Computational Fluid Dynamics and Heat Transfer

Prof. Gautam Biswas
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 12
Introduction to Finite Element Method (Preliminary Concepts)

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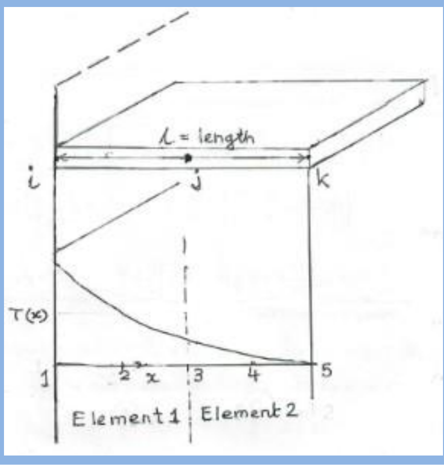
Background

- Consider heat transfer through a fin, with a uniform cross section. The figure shows the temperature profile in an element as represented by quadratic polynomial.
- Let us see how the shape functions help in describing temperature profile and its derivative.

- **Consider one-dimensional quadratic element**



Today, we will start a new topic on Finite Element Method. Before we start our topic on finite element, we will focus on a problem just to explain few basic ideas, and this can be considered as a background material. Now, consider heat transfer through a fin, now, a fin is attached to a base you can see, and we are interested in temperature profile in the fin.

So, this is a realistic fin attached to a base, but let us you know, approximate this fin as a line element. So, at one point; that means, at the base let us say, the point is i and this line, which is representing the fin, the endpoint the point is k. And temperature varies over this line element and as we know, the fin is a device for extended heat transfer.

So, basically, you know it pulls the metal or the body of interest and its temperature. You know, profile is basically parabolic or non-linear and it varies from base to the tip, and you know, basically the fin loses heat to the local ambience and that is how the temperature varies from a value say, T_i to T_k through a parabolic variation.

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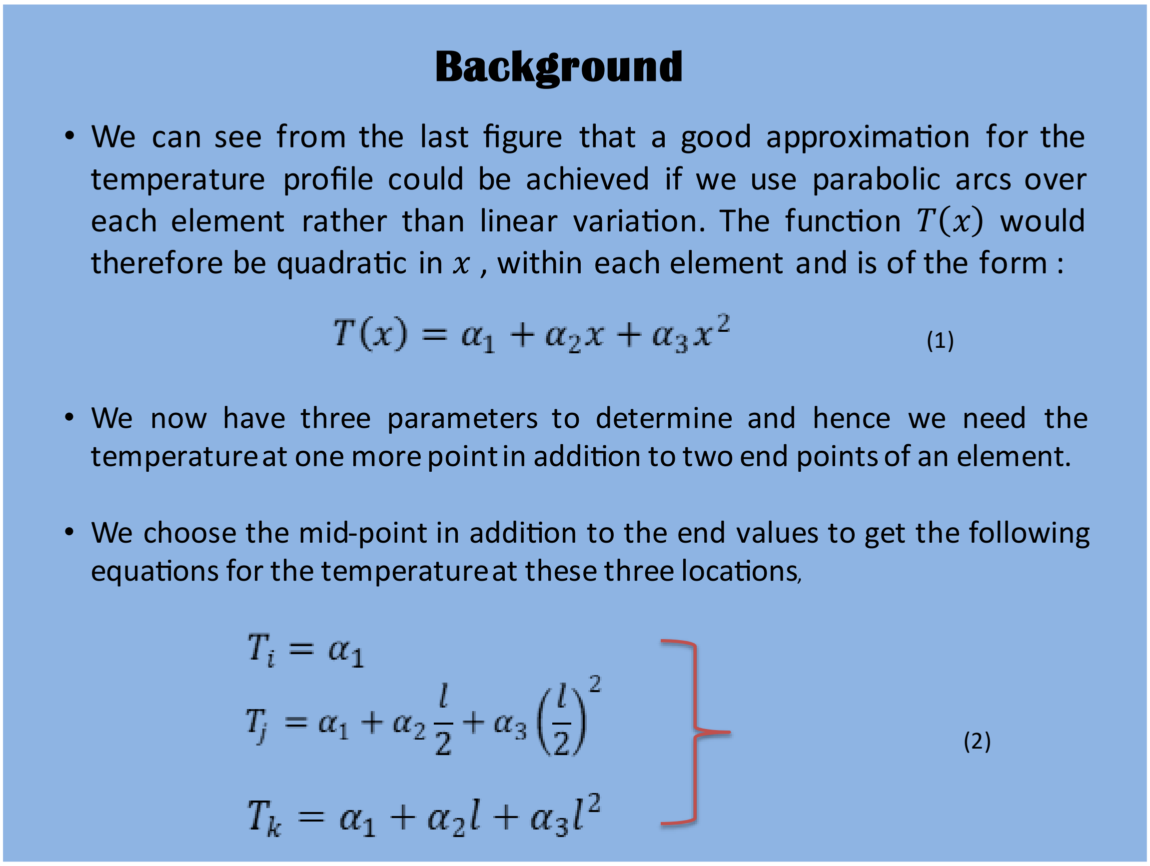

