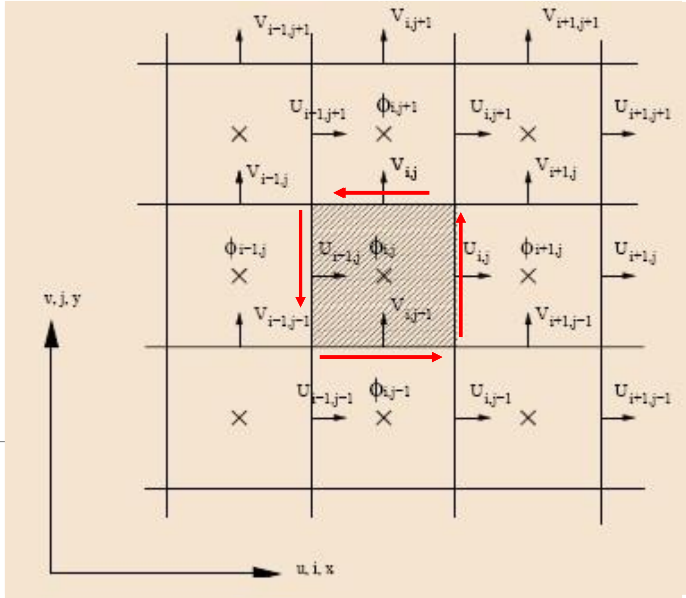

Figure 2: Control volume for continuity equation

How the integration has been done? This is line integral over the cell of interest. Now, you can see $u\,dy$. So, u this dy is positive, u_{ij} into dy, $u_{i-1,j}$ this dy is in the negative direction. So, u_{ij} minus $u_{i-1,j}$ into Δy, both are at the level of n+1. Minus $v\,dx$, now v_{ij} is defined here. But when we do the cyclic integral, dx is in the negative direction.

So, v_{ij} into dx it is in the negative direction, so minus $v_{i,j}\,dx$, but that quantity has to be multiplied by minus, so it becomes plus. Plus $v_{i,j}$ at n+1, similarly in the cyclic integral this is $v_{i,j-1}$, and this is positive direction in the positive dx. So, $v_{i,j-1}$ into dx, it is positive. But in the expression for the integration, negative sign is there, so that sign will come.

And this will be then v_{ij} minus $v_{i,j-1}$ into Δx both are at the level n + 1. So, this becomes the discretized continuity equation that is what we wrote in the previous slide if you can see this is equation 5 is the discretized continuity equation. And how this discretization is done? We have shown it because of the cyclic integral, we get this form.

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X- momentum equation

The Navier-Stokes equation in x-direction in conservative form:

$$\frac{\partial u}{\partial t}+\frac{\partial (u)^2}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial p}{\partial x}=\frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

Integrating over the u-control volume (see the figure), one can write

$$\frac{\Delta x\,\Delta y}{\Delta t}\left(u_{i,j}^{n+1}-u_{i,j}^{n}\right)+\iint\left[\frac{\partial}{\partial x}\left\{u^2-\frac{1}{Re}.\frac{\partial u}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{uv-\frac{1}{Re}\frac{\partial u}{\partial y}\right\}\right]dx\,dy+\iint\left(\frac{\partial p}{\partial x}\right)dx\,dy=0$$

Application of the finite- volume method to the x- momentum equation

$$\frac{\Delta x\Delta y}{\Delta t}\left(u_{i,j}^{n+1}-u_{i,j}^{n}\right)+\left(E^1_{i+\frac{1}{2},j}-E^1_{i-\frac{1}{2},j}\right)\Delta y+\left(F^1_{i,j+\frac{1}{2}}-F^1_{i,j-\frac{1}{2}}\right)\Delta x+\left(p_{i+1,j}^{n+1}-p_{i,j}^{n+1}\right)\Delta y=0 \qquad (6)$$

with $E^1=u^2-\frac{1}{\text{Re}}\frac{\partial u}{\partial x}$ $\quad F^1=uv-\frac{1}{\text{Re}}\frac{\partial u}{\partial y}$

Now, we look at the x-momentum equation as usual terms have been written as

$$\frac{\partial u}{\partial t}+\frac{\partial (u)^2}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial p}{\partial x}=\frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

And we perform this integration, we do this is called mass lumping. We you know over the control volume, we do the integration through $dx\,dy$. And then over the control volume, this is equivalent to if you multiply by ρ it will be in non-dimension ρ is 1. So, this is mass by Δt and $u_{ij}^{n+1}-u_{ij}^{n}$ at the level of n the $\partial u/\partial t$ is taken care of.

