

Computational Fluid Dynamics and Heat Transfer
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Lecture - 16

Vorticity – Stream Function Approach for Solving Navier-Stokes Equations
(Calculation of Pressure and Application to Curvilinear Geometry)

Good morning, everybody. Today, we will continue with Vorticity Stream Function Approach for Solving Navier-Stokes Equations.

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Some Important Observations

For high speed flows, we require to introduce an upwind bias. Therefore the difference quotients with respect to convective components become:

$$u \frac{\partial \omega}{\partial x} = u_{i,j} \left(\frac{\omega_{i,j} - \omega_{i-1,j}}{\Delta x} \right) \text{ for } u_{i,j} > 0 \quad (25)$$

$$= u_{i,j} \left(\frac{\omega_{i+1,j} - \omega_{i,j}}{\Delta x} \right) \text{ for } u_{i,j} < 0 \quad (26)$$

Increasing Accuracy

Regarding the vorticity boundary condition given by equation (14), it can be said that the condition is first order accurate. The accuracy can be increased, and we can try for a second order accurate boundary condition.



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$$= u_{i,j} \left(\frac{\omega_{i+1,j} - \omega_{i,j}}{\Delta x} \right) \text{ for } u_{i,j} < 0 \quad (26)$$

If you recall, in a last lecture we discussed about boundary conditions; obviously, governing equations, boundary conditions, and the solution strategy. Today's lecture we will take up some issues for related to improving the accuracy and then we will extend our analysis for the curvilinear coordinates.

Now, the first point that I would like to mention if we try to model high speed flows, high Reynolds number flows that is convection dominated flows then obviously, we have to go for upwinding. And I will take a representative convective term $u \frac{\partial \omega}{\partial x}$, when we discretize it for local velocity $u_{i,j}$ for $u_{i,j}$ positive it will be $u_{i,j} \omega_{i,j}$ minus $\omega_{i-1,j}$ upstream point by Δx .

And if the local velocity is negative then $\omega_{i+1,j}$ downstream point minus $\omega_{i,j}$ by Δx . So, this is simple upwinding. And if needed we can go for higher order upwinding also, something like maybe weak scheme or something like ohara scheme we can make use of.

Next point we will discuss about increasing accuracy. Now, in vorticity stream function related solver. Vorticity term is very crucial. And we have discussed that vorticity is generated on the solid wall. Why on the solid wall? Because of no slip condition there will be the velocity will be 0 at the wall and it will reach through the boundary layer the free stream velocity.

That means, there will be a gradient $\frac{\partial u}{\partial y}$ throughout the boundary layer and vorticity is $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. Because of generation of $\frac{\partial u}{\partial y}$ term, vorticity will be generated near the wall and this vorticity generation has to be very accurate while solving the flows through using stream function vorticity approach.

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If we want to increase the accuracy, we have to retain the fourth term on the right hand side of equation (13). We know from the definition of vorticity:

$$\frac{\partial \omega}{\partial y} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) - \frac{\partial^3 \psi}{\partial y^3} \left[\text{because } u = \frac{\partial \psi}{\partial y} \right]$$

or,

$$\frac{\partial \omega}{\partial y} = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^3 \psi}{\partial y^3} \left[\text{from continuity, } \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \right]$$

or,

$$\left. \frac{\partial \omega}{\partial y} \right|_{\text{at } w} = -\left. \frac{\partial^3 \psi}{\partial y^3} \right|_{\text{at } w} \left[\frac{\partial^2 u}{\partial x^2} = 0, \text{ at the wall} \right] \quad (27)$$


$$\frac{\partial \omega}{\partial t} = u \frac{\partial(\omega)}{\partial x} \frac{\partial \omega}{\partial y} + \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) - \frac{\partial^3 \psi}{\partial y^3} \quad \left[\text{because } u = \frac{\partial \psi}{\partial y} \right]$$

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or,

$$\left. \frac{\partial \omega}{\partial t} \right|_{\text{at W}} = -\left. \frac{\partial^3 \psi}{\partial y^3} \right|_{\text{at W}} \quad \left[\frac{\partial^2 u}{\partial x^2} = 0, \text{ at the wall} \right] \quad (27)$$

Now, if you recall for stream function or finding out the boundary conditions on stream function, we considered Taylor series approximation of the stream function at the near wall point. In the Taylor series approximation, we considered up to second order term and we neglected all other higher order terms and higher order derivatives.

