Computational Fluid Dynamics and Heat Transfer
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Lecture - 13
Introduction to Finite Element Method
(Galerkin Weighted Residual Method)

Good morning, everyone. Today, we will continue our lecture on Finite Element Method.

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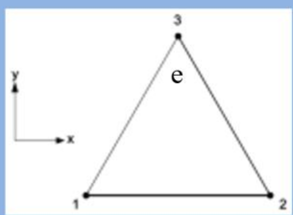
Finite Element Methods

Let us recall Eqn. (11) which we obtained via integration by parts:

$$\underbrace{\iint_{\Omega} k\left(\frac{\partial N_i}{\partial x}\cdot\frac{\partial T}{\partial x}+\frac{\partial N_i}{\partial y}\cdot\frac{\partial T}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y}_{(i)}+\underbrace{\int_{S_q} N_i\, q\,\mathrm{d}l}_{(ii)}+\underbrace{\int_{S_h} N_i\, h(T-T_f)\,\mathrm{d}l}_{(iii)}$$

$$+\underbrace{\int_{S_r} N_i\,\sigma\,\epsilon_s(T^4-T_f^4)\,\mathrm{d}l}_{(iv)}-\underbrace{\iint_{\Omega} N_i\, Q\,\mathrm{d}x\,\mathrm{d}y}_{(v)}=0 \qquad (11)$$

Where the terms have been numbered as (i) to (v) for the future reference



If we recapitulate what we did in the last class; basically, we evaluated the residue equation through integration by parts and then applying the boundary conditions and we got equation 11. And equation 11 these terms which are identified as 1, 2, 3, 4 and 5 we will remember for our future reference, because we have to evaluate all these integrals individually.

$$\underbrace{\iint_\Omega k\left(\frac{\partial N_i}{\partial x}\cdot\frac{\partial T}{\partial x}+\frac{\partial N_i}{\partial y}\cdot\frac{\partial T}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y}_{(i)}+\underbrace{\int_{S_q} N_i\,q\,\mathrm{d}l}_{(ii)}+\underbrace{\int_{S_h} N_i\,h(T-T_f)\,\mathrm{d}l}_{(iii)}$$

$$+\underbrace{\int_{S_r} N_i\,\sigma\,\epsilon_s(T^4-T_f^4)\,\mathrm{d}l}_{(iv)}-\underbrace{\iint_\Omega N_i\,Q\,\mathrm{d}x\,\mathrm{d}y}_{(v)}=0 \qquad (11)$$

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Finite Element Method

Let us recall the last slide of the last lecture

Using matrix notation, we obtained:

$$\begin{bmatrix}T_i\\T_j\\T_k\end{bmatrix}=\begin{bmatrix}x_i & y_i & 1\\x_j & y_j & 1\\x_k & y_k & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} \qquad (16)$$

Since the equation (12) can be written as:

$$T=\begin{bmatrix}x & y & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$$

We get within the element (e):

$$T=\begin{bmatrix}x & y & 1\end{bmatrix}\begin{bmatrix}x_i & y_i & 1\\x_j & y_j & 1\\x_k & y_k & 1\end{bmatrix}^{-1}\begin{bmatrix}T_i\\T_j\\T_k\end{bmatrix}=\begin{bmatrix}N_i & N_j & N_k\end{bmatrix}\begin{bmatrix}T_i\\T_j\\T_k\end{bmatrix} \qquad (17)$$

Now, again also recapitulate that, how did we define temperature at 3 vertices of the triangle T_i, T_j, T_k, this is basically given by equation 16 and also you may recall that; T is given by equation 12 which is x y i multiplied by a, b, c the column vector. Now this column vector from equation 16 we can easily see that, this is basically my inverse of a matrix x_i, y_i, 1; x_j, y_j, 1; x_k, y_k 1; these matrix inverse of a matrix multiplied by T_i, T_j, T_k column matrix.

So, if we combine this then we can say that, T is defined as N_i, N_j, N_k which is basically, a row matrix multiplied by T_i, T_j, T_k column matrix N_i, N_j, N_k this row matrix multiplied by T_i, T_j, T_k this column matrix and this is given by equation 17.

