Computational Fluid Dynamics and Heat Transfer

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Lecture - 15

Vorticity- Stream Function Approach for Solving Navier-Stokes Equations

Good morning, everybody. Today, we will start a new topic that is Vorticity-Stream Function Approach for Solving Navier-Stokes Equations. I may mention that this approach is one of the earliest successful approaches for solving Navier-Stokes equations.

(Refer Slide Time: 00:43)

Vorticity- transport Equation

Derivation of Vorticity Transport Equation

The momentum equations in x and y direction are:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad (1)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \qquad (2)$$

Let us find out x-derivative of Eqn. (2):

$$\rho\left(\frac{\partial^2 v}{\partial t\partial x}+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+v\frac{\partial^2 v}{\partial x\partial y}\right)=-\frac{\partial^2 p}{\partial x\partial y}+\mu\left(\frac{\partial^3 v}{\partial x^3}+\frac{\partial^3 v}{\partial x\partial y^2}\right) \quad (3)$$

Let us find out y-derivative of Eqn. (1):

$$\rho\left(\frac{\partial^2 u}{\partial t\partial y}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+u\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y^2}\right)=-\frac{\partial^2 p}{\partial x\partial y}+\mu\left(\frac{\partial^3 u}{\partial x^2\partial y}+\frac{\partial^3 u}{\partial y^3}\right) \quad (4)$$

Although, it does not solve Navier-Stokes equations in terms of u, v, w and p, it solves a Navier-Stokes equations in terms of derived variables that is vorticity and stream function.

Having found out vorticity and steam function in the domain u, v and p are derived out of that. So, the governing equations become vorticity and the equation for stream function.

I have already mentioned about the vorticity transport equation while discussing preliminaries, but I will mention it here again. We can see, we have written full Navier-Stokes equations in 2-dimension. Equation 1 is the x direction momentum equation and equation 2 is the y direction momentum equation.

Now, what we do? We first differentiate equation 2 with respect to x. So, we find out x derivative of equation 2 and we get equation 3. Straightforward, just differentiate equation 2 with respect to x. Then we differentiate equation 1 with respect to y and we get equation 4. Next what we do? We subtract equation 4 from equation 3.

(Refer Slide Time: 02:56)

Vorticity- transport Equation

Eqn. $(3)-$ Eqn. (4) will yield

$$\rho\left[\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right]$$
$$=\mu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right]$$

In 2D, vorticity is given by, $\omega=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$

So, $$\rho\left[\frac{\partial \omega}{\partial t}+\frac{\partial u}{\partial x}\omega+u\frac{\partial \omega}{\partial x}+\frac{\partial v}{\partial y}\omega+v\frac{\partial \omega}{\partial y}\right]=\mu\left[\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right]$$

Or, $$\rho\left[\frac{\partial \omega}{\partial t}+u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}+\omega\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]=\mu\left[\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right]$$

0

And then, we can see a new term $\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$ appearing in the governing equation as a dependent variable and all of us know that in 2-dimensional analysis on the plane where u and v velocities are describable, ω is the vorticity and vorticity is given by $\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$.

So, this term $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, we substitute by ω and finally, we get

$$\rho\left[\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}\right] = \mu\left[\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right] \tag{5}$$

this is called vorticity transport equation.

(Refer Slide Time: 04:16)

Vorticity- transport Equation

Or, $$\rho\left[\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}\right] = \mu\left[\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right] \tag{5}$$

Now we will proceed with our lecture on "Solution of N-S Equations using Vorticity-Stream Function Approach".

Equation 5 so, this special mention was needed because we will be using this equation as the governing equation.

(Refer Slide Time: 04:33)

Solution through Vorticity- Stream Function Approach

The vorticity-stream function method is one of the most popular methods for solving 2-D incompressible Navier- Stokes equation:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\ \nabla^2\ u \qquad (1)$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)\underset{\omega}{=}-\frac{\partial p}{\partial y}+\mu\ \nabla^2\ v \qquad (2)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (3)$$

We introduce stream function (ψ) and vorticity (ω) as

$$u=\frac{\partial \psi}{\partial y},\quad v=-\frac{\partial \psi}{\partial x} \qquad (4)$$

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \qquad (5)$$

Having done that, let us go to our main topic, we decided to discuss today that is solution through vorticity and stream function approach. So, again what we have done? We have written x momentum equation as equation 1, y momentum equation as equation 2, continuity equation and then, we have defined u velocity and v velocity considering the presence of stream function in the flow field.

Then, u velocity becomes $\frac{\partial \psi}{\partial y}$, v velocity becomes $-\frac{\partial \psi}{\partial x}$ equation 4 and ω, we have already defined in our preliminary discussions that ω is vorticity $\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$.

(Refer Slide Time: 05:35)

Invoking equation (4) into (5) we obtain Poisson equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \quad (6)$$

Subtract the differentiated equation (2) from differentiated equation (1) and rearrange the resulting equation, we get the vorticity transport equation as:

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \quad (7)$$

Let us express the equation (6) in terms of finite difference quotient for $\left.\frac{\partial^2 \psi}{\partial x^2}\right|_{i,\, j}$, $\left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,\, j}$, setting ar$\Delta x = \Delta y$ $\omega_{i,j}$

$$\psi_{i,\, j} = \frac{1}{4}\left[\psi_{i+1,\, j} + \psi_{i-1,\, j} + \psi_{i,\, j+1} + \psi_{i,\, j-1} + \omega_{i,\, j}\ \Delta x^2\right] \quad (8)$$

Now, in ω where we have written ω equal to $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ equation 5, if we substitute equation 4 and v is substituted as $-\frac{\partial \psi}{\partial x}$ and u is substituted as; substituted as $\frac{\partial \psi}{\partial y}$, then we will get an equation which is Poisson's equation in ψ or stream function. So,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \omega \quad (6)$$

this is our equation 6, Poisson's equation in stream function and equation 7, we have very elaborately discussed that is the vorticity transport equation.