Now, this part of the integration is basically we are doing following way. We are writing it in strong conservative form. So, we are moving viscous term to the left-hand side. So, this becomes then

$$\frac{\Delta x\,\Delta y}{\Delta t}\left(u_{i,j}^{n+1}-u_{i,j}^{n}\right)+\iint\left[\frac{\partial}{\partial x}\left\{u^2-\frac{1}{Re}.\frac{\partial u}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{uv-\frac{1}{Re}\frac{\partial u}{\partial y}\right\}\right]dx\,dy+\iint\left(\frac{\partial p}{\partial x}\right)dx\,dy=0$$

Now, this integration again if you apply Greens theorem, we will get something like you know minus of this term into dx plus of this term into dy the term within the bracket into within the curly bracket within the second bracket $\partial/\partial x$ of this term within curly bracket $dx\,dy$.

So, we will get positive this quantity which is given by $E^1 = u^2 - \frac{1}{Re}\frac{\partial u}{\partial x}$ and so E^1 into dy and here this quantity minus F^1, this we are defining $F^1 = uv - \frac{1}{Re}\frac{\partial u}{\partial y}$ into dx.

And then we perform the integration. After integration we are writing $E^1_{i+\frac{1}{2},j} - E^1_{i-\frac{1}{2},j}$. So, $\left[E^1_{i+\frac{1}{2},j} - E^1_{i-\frac{1}{2},j}\right]\Delta y$. And $\left[F^1_{i,j+\frac{1}{2}} - F^1_{i,j-\frac{1}{2}}\right]\Delta x$, that is the result of this integral. Again, this area integral should be converted into line integral using Greens theorem and evaluated. I will do it elaborately in the next slide. And this integration we can write simply $p_{i+1,j} - p_{i,j}$ simply from this integration into Δy both the pressures at the level of n+1.

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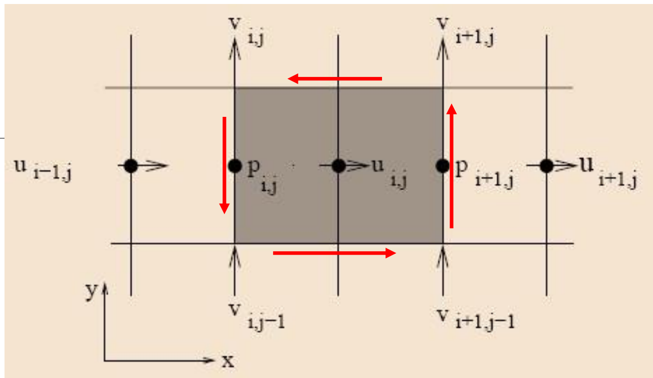
X- momentum equation

Application of the finite- volume method to the x- momentum equation

$$\frac{\Delta x \Delta y}{\Delta t}\left(u^{n+1}_{i,j} - u^n_{i,j}\right) + \left(E^1_{i+\frac{1}{2},j} - E^1_{i-\frac{1}{2},j}\right)\Delta y + \left(F^1_{i,j+\frac{1}{2}} - F^1_{i,j-\frac{1}{2}}\right)\Delta x + \left(p^{n+1}_{i+1,j} - p^{n+1}_{i,j}\right)\Delta y = 0 \qquad (6)$$

with $E^1 = u^2 - \frac{1}{Re}\frac{\partial u}{\partial x}$ $\qquad F^1 = uv - \frac{1}{Re}\frac{\partial u}{\partial y}$



Now, how this integration is performed? And this integration is performed on u control volume or on x control volume by x-momentum control volume or on u control volume; u control volume means main control volume pressure is at the center. If you recall when we discussed staggered grid we mentioned and u control volume means u is at the center.