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Finite Element Method

where $$[N_i \quad N_j \quad N_k] = [x \quad y \quad 1] \begin{bmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{bmatrix}^{-1}$$

Let $$[A] = \begin{bmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{bmatrix}$$

$$[A]^{-1} = \frac{\mathrm{Adj}[A]}{|A|}$$

$\mathrm{Adj}[A]$ = Transpose of the matrix formed by cofactors of A

$$\begin{bmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{bmatrix}^{-1} = \frac{1}{|A|} \begin{bmatrix} \mathrm{Cof}(x_i) & \mathrm{Cof}(x_j) & \mathrm{Cof}(x_k) \\ \mathrm{Cof}(y_i) & \mathrm{Cof}(y_j) & \mathrm{Cof}(y_k) \\ \mathrm{Cof}(1) & \mathrm{Cof}(1) & \mathrm{Cof}(1) \end{bmatrix}$$

Cof (a_{13}) Cof (a_{23}) Cof (a_{33})

And this is the definition of the temperature and, we have to now find out N_i, N_j, N_k. And as we can see N_i, N_j, N_k is basically, x y 1 multiplied by inverse of this matrix. So, inverse of matrix a; now very clearly, we can define x_i, y_i, 1 first row, x_j y_j 1 second row, x_k y_k 1 third row. And inverse is as usual you know given by adjoint of the matrix divided by determinant of the matrix. And adjoint of matrix A can be written as transpose of the matrix formed by the cofactors of the elements of A.

So, here if we want to find out transpose of this matrix, we have to divide adjoint by determinant of A. And adjoint of A is basically, transpose of the cofactors. So, if we identify for example, x_i as the first element of the matrix a_{11}, then this one y_i as a_{12} and 1 as a_{13}, second row a_{21}, a_{22}, a_{23}, third row a_{31}, a_{32}, and a_{33}.

So, here when we write the first row in the adjoint matrix, that will be cofactor of x_i; first row will be cofactor of xi, then cofactor of x_j and cofactor of x_k. So, we have exactly written that, second row will be we will find cofactor of these and then transpose them. So, basically second row will be cofactor of y_i, cofactor of y_j and cofactor of y_k.

And third row we have written the cofactor of 1, cofactor of 1 and cofactor of 1, but these 1 the cofactors of these numbers, at these locations the cofactors are not same basically, this 1 represents a_{13}. So, this term will be equivalent to cofactor of a_{13}, this term will be cofactor of a_{23} and this term will be cofactor of a_{33}.

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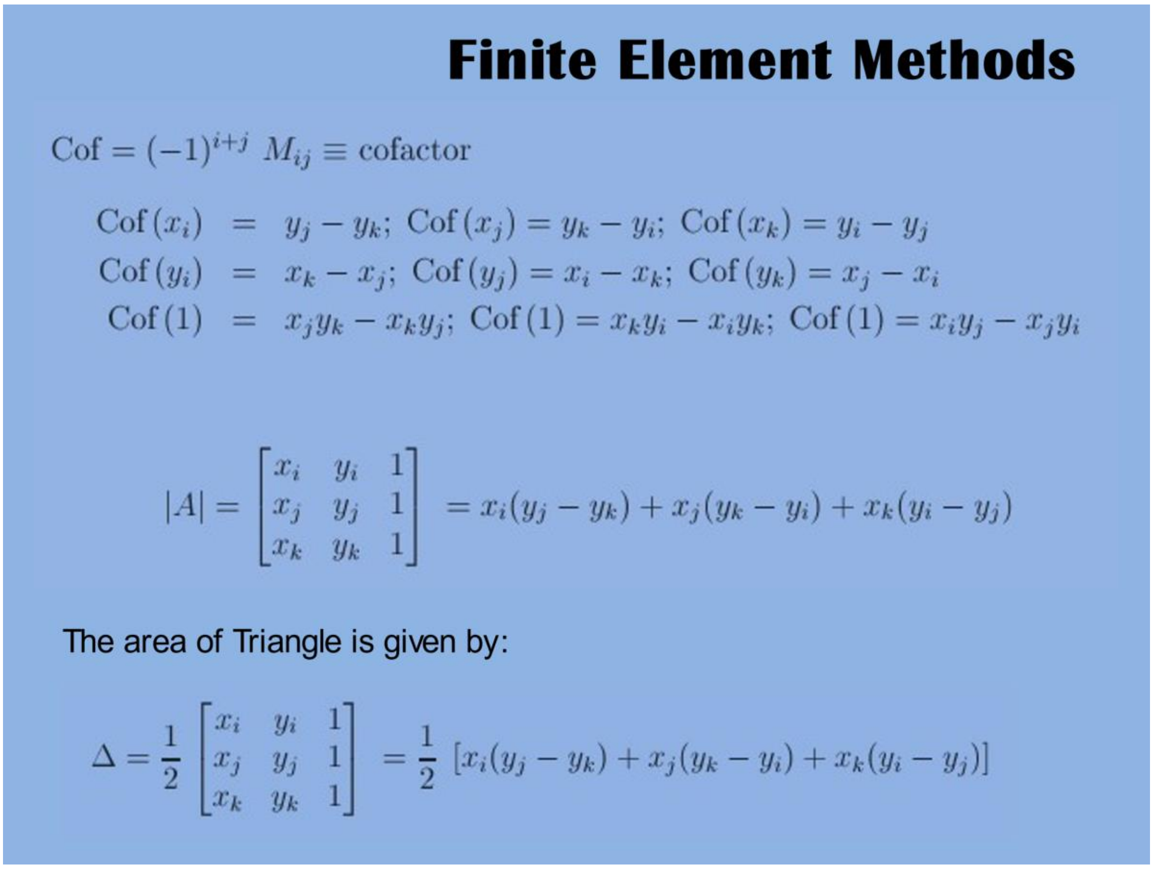