Now, here we will work with a third order term which is effectively $\frac{\partial \omega}{\partial y}$. And we can show that $\frac{\partial \omega}{\partial y}$ means $\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) - \frac{\partial^2 u}{\partial y^2}$, then we get $\frac{\partial^2 v}{\partial x \partial y}$, $\frac{\partial^2 u}{\partial y^2}$, and we can rearrange this term as $\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) - \frac{\partial^2 u}{\partial y^2}$ equal to $\frac{\partial^3 \psi}{\partial y^3}$. Here $\frac{\partial v}{\partial y}$ can be substituted by $-\frac{\partial u}{\partial x}$. So, we get finally, $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^3 \psi}{\partial y^3}$.

But $\frac{\partial^2 u}{\partial x^2}$ on the boundary on the solid boundary all u 's are 0. So, $\frac{\partial u}{\partial x}$ is also 0, $\frac{\partial u}{\partial y}$ is present strong component, but since all u 's are 0. So, gradient of u in x direction is also 0 and double derivative is also 0. So, we can neglect this term $\frac{\partial \omega}{\partial y}$ at wall is minus $\frac{\partial^3 \psi}{\partial y^3}$ at wall. Equation 27.

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$$\psi_{i,j8+1} = \psi_{i,j8} + \frac{\partial \psi}{\partial y} \Big|_{i,j8} \Delta y + \frac{1}{2} \frac{\partial^2 \psi}{\partial y^2} \Big|_{i,j8} (\Delta y)^2 + \frac{1}{6} \frac{\partial^3 \psi}{\partial y^3} \Big|_{i,j8} (\Delta y)^3 + \dots \quad (13)$$

Invoking and substituting the values for $\frac{\partial \psi}{\partial y}$ and $\frac{\partial^2 \psi}{\partial y^2}$ in equation (13), we get

$$\psi_{w+1} = \psi_w - \frac{\Delta y^2}{2} \omega \Big|_{\text{at } w} - \frac{\Delta y^3}{6} \frac{\partial \omega}{\partial y} \Big|_{\text{at } w} \quad (28)$$

Instead of Δy we shall write Δn to mean normal direction from the wall. We shall also substitute $\frac{\partial \omega}{\partial y}$ by a forward difference quotient as

$$\psi_{w+1} = \psi_w - \frac{(\Delta n)^2}{2} \omega \Big|_{\text{at } w} - \frac{(\Delta n)^3}{6} \times \left[\frac{\omega \Big|_{\text{at } w+1} - \omega \Big|_{\text{at } w}}{(\Delta n)} \right]$$



$$\psi_{w+1} - \psi_w + \frac{(\Delta n)^2}{6} \omega \Big|_{\text{at } w+1} = -\frac{(\Delta n)^2}{3} \omega \Big|_{\text{at } w}$$

$$\psi_{i,j8+1} = \psi_{i,j8} + \frac{\partial \psi}{\partial y} \Big|_{i,j8} \Delta y + \frac{1}{2} \frac{\partial^2 \psi}{\partial y^2} \Big|_{i,j8} (\Delta y)^2 + \frac{1}{6} \frac{\partial^3 \psi}{\partial y^3} \Big|_{i,j8} (\Delta y)^3 + \dots \quad (13)$$

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We now again rewrite equation 13, as you recall that $j s$ is the representative of solid surface. So, $\psi_{i,j8}$ ψ at the solid surface and this is $\psi_{i,j8} + 1$ that is ψ at the near

wall point plus $\frac{\partial \psi}{\partial y}$ into $\frac{\Delta y}{2}$ plus $\frac{\partial^2 \psi}{\partial y^2}$ into $\frac{\Delta y^2}{6}$ plus $\frac{\partial^3 \psi}{\partial y^3}$ into $\frac{\Delta y^3}{24}$.