So, equation 6 and equation 7 are the equations of interest for the solution we are going to discuss. So, what we do? We start computational strategies with equation 6. In equation 6, we substitute $\frac{\partial^2 \psi}{\partial x^2}$ in terms of its finite difference quotient through central difference discretization. Similarly, $\frac{\partial^2 \psi}{\partial x^2}$ we write central difference discretization and then, we set $\Delta x = \Delta y$, there is no harm in doing that, we simplify problem little bit.

And in usual stream function vorticity methods of solution, if we go for a final (Refer Time: 07:43) taking $\Delta x, \Delta y$ is quite easy and rational and then, since we are discretizing at the point i, j so, this ω value we take as $\omega_{i,j}$ and then, we tried to find out expression for $\psi_{i,j}$, you were, I think confident now $\psi_{i,j}$ will be created like you know central difference, it is

$$\psi_{i,j} = \frac{\psi_{i+1,j} - 2\,\psi_{i,j} + \psi_{i-1,j}}{\Delta x^2}$$

similarly, here also we get $\psi_{i,j}$. So, now, we write $\psi_{i,j}$, we transfer all the terms on the right-hand side and this is

$$\psi_{i,j} = \frac{1}{4}[\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} + \omega_{i,j}\,\Delta x^2 \qquad (8)$$

It becomes very easy to find out this expression which is as I said comprising of ψ and ω by 4 and this is equation 8.

(Refer Slide Time: 09:19)

For equation (7) we can write:

$$u_{i,\,j}\,\frac{\omega_{i+1,\,j} - \omega_{i-1,\,j}}{2\Delta x} + v_{i,\,j}\,\frac{\omega_{i,\,j+1} - \omega_{i,\,j-1}}{2\Delta y} = \nu\left[\frac{\omega_{i+1,\,j} - 2\omega_{i,\,j} + \omega_{i-1,\,j}}{(\Delta x)^2} + \frac{\omega_{i,\,j+1} - 2\omega_{i,\,j} + \omega_{i,\,j-1}}{(\Delta y)^2}\right] \qquad (9)$$

If we put $\Delta x = \Delta y$

$$\omega_{i,\,j} = \frac{1}{4}\left[\left(1 - \frac{1}{2}\,\frac{u_{i,\,j}\,\Delta x}{\nu}\right)\omega_{i+1,\,j} + \left(1 + \frac{1}{2}\,\frac{u_{i,\,j}\,\Delta x}{\nu}\right)\omega_{i-1,\,j} + \left(1 - \frac{1}{2}\,\frac{v_{i,\,j}\,\Delta x}{\nu}\right)\omega_{i,\,j+1} + \left(1 + \frac{1}{2}\,\frac{v_{i,\,j}\,\Delta x}{\nu}\right)\omega_{i,\,j-1}\right] \qquad (10)$$

We rewrite the equation (4) as:

$$u_{i,\,j} = \frac{\psi_{i,\,j+1} - \psi_{i,\,j-1}}{2\Delta y} \qquad (11)$$

$$v_{i,\,j} = -\frac{\psi_{i+1,\,j} - \psi_{i-1,\,j}}{2\Delta x} \qquad (12)$$

Now, let us look at vorticity transport equation. This equation can also be discretized at i, j point and if we do that, we will get this expression I am going back by the one slide again. So, this will be $u_{i,j}$ this will be central difference of $\omega_{i,j}$ at i, j point, this will be $v_{i,j}$, this will be central difference of ω in y direction at i, j point and ν into diffusion terms which will also be discretized by central differencing. We have exactly done that what we said, this becomes equation 9.

And equation 9 also we can now substitute $\Delta x = \Delta y$ and if we set $\Delta x = \Delta y$ from equation 9, we can easily find out an expression for $\omega_{i,j}$ and this becomes

$$\omega_{i,\,j} = \frac{1}{4}\left[\left(1 - \frac{1}{2}\,\frac{u_{i,\,j}\,\Delta x}{\nu}\right)\omega_{i+1,\,j} + \left(1 + \frac{1}{2}\,\frac{u_{i,\,j}\,\Delta x}{\nu}\right)\omega_{i-1,\,j} + \left(1 - \frac{1}{2}\,\frac{v_{i,\,j}\,\Delta x}{\nu}\right)\omega_{i,\,j+1} + \left(1 + \frac{1}{2}\,\frac{v_{i,\,j}\,\Delta x}{\nu}\right)\omega_{i,\,j-1}\right] \qquad (10)$$

Similarly, $\omega_{i,j+1}$ with a coefficient comprising of v and ν and grid size and $\omega_{i,j-1}$ with a coefficient which has $v_{i,j}$ Δx and ν. But these coefficients are not constant coefficients, these are all again dependent function, $u_{i,j}$ is related to ω, $v_{i,j}$ is related to $\omega, u_{i,j}$ is related to ω.