So, we will find out the cofactors. And as we know cofactor is having a specific sign and sign is determined by $(-1)^{i+j}$; i is counted for the row number and j is counted for the column number; for identifying a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23} etc. So, from there we determine the sign and this is minor.

So, we will calculate minor and basically, then we will sort of assign the sign and write the cofactor. So, the if we go back to the previous slide cofactor of x_i, this will be basically, it signs will be $(-1)^{i+j}$; that means, 2. So, it will be positive and minor of x_i is $(y_j - y_k)$.

So, we will see $y_j - y_k$ and minor will have no sign change, because the first element will have a positive. Similarly, we will find the second term in the first row, third term in the first row and all the cofactors of the first row, all the cofactors of the second row and all the cofactors of; all the value of the cofactors at the third row, value of the cofactors at the third row we find out.

And then, we also determine the determinant of this matrix and, determinant of this matrix is $x_i (y_j - y_k)+x_j (y_k-y_i)+x_k (y_i-y_j)$ Straightforward determinant, then area of the triangle can also be determined; it is we can basically, recapitulate or mathematics little bit and it happen so, that area of the triangle is basically, you will see what you will get is; half of the determinant, value is the value of the determinant by two. So, we have found out all these.

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Finite Element Method

Therefore,

$$[N_i \; N_j \; N_k] = \frac{[x \; y \; 1]}{|A|} \begin{bmatrix} (y_i - y_k) & (y_k - y_i) & (y_i - y_j) \\ (x_k - x_j) & (x_i - x_k) & (x_j - x_i) \\ (x_j \, y_k - x_k \, y_j) & (x_k \, y_i - x_i \, y_k) & (x_i \, y_j - x_j \, y_i) \end{bmatrix}$$

We can express N_i , N_j and N_k as:

$$\begin{aligned} N_i\,(x,y) &= \frac{x\,(y_j - y_k) + y\,(x_k - x_j) + x_j y_k - x_k y_j}{x_i\,(y_j - y_k) + x_j\,(y_k - y_i) + x_k\,(y_i - y_j)} \\ &= \frac{x\,(y_j - y_k) + x_j\,(y_k - y) + x_k\,(y - y_j)}{x_i\,(y_j - y_k) + x_j\,(y_k - y_i) + x_k\,(y_i - y_j)} \end{aligned} \qquad (18)$$

$$\begin{aligned} N_j\,(x,y) &= \frac{x\,(y_k - y_i) + y\,(x_i - x_k) + x_k y_i - x_i y_k}{x_i\,(y_j - y_k) + x_j\,(y_k - y_i) + x_k\,(y_i - y_j)} \\ &= \frac{x_i\,(y - y_k) + x\,(y_k - y_i) + x_k\,(y_i - y)}{x_i\,(y_j - y_k) + x_j\,(y_k - y_i) + x_k\,(y_i - y_j)} \end{aligned} \qquad (19)$$

And then, we find out N_i, N_j, N_k which is x y 1, which is a row matrix only single row multiplied by this matrix which has been formed through getting the transpose of the cofactors of the matrix, and divided by the determinant. Then; obviously, you know N_i will be the term will be found out, if we multiply this row versus the first column by the determinant, N_j second column into this row matrix and we get this divided by determinant. So, equation 19.

So, basically that N_i, N_j, N_k these 3 terms we can find out from this relationship, x y 1 multiplied by this matrix divided by the determinant individual terms of N_i, N_j, and N_k we have found out.