Now, in this equation clearly $\frac{\partial \psi}{\partial y}$ is u velocity and u velocity is 0, we can say directly this to 0. And $\frac{\partial^2 \psi}{\partial y^2}$ if you recall we substitute it for ω in our last lecture, so we do that. And here $\frac{\partial^3 \psi}{\partial y^3}$ is nothing but $\frac{\partial \omega}{\partial y}$.

So, we will substitute this by $\frac{\partial \omega}{\partial y}$. Exactly, we do that, ψ at near wall point equal to ψ at the wall this is 0. So, minus $\frac{\Delta y^2}{2}$ and $\frac{\partial^2 \psi}{\partial y^2}$ is minus ω . So, this is ω at the wall and $\frac{\Delta y^3}{6}$ $\frac{\partial^3 \psi}{\partial y^3}$ substitute it by $\frac{\partial \omega}{\partial y}$ at the wall. So, we now generalize this equation 28 by this consideration that see Δy we convert into Δn because if the solid surface is horizontal Δy is the near wall point in the normal direction.

But solid surface maybe horizontal maybe vertical may be inclined, then we have to consider basically near wall point in the normal direction. So, Δy is substituted by Δn normal direction from the wall and $\frac{\partial \omega}{\partial y}$ is discretized by a forward difference quotient ω at $w+1$ minus ω at w ; that means, ω if you consider i, j then ω at $i, j+1$ minus ω at i, j divided by Δn .

So, this becomes then ψ at $w+1$ means the ψ at the near wall point minus ψ at the wall plus $\frac{\Delta n^2}{6}$ ω at $w+1$ equal to these two terms together ω at w and ω at w minus $\frac{\Delta n^2}{3}$ ω at w . So, this is just algebraic steps I have narrated. There was no need of explaining it, you can understand it quite clearly.

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Finally we get:

$$\omega|_{\text{at } w} = \frac{-3(\psi_{w+1} - \psi_w)}{(\Delta n)^2} - \frac{1}{2} \omega|_{\text{at } w+1} \quad (29)$$

The above equation is second order accurate boundary condition but many a times it does not lead to stable computation for high Reynolds number flows



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Now, from there we can write then omega at the wall we can get explicit expression. So, omega at the wall minus 3 psi at the near wall point minus psi at the wall delta n square minus half omega at near wall point. So, omega at wall when we are calculating we are involving psi at the wall and psi at the near wall point normal distance between them and psi omega also at near wall point. This is omega at wall near wall point means next to the wall. And this omega obviously, we get from the previous level of calculations.

So, the above equation is second order accurate boundary condition obviously, but you know many a times it does not lead to stable computation, some problems arise and those problems can be addressed by fine tuning other boundary conditions. And also having a closer loop at the convective terms; maybe little higher order discretization is needed.

So, you know overall this is a solution strategy and strategy for increasing accuracy for you know high speed flows upwinding. We have discussed everything, but obviously, for a practical problem one has to try and see what was best.

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Application to Curvilinear Geometries

Figure 2: Flow Over a Circular Cylinder

Under the steady, incompressible flow over an infinitely long cylinder of radius R_0

Application to Curvilinear Geometries

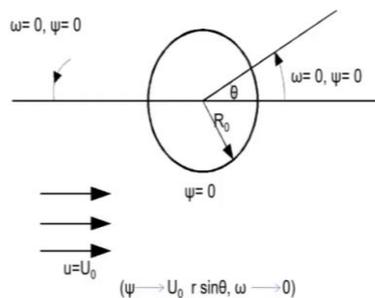


Figure 2: Flow Over a Circular Cylinder

Now, I will extend our analysis to curvilinear geometry and we will take up a circular cylinder with radius R_0 and the cylinder is very long. Why we are saying cylinder is very long? So that there is no end effect on the solution. At any cross section of the cylinder we can get the solution plane and on that solution is valid. And on this line, since this is symmetric line you know $\psi = 0$ $\omega = 0$ on this line $\psi = 0$ $\omega = 0$.