So, u is at the center, that means, the right neighbor immediate right neighbor is $p_{i+1,j}$ immediate left neighbor is p_{ij}. And this is $u_{i+1,j}$ this is $u_{i-1,j}$, and we perform the integration on this. So, as I was mentioning that you know this quantity E^1 which is $u^2 - \frac{1}{Re}\frac{\partial u}{\partial x}$, this E^1 is defined here and it is being integrated. So, E^1 which is with respect to u_{ij} its location is $i + \frac{1}{2}, j$.

So, $E^1_{i+\frac{1}{2},j}\Delta y - E^1_{i-\frac{1}{2},j}\Delta y$, and this is I told you this will be $F^1_{i,j+\frac{1}{2}}$. But when we apply Greens theorem basically this will be $-F^1_{i,j+\frac{1}{2}}\, dx$. Now, $-F^1_{i,j+\frac{1}{2}}$, F is defined here and x into dx. So, x is in the negative direction that makes F into $-F^1_{i,j+\frac{1}{2}}$ into dx positive, so $F^1_{i,j+\frac{1}{2}}\, dx$.

And as I said it is minus $F\, dx$. So, when you integrate this side, this is with positive Δx. So, $F^1_{i,j-\frac{1}{2}}$ at this location into dx, dx is positive. But this quantity F^1 has a negative sign because of Greens theorem and that makes it negative minus $F^1_{i,j-\frac{1}{2}}$, this whole quantity into Δx.

So, this is the result of integration of this E^1 and F^1 along the line integral along the along its contour of the defined cell. So, it is clear now, that means, we get this term we have already explained. These two terms we have now explained. And this $p_{i+1,j} - p_{i,j}$ into Δy we have obtained earlier we have shown in the earlier slide.

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E^1 and F^1 are axial and transverse fluxes of x-momentum. Thus

$$E^1_{i+\frac{1}{2},j} = 0.25\left(u_{i,j} + u_{i+1,j}\right)^2 - \frac{1}{Re}\frac{(u_{i+1,j} - u_{i,j})}{\Delta x}$$

$$E^1_{i-\frac{1}{2},j} = 0.25\left(u_{i-1,j} + u_{i,j}\right)^2 - \frac{1}{Re}\frac{(u_{i,j} - u_{i-1,j})}{\Delta x}$$

$$F^1_{i,j+\frac{1}{2}} = 0.25\left(v_{i,j} + v_{i+1,j}\right)\left(u_{i,j} + u_{i,j+1}\right) - \frac{1}{Re}\left(\frac{u_{i,j+1} - u_{i,j}}{\Delta y}\right)$$

$$F^1_{i,j-\frac{1}{2}} = 0.25\left(v_{i,j-1} + v_{i+1,j-1}\right)\left(u_{i,j-1} + u_{i,j}\right) - \frac{1}{Re}\left(\frac{u_{i,j} - u_{i,j-1}}{\Delta y}\right)$$

So, now, what we will do? We will simply write this term at this point. So, $E^1_{i+\frac{1}{2},j}$ means E^1 has to be written at $i + 1/2, j$ point, that means, u_{ij} is defined here $u_{i+1,j}$ is defined here.

So, mean of that, that means, $(u_{ij}+u_{i+1,j})/2$ will be u here and its square. So, we will get $(u_{ij}+u_{i+1,j})$ square and denominator will have 4 that is 0.25, 1 by 4; 0.25. Then – 1/Re, $\frac{\partial u}{\partial x}$. So, basically at this point $\frac{\partial u}{\partial x}$ means $(u_{i+1,j} - u_{i,j})/\Delta x$. So, $(u_{i+1,j} - u_{i,j})/\Delta x$.