Now, if we want to analyze the last picture, we can say that the temperature variation over; if this element from i to k can be represented by, $\alpha_1 + \alpha_2 x + \alpha_3 x^2$. This is a quadratic variation. We now, so we have basically we need temperature at one more point, in addition to i and k, j point which is located at $l/2$, we need help of that point.

So, we choose the mid-point in addition to the end values to get the following equations,

$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \tag{1}$$

then when $x = 0$, we had said the first point T_i. So, here we can say, $T_i = \alpha_1$ since x is a 0; mid-point is T_j, which is $l/2$ away from zero. So, $T_j = \alpha_1 + \alpha_2 \left(\frac{l}{2}\right) + \alpha_3 \left(\frac{l}{2}\right)^2$. And the endpoint, which is l distance away from $x = 0$, we can say $T_k = \alpha_1 + \alpha_2 l + \alpha_3 l^2$.

We have to solve these 3 equations, in order to find out α_1, α_2 and α_3. Here directly, we can see $T_i = \alpha_1$. Then we can substitute this aα_1 in second equation, by T_i and in the third equation also α_1 by T_i, then we will have two equations in T_j and T_k, two unknowns α_2 and α_3 and we can find out α_2 and α_3.

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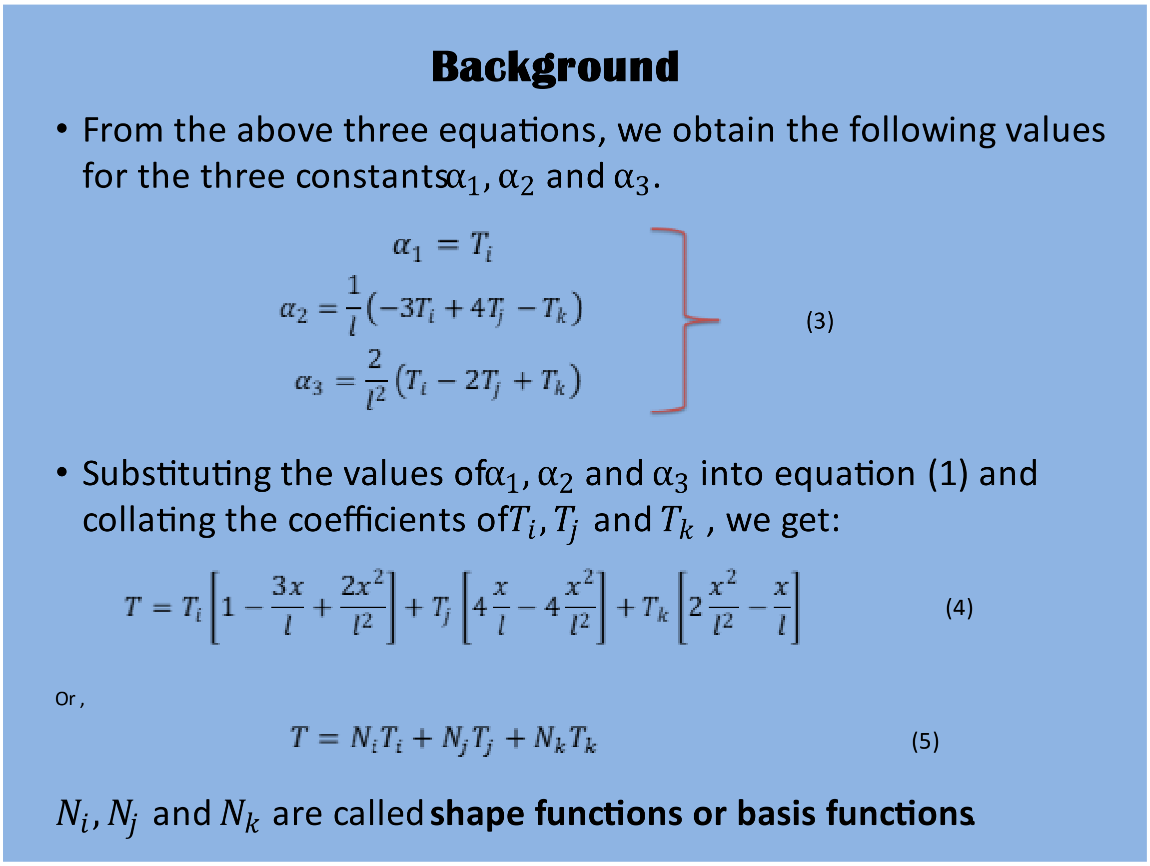