And on the solid surface again psi equal to 0 because of no slip condition there is no mass flow rate on the solid surface and psi is indicative of the mass flow rate. So, psi equal to 0 is quite rational. And you know basically psi is obtainable from the tangential velocity component which we can say here like you know $u_{\theta} = r \sin \theta$ and far away from the wall this will be you know the free stream velocity or even can be 0 velocity, and omega will be also 0.

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➤ The stream function and vorticity equations for this geometry in terms of cylindrical polar coordinates (r, θ) are

$$v_r \frac{\partial \omega}{\partial r} + \frac{v_\theta}{r} \frac{\partial \omega}{\partial \theta} = \nu \left(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \omega}{\partial \theta^2} \right) \quad (30)$$

$$\omega = \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) - \frac{1}{r} \frac{\partial v_r}{\partial \theta}$$

➤ The radial and tangential velocity components are

$$v_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = \frac{\partial \psi}{\partial r} \quad (31)$$

➤ The Poisson's Equation for stream function

$$\omega = \frac{1}{r} \frac{\partial}{\partial r} \left(-r \frac{\partial \psi}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta} \right)$$

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = -\omega \quad (32)$$

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So, if we go for the governing equation; obviously, the first equation will be the vorticity transport equation and here we will use r θ coordinate. The unsteady term $\frac{\partial \omega}{\partial t}$ we have neglected because we are considering steady flow; v_r is the radial direction velocity, and say v_r is a radial direction velocity v_θ is the tangential velocity.

Now, $v_r \frac{\partial \omega}{\partial r} + v_\theta \frac{1}{r} \frac{\partial \omega}{\partial \theta}$ equal to ν this is Laplacian ω . In cylindrical polar coordinate system this is the expression for Laplacian; that means, operator is $\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$. So, here $\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \omega}{\partial \theta^2}$.

And ω is vorticity in again cylindrical polar coordinate system, it is given by $\frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) - \frac{1}{r} \frac{\partial}{\partial \theta} (v_r)$. So, v_r is a radial component of velocity and v_θ is the tangential component of velocity and from stream function again in cylindrical polar coordinate stream definition of stream function and definition of vorticity we have used. So, from definition of stream function the r direction velocity is $\frac{1}{r} \frac{\partial \psi}{\partial \theta}$ and tangential velocity v_θ is $-\frac{\partial \psi}{\partial r}$.

Now, $-\frac{\partial \psi}{\partial r}$ and $\frac{1}{r} \frac{\partial \psi}{\partial \theta}$ these are basically tangential and radial components of velocity. And if we substitute here in the expression for ω , just we have substituted we finally, get the governing equation for ψ that is basically Poisson equation in ψ . This is also you can see Laplacian ψ $\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2}$ equal to $-\omega$.

And if you recall, it is just similar to our what we did in Cartesian coordinate that in a vorticity transport equation, the Poisson's equation for stream function and the definition of v_r and v_θ , these are the governing equations.

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The value of the stream function can be set equal to zero in these surfaces (Figure 2). Therefore, we can write:

$$\begin{aligned}\psi &= 0 \text{ on } \theta = 0, r \geq R_0 \\ \psi &= 0 \text{ on } \theta = \pi, r \geq R_0 \\ \psi &= 0 \text{ on } r = R_0, 0 \leq \theta \leq \pi\end{aligned}\quad (33)$$

In the far field, velocity becomes uniform and equal to U_0 as $r \rightarrow \infty$. In terms of stream function, we get:

$$\text{for } r \rightarrow \infty \quad \psi \rightarrow U_0 r \sin \theta \quad (34)$$


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If we consider boundary condition, we have already in the schematic we have said that you know psi will be 0 you know over the entire cylinder. So, we have written that psi is 0 theta equal to 0, when R is greater than equal to R 0, psi is 0 at theta equal to pi and psi is 0 on R equal to R 0.