In the same way, we will define E^1 at this point that means, $E^1_{i-1/2,j}$, and that means, now it will be $u_{i-1,j}$ + u_{ij} by 2 square divided by 4 so 0.25, and the other term is $\frac{\partial u}{\partial x}$ which will be $u_{ij} - u_{i-1,j}$ by Δx. So, $u_{ij} - u_{i-1,j}$ by Δx into 1 by Re, so that is how this is basically interpolation of the neighboring cells to find out $E^1_{i+1/2,j}$ and $E^1_{i-1/2,j}$.

Similarly, F^1 has to be defined here. Now, when we define F^1 here it is uv. So, u has to be interpolated from neighboring u's. This u is u_{ij}, and u just above this u is $u_{i,j+1}$. So, u

here is at this point is $u_{i,j}$+$u_{i+1,j}$ by 2. And v is $v_{i,j}$+$v_{i+1,j}$ by 2. So, these two multiplications will give, so $v_{i,j}$+$v_{i+1,j}$ by 2 $u_{i,j}$+$u_{i+1,j}$ plus 1 by 2.

So, denominator it becomes 4, 0.25 into ($v_{i,j}$ + $v_{i+1,j}$) ($u_{i,j}$, +$u_{i,j+1}$). And we also have $\frac{\partial u}{\partial y}$ at this point, that means, $u_{i,j+1}$ which is u just above this $u_{i,j+1}$-$u_{i,j}$ by Δy, so This is how F^1 is interpolated at this point minus $F^1_{i,j-\frac{1}{2}}$.

In the similar way, $F^1_{i,j-\frac{1}{2}}$ has to be interpolated here in the same way $(u_{ij} + u_{i,j-1})/2$ will give u here and $(v_{ij-1} + v_{i+1,j-1})/2$ give v here. So, multiplication of these two terms, you can seev_{ij-1}+ $v_{i+1,j-1}$ multiplied by u_{ij-1}+ u_{ij}whole divided by 4 which makes 0.25, and we get this term.

Similar way, we get 1 by Re into $\frac{\partial u}{\partial y}$ which will be $\frac{(u_{ij}-u_{i,j-1})}{\Delta y}$. So, this is how all these terms are related to E^1. At the points E^1 as you can see i+1/2, j E^1 at i-1/2, j ; F^1 at i j+1/2 F^1 at i j-1/2 those are found out.

Now, these terms are all at we are discretizing at the level n + 1. So, if we write all the terms at n + 1, then obviously, we get quadratic terms here which are making the equation or which are basically the definition of nonlinearity.

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The linearized forms of these equations are

$$E^1_{i+\frac{1}{2},j} = 0.25\left(u^n_{i,j} + u^n_{i+1,j}\right)\left(u^{n+1}_{i,j} + u^{n+1}_{i+1,j}\right) - \frac{1}{\text{Re}}\frac{\left(u^{n+1}_{i+1,j} - u^{n+1}_{i,j}\right)}{\Delta x}$$

$$E^1_{i-\frac{1}{2},j} = 0.25\left(u^n_{i-1,j} + u^n_{i,j}\right)\left(u^{n+1}_{i-1,j} + u^{n+1}_{i,j}\right) - \frac{1}{\text{Re}}\frac{\left(u^{n+1}_{i,j} - u^{n+1}_{i-1,j}\right)}{\Delta x}$$

$$F^1_{i,j+\frac{1}{2}} = 0.25\left(v^n_{i,j} + v^n_{i+1,j}\right)\left(u^{n+1}_{i,j} + u^{n+1}_{i,j+1}\right) - \frac{1}{\text{Re}}\frac{\left(u^{n+1}_{i,j+1} - u^{n+1}_{i,j}\right)}{\Delta y}$$

$$F^1_{i,j-\frac{1}{2}} = 0.25\left(v^n_{i,j-1} + v^n_{i+1,j-1}\right)\left(u^{n+1}_{i,j-1} + u^{n+1}_{i,j}\right) - \frac{1}{\text{Re}}\frac{\left(u^{n+1}_{i,j} - u^{n+1}_{i,j-1}\right)}{\Delta y}$$

The pressure term after integration is: $\left(p^{n+1}_{i+1,j} - p^{n+1}_{i,j}\right)\Delta y = 0$

Now, we do we go for linearization. This is local linearization, that means, this $(u_{i,j} + u_{i+1,j})^2$ we do not write both the terms at the level n+1. We write one $u_{i,j}+ u_{i+1,j}$ at the level n, another $u_{i,j} +u_{i+1,j}$ at n+1th level.