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Finite Element Method

$$N_k(x,y) = \frac{x(y_i - y_j) + y(x_j - x_i) + x_i y_j - x_j y_i}{x_i(y_j - y_k) + x_j(y_k - y_i) + x_k(y_i - y_j)} = \frac{x_i(y_j - y) + x_j(y - y_i) + x(y_i - y_j)}{x_i(y_j - y_k) + x_j(y_k - y_i) + x_k(y_i - y_j)} \quad (20)$$

Properties of the shape functions:

$$N_i = 1 \text{ at } (x_i, y_i) \qquad N_i = 0 \text{ at } (x_j, y_j) \text{ or } (x_k, y_k)$$

Similarly,

$$N_j = 1 \text{ at } (x_j, y_j) \qquad N_j = 0 \text{ at } (x_i, y_i) \text{ or } (x_k, y_k)$$
$$N_k = 1 \text{ at } (x_k, y_k) \qquad N_k = 0 \text{ at } (x_j, y_j) \text{ or } (x_i, y_i)$$

So, if we recall in our what we sort of studied in our first lecture on finite elements that, this N_i, N_j, N_k these are having unique properties that the point, where it is defined it will have value of 1 and other points it will have 0. So, this N_i now, at point i if we substitute x by x_i, y by y_i you will see exactly this will be 1, but at j and k point this will be 0.

Similarly, N_j will be 1 at point j, but other points it will be 0. N_k will be 1 at the coordinate x_k y_k; that means, at the point k, but at another points j and i points it will be 0. So, this is exactly the way the shape functions or the basic functions have to behave we get that behavior here.

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Finite Element Method

Within each element the equation (17) can be written as:

$$T = \sum_{j=1}^{3} N_j \, T_j$$

Under consideration:

$$N_1 = \frac{x\,(y_3 - y_2) + x_2\,(y - y_3) + x_3\,(y_2 - y)}{x_1\,(y_3 - y_2) + x_2\,(y_1 - y_3) + x_3\,(y_2 - y_1)} \quad (21)$$

$$N_2 = \frac{x_1\,(y_3 - y) + x\,(y_1 - y_3) + x_3\,(y - y_1)}{x_1\,(y_3 - y_2) + x_2\,(y_1 - y_3) + x_3\,(y_2 - y_1)} \quad (22)$$

$$N_3 = \frac{x_1\,(y - y_2) + x_2\,(y_1 - y) + x\,(y_2 - y_1)}{x_1\,(y_3 - y_2) + x_2\,(y_1 - y_3) + x_3\,(y_2 - y_1)} \quad (23)$$

During the global solution of all the nodal temperatures, the nodes may be given any global number. The correspondence between the local and global node numbers are usually stored in the arrays. Such arrays are called elemental connectivity in most of the FEM codes.

Now, we can say as we mentioned earlier, that the temperature for the element is defined

$$T = \sum_{j=1}^{3} N_j T_j$$

This j as you can see is now, any variable this we could have written N_i, T_i or N_k, T_k only thing we have to write, then i = 1, to 3 or k = 1, to 3; we have written in N_j, T_j, j = 1, to 3 which means, it will be ($N_1 T_1 + N_2 T_2 + N_3 T_3$) and now, instead of defining these points, because slowly we are progressing towards writing the algorithm.

So, these points can be written as 1, 2, 3 and the whole domain now, we can conceptualize as you know discretized domain with many elements all are triangular elements and each element has a number. And each element has 3 nodes 1, 2, 3 and all these 3 nodes 1, 2, 3 are having shape functions or basis function as N_1 N_2 N_3.

So, we are whatever we wrote for N_i, N_j, N_k simply i is substituted by 1, j is substituted by 2, and k is substituted by 3. And expressions also basically, x_i, y_i will be substituted by x_1

y_1, x_j y_j will be substituted by x_2 y_2 and x_k, y_k will be substituted by x_3, y_3. And we get equation 21, 22, 23.

$$N_1 = \frac{x\,(y_3 - y_2) + x_2\,(y - y_3) + x_3\,(y_2 - y)}{x_1\,(y_3 - y_2) + x_2\,(y_1 - y_3) + x_3\,(y_2 - y_1)} \qquad (21)$$

$$N_2 = \frac{x_1\,(y_3 - y) + x\,(y_1 - y_3) + x_3\,(y - y_1)}{x_1\,(y_3 - y_2) + x_2\,(y_1 - y_3) + x_3\,(y_2 - y_1)} \qquad (22)$$

$$N_3 = \frac{x_1\,(y - y_2) + x_2\,(y_1 - y) + x\,(y_2 - y_1)}{x_1\,(y_3 - y_2) + x_2\,(y_1 - y_3) + x_3\,(y_2 - y_1)} \qquad (23)$$

We have done this because, later, you will see that we have to also identify each element and their node numbers and connect it with the global node number; see in finite difference or finite volume you have seen if we are basically, the points are expressed as i j k or i j. So, all the variables are defined by the subscripted array i j or i j k, but in finite element the points are not defined that way, when we compute the points have the global node number.