And now, the as you recall, we defined you know u and v from stream function in equation 4, we go back and we write $u_{i,j}$ simply it is since this is $\frac{\partial\psi}{\partial y}$, we apply central difference, $\frac{\psi_{i,j+1}-\psi_{i,j-1}}{2\Delta y}$ and $v_{i,j}$ is $-\frac{\partial\psi}{\partial x}$, again we apply central difference $\frac{\psi_{i+1,j}-\psi_{i,-1,j}}{2\Delta x}$ and this is $\partial\psi/\partial x$ with a minus sign. This is equation 12. So, in our domain now, equation 8, next equation 10, equation 11 and equation 12, these are the governing equations.

(Refer Slide Time: 12:33)

Vorticity- transport Equation

Thus we have now a system of simultaneous Equations, (8), (10), (11) and (12) to be solved for $\psi_{i,j}$, $\omega_{i,j}$, $u_{i,j}$ and $v_{i,j}$. Let us discuss the solution procedure through some systematics steps

So, we now get a system of simultaneous equations given by I have already said 8, 10, 11 and 12 to be solved for $\psi_{i,j}$, $\omega_{i,j}$, $u_{i,j}$ and $v_{i,j}$. So, straight away we go to the solution strategy.

(Refer Slide Time: 13:00)

Solution procedure:

1. Divide the physical domain by a mesh system where $\Delta x = \Delta y$

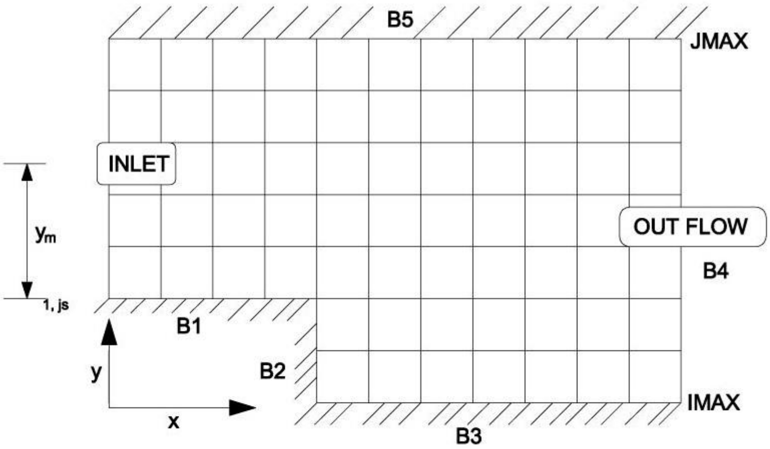

Figure : 1

Now, together with the solution strategy, we take a model geometry where we will be discussing the solve for first, then we extend it to other geometries also. This geometry is called backward facing step. You can see there is one step at the bottom wall, top wall is plane like a plane channel wall, this is inlet through which fluid comes and B4 is the outlet boundary through which the fluid leaves.

The geometry looks very simple, but this geometry can be related to many geometries of practical interest, starting from combustion to several other applications number 1 and number 2 that this geometry has been accepted as a model geometry for benchmarking the flow whenever new algorithm, new strategies, new discretization procedures are devised, results are executed on some model geometries.

You will get later one such model geometry is called lead different gravity; we will discuss that later another such model geometry is backward facing step for flow benchmarking solution benchmarking. So, you can see B1, B2, B3 these are parts of bottom wall and B5 is the top wall, B4 is outflow boundary and inlet boundary, I have not defined it to with any other nomenclature, clearly written this is inlet boundary and again, we have taken $\Delta x = \Delta y$.

This is for some simplification of understanding, you can always take Δx different than Δy accordingly, the equations have to be modified.

(Refer Slide Time: 15:34)

2. Set the known boundary conditions for u, v, ψ and ω.

3. Choose initial values of $\psi_{i,\,j}$ and $\omega_{i,\,j}$ at the interior grid points. Taking initial values for vorticity as zero is usually acceptable. Initial values for $\psi_{i,\,j}$ at each i column of points can be calculated from the axial velocity profile at that location. However, $u_{i,\,j}$ for interior points may be taken as U_∞ and $v_{i,\,j} = 0$.

4. Calculate everywhere $\psi_{i,\,j}$ using equation (8). Gauss-Seidel or overrelaxation type calculation is done, for example:

$$\psi_{i,\,j}^{k+1'} = \psi_{i,\,j}^{k'} + F\left(\psi_{i,\,j}^{k+1} - \psi_{i,\,j}^{k'}\right)$$

where, $\psi_{i,\,j}^{k'}$ is the value from previous calculation, $\psi_{i,\,j}^{k+1}$ is the most recent value, F is the overrelaxation factor and $\psi_{i,\,j}^{k+1'}$ is newly adjusted better guess.

So, what we do? How to set up the algorithm? That we know the boundary condition I mean we have not discussed so far, but we discuss shortly about the boundary conditions. Boundary conditions have to be known on all confining surfaces. So, we will have to apply boundary conditions on u, v, ψ and ω.

Choose the initial values of ψ, ω at all interior points and what I have written here, taking initial values for vorticity as zero is usually acceptable because this will be generated depending on the floor dynamics so, initial value is zero. Initial value of $\psi_{i,j}$ at each i column of points can be calculated from the axial velocity profile at that location.

That means, all i column of points that means if we take for example, $i = 2$ and here $j = 2$, but $i = 2$ and $j = 3$, $j = 4$, $j = 5$, $j = 6$ etc. can be calculated from the initial velocity profile that we take. As we have said that ψ and u are very directly linked. I will show you how that can be done little later, now we are discussing about overall strategy let us accept that ψ can be assigned that way.

So, i column of points can be calculated from the axial velocity profile at that location. However, $u_{i,j}$ for all interior points can be given any constant value say U_∞ where U_∞, if we think about entry of the flow even if it has a profile, it can be expressed in terms of U_∞.