So, you can see $u_{i,j} + u_{i+1,j}$ at the level n, $u_{ij} + u_{i+1,j}$ at the level $n + 1$, and that is what we do for the next term. And here also for F^1 expression we make $v_{ij}+v_{i+1,j}$ at the level n multiplier $u_{ij}+u_{i,j+1}$ at the level n+1. So, $v_{ij}+v_{i+1,j}$ this is at the level n, $u_{i,j}+u_{i,j+1}$ this is at the level n+1.

We do the similar you know split between nth level and n+1th level for $F^1_{i,j\,-1/2}$ expression also. And all the terms with 1 by Re, they are retained consistently at the level of n + 1, all the terms are at the level of n + 1. And p already we expressed at the level n+1 and we are retaining that. So, this is how we linearize the equations, and this particular x-momentum equation.

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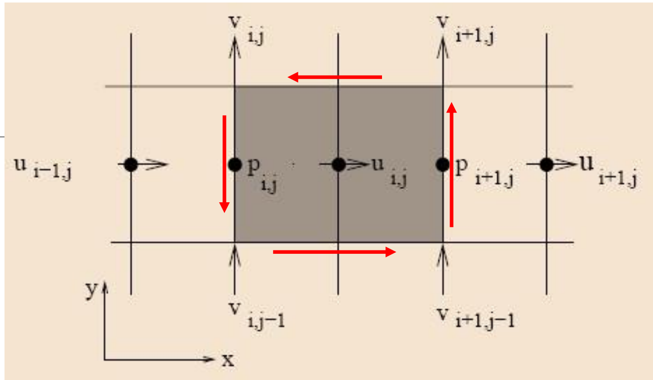
X- momentum equation

Application of the finite- volume method to the x- momentum equation

$$\frac{\Delta x \Delta y}{\Delta t}\left(u_{i,j}^{n+1}-u_{i,j}^{n}\right)+\left(E^1_{i+\frac{1}{2},j}-E^1_{i-\frac{1}{2},j}\right)\Delta y+\left(F^1_{i,j+\frac{1}{2}}-F^1_{i,j-\frac{1}{2}}\right)\Delta x + \left(p_{i+1,j}^{n+1}-p_{i,j}^{n+1}\right)\Delta y = 0 \quad (6)$$

with $E^1 = u^2 - \frac{1}{\text{Re}}\frac{\partial u}{\partial x}$ $\quad F^1 = uv - \frac{1}{\text{Re}}\frac{\partial u}{\partial y}$



So, what we did? We are if you look at it little systematically that this is integrated x-momentum equation, integration how we got we have explained. And then we have linearized it, no, sorry, then we have written the specific expression for $E^1_{i+\frac{1}{2},j}$, $E^1_{i-\frac{1}{2},j}$ $F^1_{i,j+\frac{1}{2}}$ $F^1_{i,j-\frac{1}{2}}$ in terms of the u and you know u values at the neighboring cells, and v values at the neighboring cells.