So, maybe in a domain you have maybe 30 triangles and you have maybe you know 50 points or 60 points now, these points globally 1, 2, 3, 4, 5, 6 it will go up to the you know will cover all the points in the domain by you know such count and these points, automatically you can understand will correspond to a point of in an element.

So, any global point will also be point of 2 or 3 elements common point. So, in the domain a point has identification; unique identification is global node number. Otherwise, it will have local node number depending on the element and which node of the element is basically that point.

So, this we will do as we have written, that the correspondence between local and global node numbers are usually stored in the form of arrays and this is called elemental connectivity, and we will discuss this in one of the subsequent lectures.

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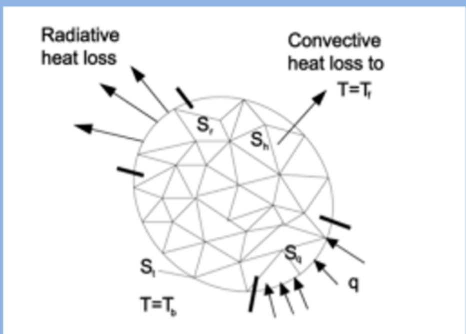
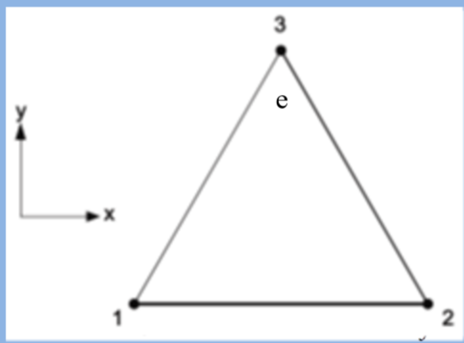
Finite Element Methods

The domain Is divided into small elements. We are using a 3-noded triangular Element (Figure) for our problem of interest.





So, as we can see, for example, here any element **e** has 1, 2, 3 say for example, this element we are talking about, if this is 1, 2, 3 these 3 points, but this point has a global nomenclature; global nomenclature means, that if we start defining 1, 2, 3, 4 this way, so, maybe this point globally, may be 10 the way data structure is arranged, but locally, maybe it is element number 8 and node number 1.

So, maybe this is element number 9 is node number 2 and it is globally maybe node number 10. So, this relationship, we have to later on, it is a little difficult bookkeeping, but through data structure we have to manage that.

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Finite Element Method

Let us consider the issue of evaluating the term (i) of equation (11) for a typical element (e):

$$\sum_{j=1}^{3}\left[\iint_{(e)} k\left(\frac{\partial N_i}{\partial x}\cdot\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\cdot\frac{\partial N_j}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right][T_j]=\left\{K^{(e)}\right\}\left\{T_j^{(e)}\right\} \quad (24)$$

where $i = 1,2,3$ and $j = 1,2,3$ for the element. Thus, the conduction contribution of an element to the residual equation at a node takes the form of a 3×3 matrix $\{K^{(e)}\}$ multiplied by $\{T_j^{(e)}\}$, a 3×1 vector. Substituting for N_1,N_2 and N_3, one can obtain all the entries of the 3×3 matrix $\{K^{(e)}\}$. For $i = 1$ and $j = 1,2,3$, we get

$$\frac{k}{4A^2}\left[\left\{(y_2-y_3)^2+(x_3-x_2)^2\right\}T_1+\left\{(y_3-y_1)\,(y_2-y_3)+(x_1-x_3)\,(x_3-x_2)\right\}T_2+\left\{(y_1-y_2)\,(y_2-y_3)+(x_2-x_1)\,(x_3-x_2)\right\}T_3\right]$$

So, now we are focused on individual element, any given element and there we have to now find out this integration, which was the first term of equation 11 that is area integral of $k\left(\frac{\partial N_i}{\partial x}.\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}.\frac{\partial N_j}{\partial y}\right)$ this is integrated over the element; that means, node 1, 2, 3 and this T_j corresponds to; obviously, T_j will also vary 1 to 3; T_1, T_2, T_3. So, this become finally, some array which is basically multiplication of a matrix, this is say 3 by 3 matrix and a column matrix; that means, this will have basically T_1, T_2, T_3 as a column.

So, 3 rows, but single column T_1, T_2, T_3. So, where i =; i = as we, where what we have explained i = 1, 2, 3 and j = 1, 2, 3 for the element, thus the conduction contribution of an element will be basically, a matrix which we can say conduction matrix at the movement. So, it is $K^{(e)}$ multiplied by a column vector which is a T_1, T_2, T_3.

Now, substituting N_1, N_2, N_3 one can obtain all the entries of this 3 by 3 matrix. So, first we have to take i = 1 where is j =1, 2, 3 then i = 2 j = 1, 2, 3, i = 3 j = 1, 2, 3 and we will be able to get basically, all the elements of this matrix.