As in the form flow, we give this condition all interior points and all interior points initial condition $v_i = 0$.

Now, if in this geometry for example, everywhere initial ω and initial ψ, initial u, initial v is there, then from the expression of ψ, if you recall what was expression of ψ? Equation 8. So, surrounding ψ's and w's we can at a point of interest, we can calculate new value of ψ. We do that, then we assign boundary conditions and check the distribution of ψ whether it is you know a plausible flow distribution and then again, we go for next iteration.

So, calculate everywhere $\psi_{i,j}$ using equation 8. Gauss-Seidel or overrelaxation type of calculation is done that means, if we start with a value say $\psi_{i,j}$ k prime distributed everywhere and then, $\psi_{i,j,k+1}$, we calculate from equation 8. So, $\psi_{i,j}$ or $i+1$, j, $i-1$, j or all points ψ, we are calling initial value as k' and then, this value which we are determining from here, we are writing $\psi_{i,j}^{k+1}$.

So, exactly that $\psi_{i,j}^{k'}$ was initial distribution and $\psi_{i,j}^{k+1}$ we calculate from there, but since we are accelerating the calculation, what we are doing that we are saying $\psi_{i,j}^{k+1'}$ prime is the most updated value so, old value plus over-relaxation factor multiplied by new value minus old value (step 4).

So, where you can see $\psi_{i,j}^{k'}$ is the value, the previous calculation on the previous level, $\psi_{i,j}^{k+1}$ is the most recent value calculated from equation 8, F is the overrelaxation factor, it is usually I will discuss it again later about overrelaxation factor, it is 1.2, 1.5, 1.3 something like that to accelerate the calculation and $\psi_{i,j}^{k+1'}$ is a newly adjusted better guess.

(Refer Slide Time: 21:21)

5. Calculate $u_{i,\ j}$ and $v_{i,\ j}$ at all the internal grid points using equation (11 and 12).

6. In a subsequent step, calculate $\omega_{i,j}$ at all interior mesh points using equation (10).

7. Apply appropriate boundary conditions (which have been discussed, in details, in a following section).

8. Go back to step (4) and calculate $\psi_{i,\ j}^{k+1}$ with the help of current value of $\omega_{i,\ j}$ and $\psi_{i,\ j}^{k}$ values, where k denotes the previous level. After, evaluating $\psi_{i,\ j}^{k+1}$ at all points. Find out $\psi_{i,\ j}^{k+1'}$ at all points which are indeed improved values. Start repeating steps 4 to 8 until desired degree of convergence is achieved.

Boundary Conditions

Bottom Wall Boundary

The nodal points which are coinciding with the solid wall we can directly put $u_{i,j} = 0$ and $v_{i,j} = 0$. Since the line B1-B2-B3 is a streamline, any constant value of ψ on it is acceptable. The usual choice is $\psi_{i,j} = 0$.

Then, after finding out as I said new $\psi_{i,j}$ we calculate from there, again $u_{i,j}$ and $v_{i,j}$ using equation 11 and 12 at everywhere and then, we calculate $\omega_{i,j}$ all interior points using equation 10. Apply appropriate boundary condition which are to be discussed now and we check the velocity field whether it has converged.

So, have we next step 8 is a final step. Go back to step 4 and calculate $\psi_{i,j}^{k+1}$ with the help of current value of $\omega_{i,j}$ and $\psi_{i,j}^{k}$ values where k denotes the previous level. After evaluating $\psi_{i,j}^{k+1'}$ at all points, find out $\psi_{i,j}^{k+1'}$ which are indeed improved values.

Start repeating step 4 to 8. So, 4 to 8 till we get convergence that means, there is no more change in ψ values or we have to give some convergence criteria and we have to check if it does not meet that criterion, we go back to again step 4 and come back from step 4 I mean execute step 4, 5, 6, 8. This is the way in iterative format, we go for the solution.

Almost, it is like unsteady solving, unsteady situations which we have done earlier through iterative procedure till we get steady values. Here of course, there is no concept of unsteadiness, we are not marching in the time direction, but the way flow field is evolving almost like marching in the time direction. That is why such procedures are called pseudo time marching; pseudo time marching not actual time marching.

And repetitively as we said, we calculate ψ, from ψ we find out u, v, from u, v and ψ, we find out new ω, we apply the boundary conditions and check whether ψ has changed on not through some comparison ψ at the previous level and ψ at the new level or ω at the previous level and ω at the new level and till we get the desired convergence, we keep on doing the iterations.

Now, we have to discuss the boundary conditions. First, set a boundary condition which we will discuss is the bottom wall boundary. Bottom wall boundary again I am going back by few slides to that geometry. Bottom wall boundary means B1, B2, B3. Now, very straightforward all these nodal points u is 0, v is 0 that is perfect representation of no-slip boundary condition and impervious boundary condition.

Then, we will also say $\psi = 0$ because ψ and the initial at the bottom wall because ψ if you remember your basic fluid mechanics, ψ in a domain is indicative of mass flow rate. So, ψ value at the bottom wall taking as 0 is quite rational because everywhere u is 0.

So, we can directly put $u_{i,j} = 0$, $v_{i,j} = 0$. Since the line B1-B2-B3 is a streamline, any constant value of ψ on it is acceptable. The usual choice is $\psi_{i,j} = 0$.