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The equation (6) can be expanded as:

$$\left[\frac{\Delta x\Delta y}{\Delta t}+0.25(u_{i,j}^{n}+u_{i+1,j}^{n})\Delta y-0.25(u_{i-1,j}^{n}+u_{i,j}^{n})\Delta y+0.25(v_{i,j}^{n}+v_{i+1,j}^{n})\Delta x\right.$$

$$\left.-0.25(v_{i,j-1}^{n}+v_{i+1,j-1}^{n})\Delta x+\frac{\Delta y}{Re\Delta x}+\frac{\Delta y}{Re\Delta x}+\frac{\Delta x}{Re\Delta y}+\frac{\Delta x}{Re\Delta y}\right]u_{i,j}^{n+1}$$

$$+u_{i+1,j}^{n+1}\left\{0.25(u_{i,j}^{n}+u_{i+1,j}^{n})\Delta y-\frac{\Delta y}{Re\Delta x}\right\}+u_{i-1,j}^{n+1}\left\{-0.25(u_{i-1,j}^{n}+u_{i,j}^{n})\Delta y-\frac{\Delta y}{Re\Delta x}\right\}$$

$$+u_{i,j+1}^{n+1}\left\{0.25(v_{i,j}^{n}+v_{i+1,j}^{n})\Delta x-\frac{\Delta x}{Re\Delta y}\right\}+u_{i,j-1}^{n+1}\left\{-0.25(v_{i,j-1}^{n}+v_{i+1,j-1}^{n})\Delta x-\frac{\Delta x}{Re\Delta y}\right\}$$

$$-\frac{\Delta x\Delta y}{\Delta t}u_{i,j}^{n}+\Delta y\left(p_{i+1,j}^{n+1}-p_{i,j}^{n+1}\right)=0$$

Then we are linearizing it. And then we write the entire x-momentum equation. So, you can see that you know if we expand write term by term, this entire you know group of terms is basically associated with u_{ij}^{n+1}. And these are basically grid dimensions, Reynolds number, and u and v values at nth level which are known with u_{ij}^{n+1}.

So, if u_{ij}^{n+1}, if this is point of interest, we can call u_p^{n+1} point of interest; instead of i, j if we call it p, then this is basically eastern neighbor $i+1, j$ at the level n+1, and these this is the basically coefficient of $u_{i+1,j}^{n+1}$. And these are also known values either values of u's or v's at nth level or grid dimension geometrical dimensions.

Then the western neighbor $u_{i-1,j}$ at n+1, again the coefficient is known these are values at nth level u or v values at nth level and grid dimensions. Then northern neighbor u_{ij} plus 1 again at the level of n+1, and this is the coefficient associated with the u_{ij+1}, u_{ij-1} southern neighbor and this is the coefficient associated with it.

This term is associated with u_{ij} at nth level. And everything is known Δx Δy Δt u_{ij} at nth level. And this is basically $p_{i+1,j}$- $p_{i,j}$ both are at the level n+1 multiplied by Δy. So, this is you know term by term, every term has been written here, and we have expressed it in terms of $u_{i,j}$ at the level n+1 its coefficient, $u_{i+1,j}$ its coefficient $u_{i-1,j}$ – its coefficient; $u_{i,j+1}$ at the level n+1 – its coefficients coefficient, u_{ij-1} at the level n+1 and the coefficient.

So, we can say if this is up, this is u_e – eastern neighbor, this is u_w, this is u north, this is u south. So, point of interest u at the point of interest u_{ij} with the coefficient. If we you know consider this coefficient associated with u_{ij}, then these are the coefficients associated with its eastern neighbor, western neighbor, northern neighbor, and southern neighbor. This term is outside these coefficients of u at and its neighbors at n+1 level. This is u_{ij} at nth level plus the pressure gradient term.

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Thus the equation (6) can be written as:

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^u_{i,j}\right) u^{n+1}_{i,j} + \sum a^u_{nb} u^{n+1}_{nb} + b^u + \Delta y \left(p^{n+1}_{i+1,j} - p^{n+1}_{i,j}\right) = 0 \qquad (7)$$

In the above equation, $\sum a^u_{nb}\ u^{n+1}_{nb}$ signifies all the convective and diffusive contribution from the neighboring nodes $\left(u^{n+1}_{i+1,j}, u^{n+1}_{i-1,j}, u^{n+1}_{i,j+1}, u^{n+1}_{i,j-1}\right.$ and their coefficients $\left.\right)$. The coefficients $a^u_{i,j}$ and a^u_{nb} contain grid sizes, and the solution of u and v at n th time level. The term b^u equals $-\Delta x \Delta y\ u^n_{i,j} / \Delta t$. In the earlier slide, equation (6) has been written term by term so that $a^u_{i,j}$, a^u_{nb} and b^u in equation (7) can be clearly determined.