So, we finally, get

$$\alpha_1 = T_i$$

$$\alpha_2 = \frac{1}{l}(-3T_i + 4T_j - T_k)$$

$$\alpha_3 = \frac{2}{l^2}(T_i - 2T_j + T_k) \qquad (3)$$

So, α_1, α_2, and α_3 these are found out. Now, we can substitute α_1, α_2, and α_3 in this you know, equation 1. And we can sort of reorganize the values, in terms of coefficient of T_i, coefficient of T_j and coefficient of T_k.

So, then we can write T equal to; if we after substitution α_1, α_2, and α_3. If we reorganize the equation and try to find out coefficients of T_i, T_j and T_k we will find out equation 4, that

$$T = T_i\left[1 - \frac{3x}{l} + \frac{2x^2}{l^2}\right] + T_j\left[4\frac{x}{l} - 4\frac{x^2}{l^2}\right] + T_k\left[2\frac{x^2}{l^2} - \frac{x}{l}\right] \quad (4)$$

So, this quantity (first bracket) can be called now, as N_1, this quantity second quantity, this quantity can be called as N_j and the coefficient of T_k can be called as N_k. So, coefficient of T_i is N_i, coefficient of T_j is N_j and coefficient of T_k is N_k. This N_i, N_j and N_k are called shape functions or basis functions.

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Background

- Hence **the shape functions**for a one-dimensional quadratic element are obtained from equation (4) as follows:

$$\left.\begin{aligned} N_i &= \left[1 - \frac{3x}{l} + \frac{2x^2}{l^2}\right] \\ N_j &= \left[4\frac{x}{l} - 4\frac{x^2}{l^2}\right] \\ N_k &= \left[2\frac{x^2}{l^2} - \frac{x}{l}\right] \end{aligned}\right\} \quad (6)$$

- The **shape functions** are employed to represent the nature of the solution within each element.
- The first derivative of temperature can now be written as:

$$\frac{dT}{dx} = \frac{dN_i}{dx}T_i + \frac{dN_j}{dx}T_j + \frac{dN_k}{dx}T_k \quad (7)$$

$$\frac{dT}{dx} = \left[\frac{4x}{l^2} - \frac{3}{l}\right]T_i + \left[\frac{4}{l} - \frac{8x}{l^2}\right]T_j + \left[\frac{4x}{l^2} - \frac{1}{l}\right]T_k \quad (8)$$

Hence, shape functions of a one-dimensional quadratic element are finally, obtained as

$$N_i = \left[1 - \frac{3x}{l} + \frac{2x^2}{l^2}\right]$$

$$N_j = \left[\frac{4x}{l} - \frac{4x^2}{l^2}\right]$$

$$N_k = \left[\frac{2x^2}{l^2} - \frac{x}{l}\right]$$

The shape functions are employed to represent the nature of the solution within each element. We have taken here, as one element, in the schematic; you recall we took one element, we can take two elements also then I mean the calculation has to be changed slightly.

But you know, if we take one element we can say, $T = N_i\, T_i + N_j\, T_j + N_k\, T_k$. And we can also find out derivative of this expression for temperature, $\frac{dT}{dx}$ which will be given by.

$$\frac{dT}{dx} = \frac{dN_i}{dx}T_i + \frac{dN_j}{dx}T_j + \frac{dN_k}{dx}T_k \tag{7}$$

So, $N_i dx$ we can find out, $N_j dx$ we can find out, and $N_k dx$ we can find out. So, it is basically,

$$\frac{dT}{dx} = \left[\frac{4x}{l^2} - \frac{3}{l}\right]T_i + \left[\frac{4}{l} - \frac{8x}{l^2}\right]T_j + \left[\frac{4x}{l^2} - \frac{1}{l}\right]T_k \tag{8}$$

So, $\frac{dT}{dx}$ can be written in this way, this can be also expressed in a compact form, if we say $\frac{dT}{dx}$ as 'q', then we can express this expression in terms of multiplication of a row matrix and column matrix.