And far field velocity becomes uniform and equal to U_0 as r tends to infinity. In terms of stream function, we get as r tends to infinity psi tends to $U_0 r \sin \theta$, $U_0 r \sin \theta$.

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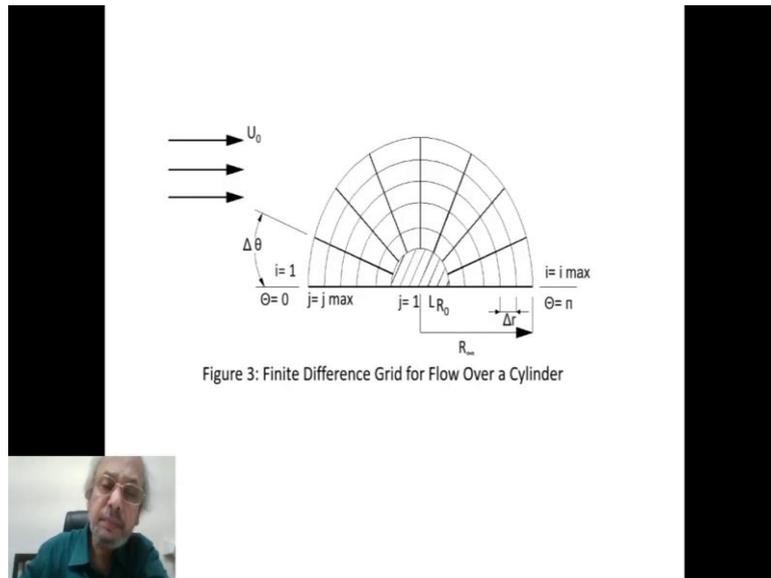


Figure 3: Finite Difference Grid for Flow Over a Cylinder

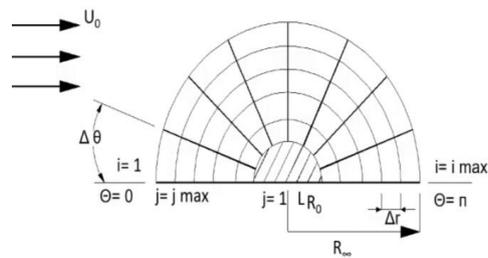


Figure 3: Finite Difference Grid for Flow Over a Cylinder

And this is with respect to you know this grid you can see the R_0 is the radius of the cylinder R_∞ is far away from the solid body I mean I have just given example schematically this R_∞ can be a very large. And the grids are in theta direction counted by i , i equal to 1, i equal to 2, i equal to 3. Similarly, i equal to i max it is from theta equal to 0 to theta equal to π and j 's are j equal to 1, j equal to 2, j equal to 3, j equal to 4 upto we will cover entire R_∞ .

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Usually, a large cylindrical surface ($r = R_\infty$) is considered for implementing the far field condition (Figure 3). Therefore, the far field boundary condition (Equation 34) is approximated as:

$$\psi = U_0 R_\infty \sin \theta \text{ at } r = R_\infty \quad (35)$$

For the vorticity, the far field boundary condition is given by

$$\omega = 0 \text{ at } r = R_\infty \quad (36)$$

Since the velocity is uniform and velocity gradients are zero, the above-mentioned condition is justified. On the axis ($\theta = 0$ and π), $\partial v_r / \partial \theta = 0$ due to axisymmetry and $\psi = 0$ for all r .

is, from equations (30) and (32) it can be seen that

$$\omega = 0 \text{ on } \theta = 0, r \geq R_0 \quad (37)$$

and $\omega = 0 \text{ on } \theta = \pi, r \geq R_0$

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Thus, from equations (30) and (32) it can be seen that

$$\omega = 0 \text{ on } \theta = 0, r \geq R_0$$

$$\omega = 0 \text{ on } \theta = \pi, r \geq R_0 \quad (37)$$

So, as I said that r equal to R infinity is considered for implementing far field condition this figure. So, ψ there will be U naught because it is as I said from the tangential component of velocity, U naught R infinity $\sin \theta$ at r equal to R infinity. For the

vorticity, the far field boundary condition there will be no vorticity, we will assume that in the far field you know only free stream is prevalent, so vorticity is not there.