So, by taking i = 1, j = 1, 2, 3 and substituting for N_i, N_j in this equation, we know what is N_i what is N_j; here it will be N_1, N_2, N_3 and so, first we keep i as 1 and vary j as you know 1, 2, 3 i as 1 j as 1, 2, 3 and we take we after substituting that and integrating we will get this row, which is basically $(y_2 - y_3)^2+(x_3 - x_2)^2$ into T_1 similarly, in expression into T_2 another expression involving x and y into T_3.

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Finite Element Method

Which yields

$$\frac{k}{4A^2}\left[\left(x_{32}^2 + y_{23}^2\right) T_1 + \left(y_{31}\, y_{23} + x_{13}\, x_{32}\right) T_2 + \left(x_{21}\, x_{32} + y_{12}\, y_{23}\right) T3\right] \quad (25)$$

Next, for $i = 2$ and $j = 1, 2, 3$, we get

$$\frac{k}{4A^2}\left[\left\{\left(y_3 - y_1\right)\left(y_2 - y_3\right) + \left(x_1 - x_3\right)\left(x_3 - x_2\right)\right\} T_1 + \left\{\left(y_3 - y_1\right)^2 + \left(x_1 - x_3\right)^2\right\} T_2 + \left\{\left(y_3 - y_1\right)\left(y_1 - y_2\right) + \left(x_1 - x_3\right)\left(x_2 - x_1\right)\right\} T_3\right]$$

which yields

$$\frac{k}{4A^2}\left[\left\{y_{31}\, y_{23} + x_{13}\, x_{32}\right\} T_1 + \left\{x_{13}^2 + y_{31}^2\right\} T_2 + \left\{y_{31}\, y_{12} + x_{13}\, x_{21}\right\} T_3\right] \quad (26)$$

Now, if we so, we have reorganize that we can clearly see its you know, basically T_1 we have together with its coefficient T_2 together with this its coefficients and T_3. Now, i = 2 and j = 1, 2, 3. If we again evaluate, we will get finally, you know some T_1 with its coefficients, T_2 with its coefficient coefficients and T_3 with its coefficients you know, basically equation 26 and in the same way; i = 3 and j = 1, 2, 3 will give us equation 27.

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Finite Element Method

Finally for $i = 3$ and $j = 1, 2, 3$, we get

$$\frac{k}{4A^2}\left[\{(y_1 - y_2)(y_2 - y_3) + (x_2 - x_1)(x_3 - x_2)\} T_1 + \{(y_1 - y_2)(y_3 - y_1)\right.$$
$$\left.(x_2 - x_1)(x_1 - x_3)\} T_2 \left\{(y_1 - y_2)^2 + (x_2 - x_1)^2\right\} T_3\right]$$

which yields,

$$\frac{k}{4A^2}\left[\{y_{12}\, y_{23} + x_{21}\, x_{32}\} T_1 + \{y_{12}\, y_{31} + x_{13}\, x_{21}\} T_2 + \left\{y_{12}^2 + x_{21}^2\right\} T_3\right] \qquad (27)$$

The final expression for the term (i) of equation (11) is:

$$\frac{k\left\{A^{(e)}\right\}}{4\left\{A^{(e)}\right\}^2}\begin{bmatrix} x_{32}^2 + y_{23}^2 & y_{31}y_{23} + x_{13}x_{32} & x_{21}x_{32} + y_{12}y_{23} \\ y_{31}y_{23} + x_{13}x_{32} & x_{13}^2 + y_{31}^2 & y_{31}y_{12} + x_{13}x_{21} \\ y_{12}y_{23} + x_{21}x_{32} & y_{12}y_{31} + x_{13}x_{21} & y_{12}^2 + x_{21}^2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

So, now, if we reorganize 25, 26 and 27 we can write in this way. So, k multiplied by area of the triangle divided by 4 multiplied by area of the triangle means, area of the element triangular element square and these are the different terms of the matrix. Equivalent to we can say a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}. So, all these are all 9 terms the matrix into T_1, T_2, T_3. So, we get the complete evaluation of a first term of equation 11.