(Refer Slide Time: 27:05)

Bottom Wall Boundary

At no- slip boundaries, is $\omega_{i,j}$ produced. It is the diffusion and subsequent advection of the wall produced vorticity which governs the physics. Using boundary B1 as an example, we expand b$\psi_{i,\ js+1}$ series as:

$$\psi_{i,\ js+1} = \psi_{i,\ js} + \left.\frac{\partial \psi}{\partial y}\right|_{i,\ js} \Delta y + \frac{1}{2} \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,\ js} (\Delta y)^2 + \frac{1}{6} \left.\frac{\partial^3 \psi}{\partial y^3}\right|_{i,\ js} (\Delta y)^3 + ... \quad (13)$$

But $\left.\frac{\partial \psi}{\partial y}\right|_{i,\ js} = u_{i,\ js} = 0$ by no-slip condition and $\left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,\ js} = \left.\frac{\partial u}{\partial y}\right|_{i,\ js}$

Again, $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

Along the wall, $\left.\frac{\partial v}{\partial x}\right|_{i,\ js} = 0$ [because v = constant = 0].

Thus, $\omega_{i,\ js} = -\left.\frac{\partial u}{\partial y}\right|_{i,\ js}$

Substituting this into (13) and solving for $\omega_{i,\ js}$ with $\psi_{i,\ js} = 0$ gives

$$\omega_{i,\ js} = \frac{-2\psi_{i,\ js+1}}{(\Delta y)^2}$$

So, we have given boundary I have discussed boundary conditions on u, boundary condition on v, boundary condition on ψ, now, we will discuss boundary condition on ω on the bottom wall. So, bottom wall is still continuing. At no-slip boundaries, $\omega_{i,j}$ is produced.

These are very remarkable point even if you go for real analysis of the analytical method through analytical methods. If you recall what you studied in boundary layer that because of boundary layer formation, the u velocity is set with a gradient at the wall. So, u velocity gradient is $\frac{\partial u}{\partial y}$.

Since $\frac{\partial u}{\partial y}$ is being created, this will finally, culminate into a non-zero term of $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ within the boundary layer and there is a possibility, or this is how the vorticity is generated. So, that physical discussion may be pertaining to viscous flow theory or basic fluid mechanics, I have just mentioned about it, but you know we utilize that physics.

At no-slip boundaries, $\omega_{i,j}$ is produced. It is a diffusion and subsequent advection of the wall produced vorticity which governs the physics. So, this vorticity how it is diffused and convicted, it that governs the physical situation even when you may read turbulence or if we get it up to get an opportunity to do the turbulence modeling.

You will see that is generation of vorticity through the solid wall; due to the solid wall, we play a very important role in turbulence modelling, understanding turbulence, where vorticity is a very important parameter. Anyway, here we are not going to that extent, we have just, we are just mentioning it, but we will calculate correct vorticity at the wall.

So, now, see if we take B1 the just entry part of the backward facing state for this channel we can expand $\psi_{i,\ js+1}$. What is js+1? Again, go back to the boundary, see if we say that this point i = 1 and j=js right and this is js+1, this is js+2 so, there what we wrote? $\psi_{i,js+1}$, we express in terms of $\psi_{i,js}$ through a Taylor series.

So, the

$$\psi_{i,\ js+1} = \psi_{i,\ js} + \left.\frac{\partial \psi}{\partial y}\right|_{i,\ js} \Delta y + \frac{1}{2}\left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,\ js} (\Delta y)^2 + \frac{1}{6}\left.\frac{\partial^3 \psi}{\partial y^3}\right|_{i,\ js} (\Delta y)^3 + \ldots.. \qquad (13)$$

Now, what is $\frac{\partial \psi}{\partial y}|_{i,js}$? That means, on this surface. This surface ψ means ψ is 0, we have already said. So, let us go back $\frac{\partial \psi}{\partial y}|_{i,js} = u_{i,js}$= 0 by no-slip boundary condition and also, we know that $u = \frac{\partial \psi}{\partial y}$. So, we can find out double derivative, $\frac{\partial^2 \psi}{\partial y^2}|_{i,js} = \frac{\partial u}{\partial y}|_{i,js}$ we leave it there, we are not giving any value right.

Now, we are saying we are just setting $\frac{\partial \psi}{\partial y}$ and u at i, js that means, on the surface is 0 and we are mentioning that this is due to no-slip condition and we are mentioning that $\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial u}{\partial y}$ because u is $\frac{\partial \psi}{\partial y}$. Then again, ω is $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. Along the wall, $\frac{\partial v}{\partial x}$ is 0 since v is 0 all along. So, along the same wall, $\frac{\partial v}{\partial x}$ is 0.

Now, in this equation ω, if we substitute $\frac{\partial v}{\partial x}$ is 0, then ω at i, js that means, at the wall becomes $-\frac{\partial u}{\partial y}|_{i,js}$ and $\frac{\partial u}{\partial y}$ is prevalent there because it is the you know within the boundary layer from the first point and this is the velocity gradient, it is present at the wall so, $-\frac{\partial u}{\partial y}|_{i,js}$.

Now, $-\frac{\partial u}{\partial y}|_{i,js}$, we have written if you can see just two steps before $\frac{\partial^2 \psi}{\partial y^2}|_{i,js}$is $\frac{\partial u}{\partial y}|_{i,js}$. So, ω is basically if you say $-\frac{\partial u}{\partial y}|_{i,js}$, we can also say it is $-\frac{\partial^2 \psi}{\partial y^2}|_{i,js}$. Now in equation 13, $\frac{\partial^2 \psi}{\partial y^2}$ we substitute by ω and if we substitute by ω, this will be $-\omega$, then we can write you know we can transfer other terms on the left-hand side, $\frac{\partial \psi}{\partial y}$ is 0 so, $\omega_{i,\, js} = \frac{-\psi_{i,js+1}}{(\Delta y)^2}$.