So, we can write as we said earlier u_{ij} at n+1 and its coefficients, first term is that u_{ij} at the level n+1 and all the coefficients. These coefficients we are saying only this we are calling this term as Δx, Δy, delta t, but all other terms together we are giving a nomenclature, a_{ij}^u, that means, the coefficient of u_{ij} that is known as a_{ij}^u coefficient of u_{ij} at the level of n+1. So, this is this plus $\Delta x \Delta y / \Delta t$.

Then we will see all these neighbors this term is gone. So, I mean we have accounted for then all these neighbors and their coefficients we are writing as summation of u_{nb} is u at the neighboring points i+1,j; i-1,j; i j+1; i j-1 east, west, north, south, all are at n+1th level and their coefficients again coefficients are called a is a coefficient, a neighbors coefficients of u. b^u is basically this term Δx Δy into delta Δx into Δy by Δt into u_{ij} at nth level plus Δy into $p_{i+1,j}$-$p_{i,j}$ both are at the level n+1 plus pressure gradient term.

In the above equation, this summation of $a_{nb}u_{nb}$, all u_{nb}s are at the level of n+1 signifies all the convective and diffusive contribution from neighboring nodes that is I have already explained $u_{i+1,j}$, $u_{i-1,j}$, $u_{i,j+1}$, $u_{i,j-1}$, all are at the level of n+1 and they are coefficient. The coefficients a_{ij}^u and a_{nb}^u, that means, these coefficients and this coefficient contain grid sizes and the solution of u and v at the nth level. We have seen that when we have written term by term.

The term b equals to minus Δx into Δy into $u_{i,j}$ at nth level divided by Δt that also we have seen. This is basically b. And in the earlier slide; that means, equation 6 has been written term by term, so that a_{ij}^u, a_{nb}^u , and b^u in equation 7 can be clearly determined. This term b^u term will again have a loop this is u_{ij} at the level n multiplied by this. This came from temporal discretization.

And so b^u equals to minus Δx Δy u_{ij} at the level n divided by delta t. So, this is b^u. And a_{nb}^u, and b^u in equation 7, it can clearly be seen from the previous slide. So, this slide gives you know meaning of this compact form I mean all the individual terms in this compact form.

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Thus the equation (7) is re-written together with the u-control volume as:

$$\left(\frac{\Delta x \Delta y}{\Delta t} + a^u_{i,j}\right) u^{n+1}_{i,j} + \sum a^u_{nb} u^{n+1}_{nb} + b^u + \Delta y \left(p^{n+1}_{i+1,j} - p^{n+1}_{i,j}\right) = 0 \qquad (7)$$

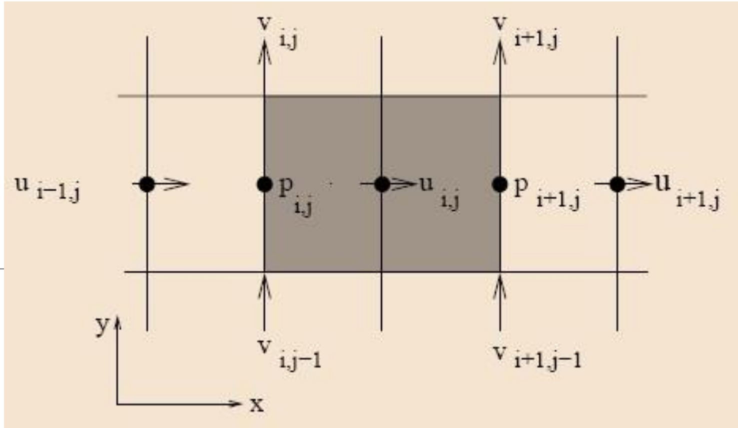

Figure 3: control volume for x-momentum equation

So, this is equation 7 final form of discretized x-momentum equation. Again, we have rewritten u_{ij} at the level $n+1$ into all u neighbors of u_{ij} eastern, western, northern, and southern neighbors at $n+1$ time level b^u plus Δy into pressure gradient. This is equation 7 that was found through integration of x-momentum equation on this control volume defined by dx into dy.