Since, the velocity is uniform and velocity gradients are 0 the above mentioned condition is justified. On the axis, theta equal to 0 and π del v r del theta equal to 0 due to axisymmetry.

So, on the axis, you know del v r del theta is 0 on the axis because theta is variation in this direction. So, if the lower half is exactly symmetry. So, del v r del theta gradient in theta direction is a 0 and psi equal to 0 at all r, this is on the axis on the again on the axis all r psi equal to 0.

Now, in equation 30 and 32; that means, I am going back few slides equation 30 and 32, we will substitute for v r and v theta at the point theta equal to 0 and theta equal to pi. If we do that, we will get omega equal to 0, at theta equal to 0 and omega equal to 0 at theta equal to pi.

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Starting with equation (32) and incorporating the impervious and no-slip conditions of $v_r = 0$ and $v_\theta = 0$, the expression for the wall vorticity reduces to :

$$\frac{\partial^2 \psi}{\partial r^2} = -\omega \tag{38}$$

Now, the wall vorticity can be calculated using the image point method. Considering a curvilinear grid as shown in Figure 3, the derivatives of stream function or vorticity can be replaced by their discrete forms as:

$$\frac{\partial \omega}{\partial \theta} = \frac{\omega_{i+1,j} - \omega_{i-1,j}}{2\Delta\theta}; \frac{\partial \omega}{\partial r} = \frac{\omega_{i,j+1} - \omega_{i,j-1}}{2\Delta r}$$

$$\frac{\partial^2 \omega}{\partial \theta^2} = \frac{\omega_{i+1,j} - 2\omega_{i,j} + \omega_{i-1,j}}{\Delta\theta^2}; \frac{\partial^2 \omega}{\partial r^2} = \frac{\omega_{i,j+1} - 2\omega_{i,j} + \omega_{i,j-1}}{\Delta r^2} \tag{39}$$



Early, expressions for stream function derivatives can also be obtained.

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Then, again we will take up going back by again few slides equation 32. In the equation 32, if we set the conditions v_r equal to 0 and v_θ equal to 0 when corresponding stream functions, we reappropriate we will get that $\nabla^2 \psi = -\omega$. So, $\nabla^2 \psi = -\omega$, this becomes the you know form of the equation at basically impervious and no slip condition; that means, it is on the wall.

Now, the wall vorticity can be calculated using the image point method that we mentioned while deriving basically equation 14, you may have a look into it. You have followed say you can be same strategy can be followed considering a curvilinear grid as shown in figure 3. I mean I have explained in the radial direction these are Δr s and j equal to 1 to j_{max} and theta direction these are $\Delta\theta$ s and i equal to 1 to i_{max} .

So, i equal to 1 to i_{max} in theta direction and in the radial direction j equal to 1 to j_{max} . In the radial direction it is Δr is a grid size in the theta direction $\Delta\theta$ is the grid size. So, now the derivative such as $\frac{\partial \omega}{\partial \theta}$, simply we have applied central differencing. Here $\omega_{i+1,j} - \omega_{i-1,j}$ divided by twice $\Delta\theta$.

$\frac{\partial \omega}{\partial r}$ $\omega_{i,j+1} - \omega_{i,j-1}$ in the r direction; that means, $\omega_{i,j+1} - \omega_{i,j-1}$ by twice Δr . $\frac{\partial^2 \omega}{\partial \theta^2}$ similarly central difference $\omega_{i+1,j} - 2\omega_{i,j} + \omega_{i-1,j}$ by $\Delta\theta^2$; $\frac{\partial^2 \omega}{\partial r^2}$ is basically $\omega_{i,j+1} - 2\omega_{i,j} + \omega_{i,j-1}$ simply central difference divided by Δr^2 . Similarly, if I mean the derivatives $\frac{\partial \psi}{\partial \theta}$, $\frac{\partial \psi}{\partial r}$, $\frac{\partial^2 \psi}{\partial \theta^2}$, and $\frac{\partial^2 \psi}{\partial r^2}$ exactly this way we can substitute.