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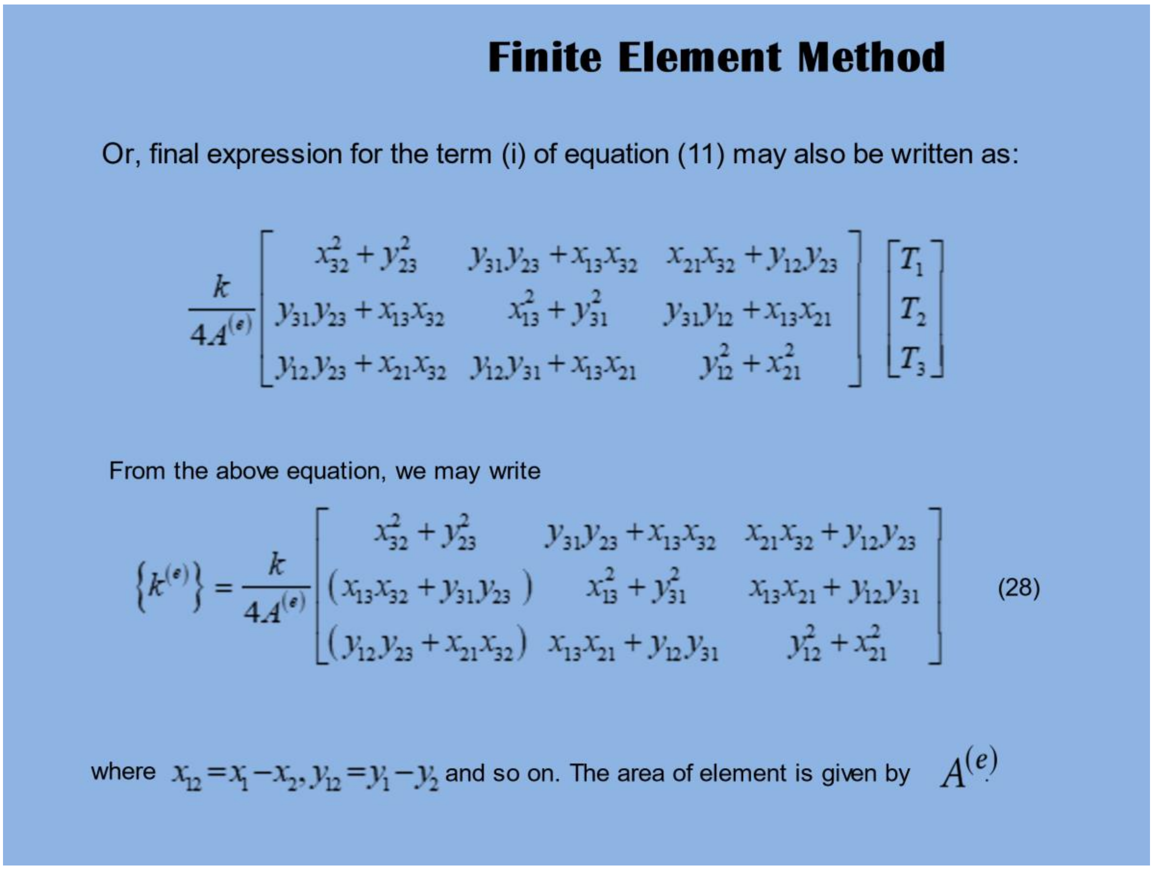

So, final expression of term one of equation 11 as we said, we can write it in this way. And here if you recall that, we wanted to describe this as k elemental matrix into T column matrix. So, this k elemental matrix now, comes in this form. k by 4 into area of triangle and these are the different terms in the matrix.

$$\left\{k^{(e)}\right\} = \frac{k}{4A^{(e)}}\begin{bmatrix} x_{32}^2+y_{23}^2 & y_{31}y_{23}+x_{13}x_{32} & x_{21}x_{32}+y_{12}y_{23} \\ (x_{13}x_{32}+y_{31}y_{23}) & x_{13}^2+y_{31}^2 & x_{13}x_{21}+y_{12}y_{31} \\ (y_{12}y_{23}+x_{21}x_{32}) & x_{13}x_{21}+y_{12}y_{31} & y_{12}^2+x_{21}^2 \end{bmatrix} \quad (28)$$

So, and obviously, we know what are the meanings of this you know, quantities like x_{32} or y_{23}, we have written it also here, x_{12} is x_1-x_2, y_{12} is y_1-y_2 and so on. And area of the element is given by $A^{(e)}$. And so, this is the $k^{(e)}$ matrix 3 by 3 multiplied by the column vector T_1, T_2, T_3 is basically, the final outcome of the integration of the first term of equation 1.