So, exactly, we have, but $\psi_{i,js}$ is again 0 we substitute that because on the wall ψ is 0. So, then ω becomes $\omega_{i,js}$ equal to minus twice, 2 has been transferred to left-hand side twice $\psi_{i,js}$ plus 1 divided by delta y square, this has been 0 and if you properly take care of the sign, you will get $\omega_{i,js}$ is minus twice psi i, js plus 1 by delta y square.

$\psi_{i,js+1}$ is known from the velocity profile, we have to learn how to evaluate ψ and then ω will be prescribed. So, this is how at the bottom wall we prescribe u, v, ψ and ω.

(Refer Slide Time: 36:05)

More general form, regardless of the wall orientation or value of ψ at the boundary, it can be written as

$$\omega\big|_{\text{at w}} = \frac{-2(\psi_{\text{w}+1} - \psi_{\text{w}})}{(\Delta n)^2} \qquad (14)$$

where Δn is the distance from (w + 1) to (w) in the normal direction

Upper Boundary

The upper boundary B5 in figure is having the usual no-slip and impervious conditions for velocity components, i.e., $u_{i,\, j} = 0$, $v_{i,\, j} = 0$. For $\omega_{i,\, j}$ Eqn. (14) will apply

Now, we go to or little more that if you look at the geometry, we have taken at backward phase stream step I mean this is horizontal, this is horizontal, this is vertical surface.

So, to generalize this, we can say that basically, this is instead of saying minus twice $\psi_{i,j+1}$ by Δy^2, we can say minus twice ψ what the neighboring point of the wall that means, if wall point is considered as a no-slip boundary so,

$$\omega\big|_{\text{at w}} = \frac{-2(\psi_{\text{w}+1} - \psi_{\text{w}})}{(\Delta n)^2} \tag{14}$$

where n is a normal distance from the wall, I mean point on the wall and immediate neighboring point.

So, Δn instead of Δy, we are writing Δn, Δn is the distance from $w + 1$ to w in the normal direction. So, and this is equation 14 and that is how all the bottom boundary conditions; bottom wall boundary conditions have been explained. Now, we go for upper boundary condition of a top wall that means, again go back to the geometry B5.

Now, when we discuss upper boundary, B5 the figure is having the usual no-slip and impervious condition that means, $u_{i,j} = 0$, $v_{i,j} = 0$, this is no-slip, this is impervious condition and now, since $\omega_{i,j}$ to the near wall point from the wall can be calculated by knowing the normal distance up to the first grid that means, grid thickness, we can apply that, calculate that. So, $\omega_{i,j}$ equation 14 will apply. So, on the top wall, $u_{i,j}$, $v_{i,j}$, $\omega_{i,j}$ we can prescribe.

(Refer Slide Time: 38:52)

Upper Boundary

The value of $\psi_{i,\ j}$ at the upper wall is constant and may be evaluated by integrating the u velocity profile at the inlet. Integration may be performed through Simpson's rule to get

$$\psi_{i,\ j}\big|_{\mathrm{w}} = \int_{y \text{ at } j=\ js}^{y \text{ at } j=JMAX} u(y)_{\text{inlet}}\ dy \qquad (15)$$

Upper boundary can be modeled as infinite boundary for external flows.

Thoman and Szewczyk used a treatment which specifies the far-field condition of $\omega_{i,\ j} = 0$ with $\frac{\partial \psi}{\partial y} = u = U_\infty$ and $\frac{\partial \psi}{\partial x} = v = 0$. Thus $u = U_\infty$ was applied through a Neumann condition at the boundary along $B5$ as

$$\psi_{i,\ ju} = \psi_{i,\ ju-1} + U_\infty\ \Delta y \qquad (16)$$

where $B5$ is considered at $j = ju$

Now, problem will be with ψ not the problem, but you have to understand what ψ is. The value of $\psi_{i,j}$ at the upper wall again constant throughout the wall it will be same is constant and may be evaluated by integrating u velocity profile at the inlet, this is from $u = \frac{\partial \psi}{\partial y}$. So, $\psi = \int u\, dy$. So, u velocity profile at the inlet. Integration may be performed through Simpson's rule or you can use you know Gaussian-quadrature or you can use Trapezoidal method anything, but if you use Simpson's one-third rule, it is also quite accurate. So, $\psi_{i,j}$ at w that means, upper wall equal to y at js bottom wall here and y at j = JMAX. JMAX is the last point in j direction which is at the wall. u(y) inlet dy equation 15.

$$\psi_{i,\ j}\big|_{\mathrm{w}} = \int_{y \text{ at } j=\ js}^{y \text{ at } j=JMAX} u(y)_{\text{inlet}}\ dy \qquad (15)$$

So, if the upper boundary is a confining boundary like, what we have taken u, v should be set equal to 0, ω can be found out from equation 14 and ψ can be found out from equation 15. Now, I will just mention since you know it is a general discussion, even though we have taken a specific boundary where top wall is a rigid wall there may be external flows.

We may consider modeling of external flows also we using stream function vorticity method and that is what if you recall my introductory lecture was the first successful computation of Navier-Stokes equations done by Fromm and Welge in Los Alamos and Thoman and Szewczyk in University of Rotterdam probably. I am not recalling exactly the university where they worked Thoman and Szewczyk.

So, they modeled both these groups modeled basically flow past a circular cylinder. Now, circular cylinder when you some such flow, it may be circular cylinder, it may be square cylinder, it may be aero-foil blade basically you know boundary is well-defined solid boundary, but other side is open, its extending to infinity.