This is u control volume on which we performed the x-momentum equation to get equation 7. We have already gotten integrated continuity equation and which was equation 5. And this is equation 7 – integrated x-momentum equation.

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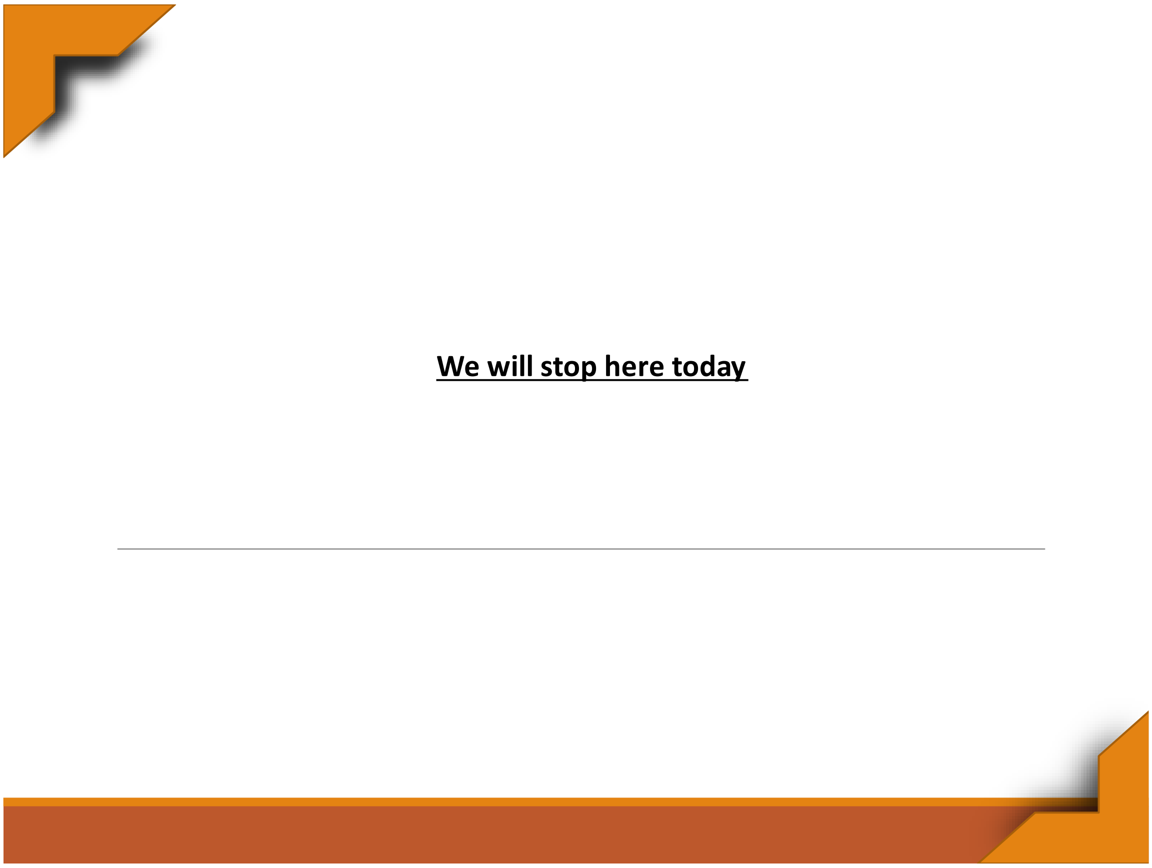

We will stop here today. And remaining part of the lecture, we will cover next day on the next day, and we will work with y-momentum equation, pressure solver and the algorithm.

Thank you very much. Thank you.