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Vorticity- Stream Function Approach

Similarly, expressions for stream function derivatives can also be obtained. Substituting these expressions into the governing equations (30) and (32), the nodal equations of all the interior nodes ($i = 2, \dots, imax - 1, j = 2, \dots, jmax - 1$) are derived. Finally, the set of discretized equations and boundary conditions are solved by iterative methods. For high Reynolds number flows, upwinding can also be implemented based on the magnitudes of u_r and u_θ , in a similar manner as described earlier.

The important fact to be kept in mind while simulating flows in curvilinear geometries is that due to curvature, certain terms in the governing equations or boundary conditions may take a $0/0$ form. In such cases, one can either resolve the $0/0$ form using L'Hospital rule (if possible) or employ a local Cartesian mesh at the point of singularity.



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And again, this is the grid and the solution strategy is same basically some initial condition for appropriate initial condition for v_r , v_θ , ω , and ψ they are given, then the boundary conditions are properly applied we have discussed about the boundary conditions on all confining surfaces. That means, here the confining surface means this is one surface and r equal to R infinity is another surface. And this is one surface and this line is also one surface.

So, basically that $\theta = 0$ will be described by one surface and $\theta = \pi$, this is another stream and you know $j = j_{\max}$ for all the θ s that is $j = j_{\max}$ is the boundary at $r = R$ and θ will vary from $\theta = 0$ to $\theta = \pi$, r will vary from $r = 1$ to $r = R$, θ will vary from $\theta = 0$ to $\theta = \pi$.

Now, this is what I have again repeated actually that a way equation 39 is discretized in the same way expression for stream functions can be obtained and discretized form can be obtained. Substituting these expressions into governing equations 30 and 32 the Nodal equations of all the interior nodes $i = 2$ to $i = i_{\max} - 1$, $j = 2$ to $j = j_{\max} - 1$ are derived.

Finally, the set of discretized equations and boundary conditions are solved by iterative method. Just the way we set up the solution algorithm for Cartesian coordinate method here it will be the same. Only thing quantities are v_r , v_θ , and ψ in cylindrical polar coordinate and ω_n cylindrical polar coordinate. For high Reynolds number flows upwinding can be implemented based on the magnitude of v_r and v_θ in a similar manner as described earlier.

The important fact to be kept in mind while simulating flows in curvilinear geometries, it is to be always remembered that because of curvature this $1/r$ term certain terms in the governing equations or boundary conditions may take $0/0$ form. In such cases, one can either resolve the $0/0$ form using L'Hospital rule.

If L'Hospital rule can be applied or what one can do locally by taking a few points a Cartesian mesh can be I mean described to get rid of the point of singularity. This is a very special technique I will mention in some of the in one of the subsequent lectures.

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Calculation of Pressure

Upon completing the steps enumerated earlier, the velocity components are determined at each grid point. In order to determine the pressure at each point, it is necessary to derive Poisson equation for pressure. This equation is derived by differentiating equation (1) by x and equation (2) by y and then adding them

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\partial y}\right) + v\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial y^2}\right) = -\frac{1}{\rho}\nabla^2 p + v\left[\frac{\partial}{\partial x}(\nabla^2 u) + \frac{\partial}{\partial y}(\nabla^2 v)\right] \quad (40)$$

By making use of continuity equation (40) can be reduced to

$$\nabla^2 p = 2\rho\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \quad (41)$$

or,

$$\nabla^2 p = 2\rho\left[\left(\frac{\partial^2 \psi}{\partial x^2}\right)\left(\frac{\partial^2 \psi}{\partial y^2}\right) - \left(\frac{\partial^2 \psi}{\partial x\partial y}\right)^2\right] \quad (42)$$

have obtained a Poisson equation for pressure which is analogous to equation (6). Poisson equation for pressure can be solved over entire domain using appropriate boundary conditions at the confining surfaces.