And in order to calculate accurately the flow field, you set up your boundary condition at the other extreme away from the wall as far as possible, it should be unparted boundary. So, I have mentioned what Thoman and Szewczyk, they use the treatment which specifies far field condition.

In the far field, if it is undisturbed field, we can say $\omega_{i,j} = 0$ with $\frac{\partial \psi}{\partial y} = u = U_\infty$ may be uniform stream may be considered which is flowing with the U_∞ velocity and $\frac{\partial \psi}{\partial x} = v = 0$.

So, unparted stream only stream wise velocity is there which is U_∞, v component is also 0, it is quite far from the solid boundary and $u = U_\infty$ was also utilized through a Neumann type of condition that means setting up $\frac{\partial \psi}{\partial y} = 0$.

So, instead of confining boundary, it becomes a free boundary, then we can follow the strategy of Thoman and Szewczyk, but it should be sufficiently away from the solid one. $\omega_{i,j} = 0$, $\frac{\partial \psi}{\partial y} = u = U_\infty$ so, $u = U_\infty$ $v = 0$ and ψ value considering $\frac{\partial \psi}{\partial y} = u = 0$ and that equal to U_∞.

So, between just two neighboring layers in the far-field that means, why I am saying neighboring layer? Top most layer nomenclature is ju just you know one layer below the grids j direction nomenclature is ju - 1. So, $\psi_{i,ju} = \psi_{i,ju-1} + U_\infty \Delta y$. Here, Δx Δy are same so, but for generality I have written Δy is equation 16 where B5 is considered at j = ju what I said that means, the top most boundary is at j = j u.

But confining flow, your geometry is fixed, if it is basically external flow, then it should be depending on your computer or a capacity number of grids you can take, but it should be located at a faraway location as far as possible from the solid surface, solid port.

(Refer Slide Time: 45:20)

Inlet Boundary

For the axial velocity (u), uniform or parabolic or any possible profile can be taken. Most widely used conditions are:

$$u_{1,\ j} = 1.5\ U_{av} \left[1 - \left(\frac{y_m - y}{y_m}\right)^2\right]_{j=js \ \text{ to } \ JMAX} \quad (17)$$

For normal velocity (v), Fromm and Harlow recommended:

$$v_{1,\ j} = 0 \quad \text{for} \quad j = js \ \text{ to } JMAX \quad (18)$$

The stream function $\psi_{1,\ j}$ can be obtained from the axial velocity profile at the inlet as

$$\psi_{1,\ j} = \int_{y \text{ at } js}^{y \text{ at } j} u(y)\Big|_{\text{inlet}}\ dy \quad (19)$$

Now, inlet condition. So, for the actual velocity u, uniform or parabolic or any possible profile can be taken. So, inlet profile when you model any such flow has to be known. If it is not known, you can say uniform velocity profile or a parabolic velocity profile or any profile that you choose and if you know from elsewhere like if it is you are doing a parallel experiment, experimentally obtained velocity profile, you can map that. So, it is up to you, you have to prescribe it.

But I have just given parabolic condition u equal to U_∞ is very easy. Instead, if it is a parabolic profile, we you can see, we have written $u_i = 1$ that means, at the first a column of grid points, j will vary from j minimum to j maximum, j is the I mean is a variable which is varying

$$u_{1,\ j} = 1.5\ U_{av} \left[1 - \left(\frac{y_m - y}{y_m}\right)^2\right]_{j=js \text{ to } JMAX} \quad (17)$$

This is y_m means up to the middle of the channel right, y_m not written a specifically forgotten to write y_m is the mid height of the channel right. So, j = js to JMAX, it makes a parabolic velocity profile with the central line velocity as 1.5 into Uav or 3/2 into Uav and this will give the variation in y direction parabolic variation equation 17.

Now, we have to specify the normal velocity that means, v velocity and we can follow the recommendations of Fromm and Harlow I have already mentioned. They were the first to model successfully using stream function vorticity approach and here, $v_{i,j} = 0$ of course, at the inflow plane, i = 1 so, j will vary from js and solid surface to JMAX and if on all j points, it will be 0, $v_{1,j} = 0$ that is equation 18.

Next, we will work with the stream function and as we have already said stream function can be obtained from the axial velocity profile, integration of the axial velocity profile axial velocity has been defined by equation 17. Here again, ψ at i = 1 for different js j = 1, j = 2, j = 3, j = 4 up to JMAX rather j = js to JMAX, we will integrate it from y = js to specified j.

Whatever is the value of j, j = 3, j = 4 up to that is specified j we will integrate the velocity profile and find out the stream function for the entire inflow plane and this inflow plane it is defined by ψ_1, i is fixed at 1 and different j values obtainable from this integration; j = js to any specified j and this is given by equation 19.

(Refer Slide Time: 50:12)

Vorticity ($\omega_{i,j}$) also depends on inlet velocity profile. Pao and Daugherty (1969) used uniform axial velocity profile, $v = 0$, and then specified $\omega_{1,j} = 0$. Greenspan (1969) fixed up $\psi_{1,j}$ from axial velocity profile similar to what has been given by (17) and assumed $\partial v/\partial x = 0$, which results in

$$\omega_{1,j} = -\frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial u}{\partial y} \qquad (20)$$

Vorticity also depends on inlet velocity profile. Now, different people considered vorticity, what should be the inlet vorticity on inlet plane a different way, Pao and Daugherty, they used uniform axial velocity profile that means, $u = U_\infty$, $v = 0$ and then specify it $\omega_{1,j} = 0$. So, inlet plane, they prescribed initial ω and then also as boundary condition ω as 0.