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Finite Element Methods

The fifth term of the equation (11) for typical element (e) is

$$\iint_{(e)} N_i Q \, dx \, dy = Q \iint_{(e)} N_i \, dx \, dy \quad \text{with } i = 1,2,3 \qquad (29)$$

For $i = 1$

$$N_1 = \frac{x(y_3 - y_2) + x_2 (y - y_3) + x_3 (y_2 - y)}{x_1 (y_3 - y_2) + x_2 (y_1 - y_3) + x_3 (y_2 - y_1)}$$

$$= \frac{Q}{2A^{(e)}} \iint \{x(y_2 - y_3) + x_2 (y_3 - y) + x_3 (y - y_2)\} dx \, dy$$

$$= \frac{Q}{2A^{(e)}} \left[(y_2 - y_3) \iint x \, dx \, dy + (x_3 - x_2) \iint y \, dx \, dy + (x_2 y_3 - x_3 y_2) \iint dx \, dy \right]$$

Now, we will go to go for the 5th term. If you are not confused from here again, we will go back to the equation of interest that means equation 11. So, please setting down here, but we will refer to equation 11 without any ambiguity. So, this was equation 11, which we are evaluating and first term has been already evaluated. We have shown second third fourth term we will evaluate little later, but evaluating fifth term is very easy, that is why we are taking it up today.

$$\iint_{(e)} N_i Q \, dx \, dy = Q \iint_{(e)} N_i \, dx \, dy \quad \text{with } i = 1,2,3 \qquad (29)$$

So, we will now go for evaluation of the fifth term, which is over the element N_i Q dxdy, where i will vary from 1, to 3; that means you know, if we take N_1 it is we can recapitulate the expression x (y_3-y_2) + $x_2(y-y_3)$ + $x_3(y_2-y)$ and this is basically (denominator), the value of the determinant of the matrix A if you recall.

So, now N_1 is this and we when we are integrating; that means, this is basically twice the elemental area going out of the integration, and we can re arrange the integration this way, and then we can perform the integration and we get a value Q.

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Finite Element Method

$$= \frac{Q}{2A^{(e)}}\left[\left(y_2 - y_3\right)A^{(e)}x_c + \left(x_3 - x_2\right)A^{(e)}y_c + \left(x_2y_3 - x_3y_2\right)A^{(e)}\right]$$

$$= \frac{Q}{2}\left[\frac{\left(x_1 + x_2 + x_3\right)}{3}\left(y_2 - y_3\right) + \frac{\left(y_1 + y_2 + y_3\right)}{3}\left(x_3 - x_2\right) + \left(x_2y_3 - x_3y_2\right)\right]$$

$$= \frac{Q}{2.3}\left[\begin{array}{l} x_1y_2 + x_2y_2 + x_3y_2 - x_1y_3 - x_2y_3 - x_3y_3 + x_3y_1 \\ + x_3y_2 + x_3y_3 - x_2y_1 - x_2y_2 - x_2y_3 + 3x_2y_3 - 3x_3y_2 \end{array}\right]$$

$$= \frac{Q}{2.3}\left[x_1\left(y_2 - y_3\right) + x_2\left(y_3 - y_1\right) + x_3\left(y_1 - y_2\right)\right]$$

$$= \frac{Q}{2.3}2A^{(e)} = \frac{QA^{(e)}}{3}$$

This is again little involved algebraic steps for forming integration substitute substituting the values and rearranging the values. But if we do that systematically, we will get Q divided by 2 into 3 and this is nothing, but again area of the triangular element. And Q is basically volumetric heat generation rate. And if we go for N = 2 and N = 3.

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Finite Element Method

Similarly for i=2 and 3 can be evaluated and we can write:

$$\iint_{(e)} N_i Q \, dx \, dy = QA^{(e)} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \left\{G^{(e)}\right\} \qquad (30)$$

where $A^{(e)}$ is the area of the element and $\{G^{(e)}\}$ is the elemental (energy) generation vector,

So, then we will for each term for each i = 1; that means, N_1, N_2, N_3 for like, i = 1 we have got Q $A^{(e)}$ / 3 similarly, i = 2 also we will get Q $A^{(e)}$ / 3, i = 3 also we will get Q $A^{(e)}$ / 3. So, if we put them together this elemental integration will be Q into $A^{(e)}$ one third, one third, one third.

And this is an array also basically, column matrix; that means, we will have 3 rows, but a single column vector and $A^{(e)}$ is area of the element and $G^{(e)}$ is the elemental generation vector or here we can say, it is heat generation vector. So, this is elemental heat generation or elemental generation vector that is equation 30.

So, with equation 30, then we have evaluated the fifth term of this integral and first term of this integral also we have evaluated, which is you know this k into again T_1, T_2, T_3 the column vector. So, these two integrations are over, we will perform remaining integrals and then talk about the strategy of assembling them for an element then also it has to represent the complete domain.

So, we have to then establish the establish basically the elemental connectivity, global, local node number connectivity etcetera. So, evaluation of remaining terms of the integral and setting up the final matrix form that is to be evaluated to find out the temperatures in the entire domain, we will take up in the next class. We stop here today. Thank you very much.

Thank you.