Calculation of Pressure

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$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\partial y}\right) + v\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial y^2}\right) = -\frac{1}{\rho}\Delta^2 p + v\left[\frac{\partial}{\partial x}(\Delta^2 u) + \frac{\partial}{\partial y}(\Delta^2 v)\right] \quad (40)$$

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Now, we have to discuss another important point. So, that means, whatever we have discussed in Cartesian coordinate for example, having solved all the equations we will get distribution of stream function, distribution of vorticity, and u velocity, and v velocity.

In cylindrical polar coordinate system, we also get distribution of omega vorticity distribution of stream function ψ and v_r ; that means, radial velocity and v_θ ; that means, tangential velocity. So, on the domain of interest we will get these variables, but since we did not handle pressure at all, we eliminated pressure by solve by formulating the problem based on vorticity transport equation.

So, pressure is not obtainable from these values. We get only velocity component stream function and vorticity. But complete solution in a domain demands pressure value also and that is very important. So, having solved that after getting converge flow field from there we can determine the pressure by solving the pressure equation which is Poisson's equation for pressure and how that is formed I have discussed it here.

So, you consider x, I mean I am giving an example through Cartesian coordinate. I have written differentiate equation 1; that means, x momentum equation by x. Earlier we for forming vorticity we differentiated with respect to y, but now we are differentiating with respect to x.

And y direction momentum equation we are differentiating with respect to y, earlier for forming vorticity transport equation the y momentum equation we differentiated with respect to x. But here we are differentiating with respect to y. That means, x momentum equation we differentiate with respect to x. y momentum equation we differentiate with respect to y, and then add if we add, we will it is very clear cut you know substitution all of you can do it please practice it.

Differentiate right x direction momentum equation with respect to x, right y direction momentum equation differentiate with respect to y and add these two equations. If you add you will get $\frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial y} + 2 \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}$ and, right hand side will clearly get basically Laplacian p.

So, that is basically you know $\frac{1}{\rho} \nabla^2 p = \text{grad}^2 u + \nabla^2 v$. We have given a number 40. So, by making use of continuity equation again here you can substitute continuity equation appropriately, and we will see a massive reduction of the terms and you will get $\nabla^2 p = 2\rho \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right)$.

And now in the domain you know what is the value of u , what is the value of v with respect to ψ , $\frac{\partial \psi}{\partial y} = u$ and $-\frac{\partial \psi}{\partial x} = v$. So, you can substitute, do the substitution and you will get $\nabla^2 p = 2\rho \left(\frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} - \left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right)$. And all these values are known, local ψ values at every location these are known.

From here this unknown pressure if you discretize it in a x direction and y direction, through central differencing and right-hand side is known and then you shall you run from $i = 1$ to i_{\max} and $j = 1$ to j_{\max} , but effectively $i = 2$ to $i_{\max} - 1$ and $j = 2$ to $j_{\max} - 1$ then you will get basically a pentadiagonal matrix.

And you can solve that matrix we have obtained now Poisson equation for pressure which is analogous to again another Poisson equation for vorticity or stream function. So, the Poisson equation, no, Poisson equation for stream function, we did not get Poisson equation for vorticity. Vorticity equation we got basically vorticity transport equation from there we calculated the expression from vorticity.

But basically, we made use of the Poisson's equation for stream function anyway whatever we did you can have the relook into it. But here as I am saying its Poisson equation for pressure and we can solve it by there we can get this form basically by breaking into central difference in x direction and central difference in y direction.

And then you know if we run from $i = 1$ to i_{\max} and $j = 1$ to j_{\max} , we get a matrix and there we will see that diagonal and two of the super diagonal elements and two of the sub diagonal elements are present other terms will be 0. And right-hand side is explicitly known because all the values are known, by solving that we will be able to determine pressure in the flow domain.

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Vorticity- Stream Function Approach For Solving Navier-Stokes Equations

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And if you are interested these are the differences I used for teaching purpose also and preparing the lectures. So, you can look at these references if you find them interesting.

Thank you very much. Thank you.