Greenspan say great mathematician, fixed up $\psi_{1,j}$ from the axial velocity profile similar to what has been given by equation 17. So, $\psi_{i,j}$ from the axial velocity profile, this I have discussed already and then, assumed $\frac{\partial v}{\partial x} = 0$. So, if you take you know this velocity profile from where you calculate $\psi_{1,j}$ and j is varying 1, 2, 3, 4, 5 so, basically you are calculating all ψ values and then, you can say he did rather Greenspan did $\frac{\partial v}{\partial x} = 0$.

So that ω is $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, ω is everywhere, $\frac{\partial^2 \psi}{\partial y^2}$ at the inlet plane given by equation 20.

(Refer Slide Time: 52:23)

Outflow Boundary

For axial and normal velocities, we can impose less restrictive type conditions as:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \qquad (21)$$

Thoman and Szewczyk developed outflow boundary conditions through setting

$$\frac{\partial \omega}{\partial x} = 0 \qquad (22)$$

Then, from $\frac{\partial v}{\partial x} = 0$, they derived $\frac{\partial^2 \psi}{\partial x^2} = 0$. For constant Δx, at $i = IMAX$, this gives:

$$\omega_{IMAX,\, j} = \omega_{IMAX-1,\, j} \qquad (23)$$

$$\psi_{IMAX,\, j} = 2\ \psi_{IMAX-1,\, j} - \psi_{IMAX-2,\, j} \qquad (24)$$

Now, last set of boundary conditions, this is given by you know again, we can we have to ensure that the flow should leave the domain without getting disturbed that means, if we can say something like $\frac{\partial u}{\partial x} = 0$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial \omega}{\partial x} = 0$, we are giving minimum disturbance because it is elliptic equation, you have to give boundary condition on all confining surfaces, but in boundary condition is not known.

And in boundary condition has to be determined, but for solving the flow equations, you have to apply. So, you apply physically, such a way that in boundary condition possess minimum disturbance to the flow filled in the interior domain and that is how we have gone we what we did? We set $\frac{\partial u}{\partial x} = 0$, $\frac{\partial v}{\partial x} = 0$.

Thoman and Szewczyk developed outflow boundary condition for ω by setting $\frac{\partial \omega}{\partial x} = 0$ that means, the wave velocity is coming $\frac{\partial u}{\partial x}$ in the from the interior domain, it will continue. The way v velocity has been developed in the interior domain; it will continue.

The way vorticity has been generated and vorticity is you know interacting, it will continue, it will have a smooth exit. So, exit boundary condition is needed, some value has to be given but the value will pose least disturbance to the interior flow field.

Then from $\frac{\partial v}{\partial x} = 0$, they derive $\frac{\partial^2 \psi}{\partial x^2} = 0$ that means, if we take Δx constant, you can take variable Δx also, but let us assume Δx = constant, then $\frac{\partial^2 \psi}{\partial x^2}$ at the point i = IMAX that means, at the end point in i direction for all js we have to prescribe that.

If we go for you know the setting up these conditions like $\frac{\partial \omega}{\partial x} = 0$ or $\frac{\partial \psi}{\partial x}$ =0, $\frac{\partial^2 \psi}{\partial x^2}$ = 0 So, what has been done this is exactly you can see that$\frac{\partial \omega}{\partial x} = 0$, ω at IMAX, j is throughout you know the column any all js, $\omega_{\text{IMAX}-1,\text{j}}$ that means, just the previous point for all js divided by Δx 0 so, which finally, culminates in $\omega_{\text{IMAX, j}} = \omega_{\text{IMAX-1, j}}$ this is $\frac{\partial \omega}{\partial x} = 0$.

And how to set $\frac{\partial^2 \psi}{\partial x^2}$ = 0? We apply central difference on ψ at the end point I mean the point at the exit boundary that means, all these points j will vary, i = IMAX, but we apply central difference. Central difference means we will involve IMAX, IMAX $-$ 1 and IMAX $-$ 2 because, usually central difference you will have two points on either side of the point of interest and this is called second order differencing in you know one side that means, it is called basically one-sided central difference.

So, we will IMAX, j is you know the varying parameter. So, IMAX $=2\psi_{IMAX-1,j} - \psi_{IMAX-2,j}$, if you think that j is same at any j, you will consider this condition you know j = 1, j = 2, j = 3 it will be same, but what we have written is $\psi_{IMAX,j} - 2\psi_{IMAX-1,j} - \psi_{IMAX-2,j}$ by $\Delta x^2 = 0$. Then from there, we can get this, this is called one-sided central difference.

So, taking IMAX as the point where we are going to determine ψ and we are making use of ψ's from the immediate interior points and generating this value so that it poses minimum disturbance through the flow field. So, u again, u and v will just follow what we have written for ω, IMAX , j equal to IMAX -1, for ω, for u, for v.

So, that is decided and for ψ, we apply second order derivative, this $\frac{\partial^2\psi}{\partial x^2} = 0$, one-sided central difference. We are been located at IMAX and involving IMAX-1 and IMAX-2, both the points from the interior domain and generate this value.

(Refer Slide Time: 59:46)

Vorticity- transport Equation

We will stop here today. Rests will be discussed in the next lecture

So, this is how you know the geometry of interest that we have taken can be solved and the first part of stream function vorticity method is over, remaining part we will do in subsequent lecture. We will stop here for today, thank you very much, we will meet again in the next lecture.

Thank you.