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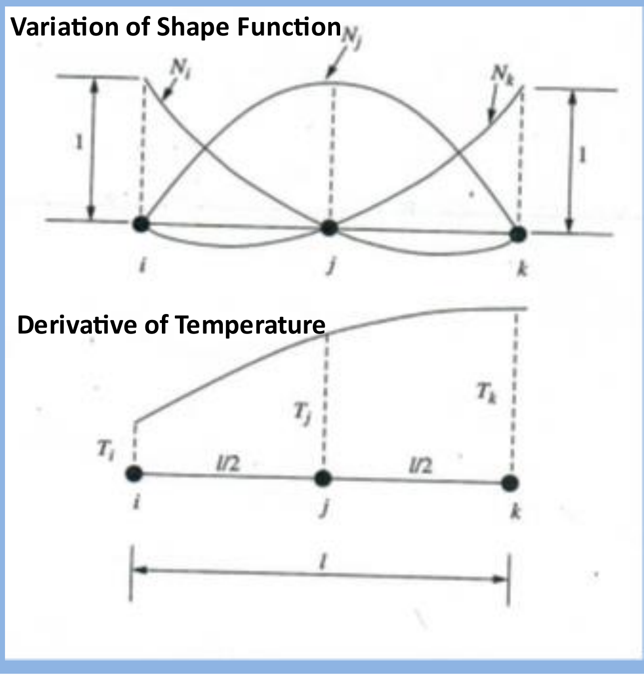
Background

- The variation of temperature gradients and shape functions of a typical quadratic element are shown in the below figure:



We will come to that little later; now, N_i the value which i have got, N_j the value which we have got an N_k we are plot it that. And $\frac{dT}{dx}$ we are plot it. $\frac{dT}{dx}$ we have seen and this is dN_i /dx into T_i, dN_j /dx into T_j, dN_k /dx into $T_k$that we have plotted.

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Background

- In matrix form,

$$q = [B]\{T\} \tag{9}$$

- The $[B]$ matrix is given as:

$$[B] = \left[\left(\frac{4x}{l^2} - \frac{3}{l}\right)\left(\frac{4}{l} - \frac{8x}{l^2}\right)\left(\frac{4x}{l^2} - \frac{1}{l}\right)\right] \tag{10}$$

- Equation (6) shows that $N_i = 1$ at i and 0 at j and k , $N_j = 1$ at j and 0 at i and k and $N_k = 1$ at k and 0 at i and j.
- It can be verified easily that within an element the summation over the shape functions is equal to unity,that is,

$$\sum_{i=1}^{3} N_i = 1 \tag{11}$$

- It can be shown that the sum of the above three shape functions anywhere in the domain,$0 \leq x \leq l$ is always unity.

We will explain these plots little later; but, let us look at what I was saying that, dT/dx can be expressed as multiplication of a row matrix and column matrix. This column matrix is understood T_i, T_j, T_k and B matrix is again if we look at this term, B matrix is having one row comprising of three elements. First element is $\frac{4x}{l^2} - \frac{3}{l}$, second element is $\frac{4}{l} - \frac{8x}{l^2}$ and third element is $\frac{4x}{l^2} - \frac{1}{l}$.

So, this is B row and T is a column T_i, T_j, T_k, so, dT/dx we can write in a compact way. Now, we will go if we plot N_i, N_j and N_k over the element. See, N_i is given by this. So, at $x = 0$ it is 1, but at all other x you will see, it is 0, all other points all other x is it is 0 that means, at j point it is 0, and k point it is 0. So, that is how, N_i is 1 at point i, at point j it is 0, point k it is 0.

Similarly, variation of N_j if we look, N_j at $l = 1/2$ it is 1, but $x = 0$ and $x = l$ it is 0. So, exactly that is what we can see, N_j at $x = 1/2$; that means, at j point it is 1, but it is 0 at i point, it is 0 at k point. Similarly, N_k at $x = 1$ it is 1, but $x = 0$ it is 0, $x = l/2$ also it is 0.

So, N_k is 0 at point i, 0 at point j, but it is 1 at point k. So, this is how within the element shape functions, value of the shape functions varies.

And this is a derivative of dT/dx variation over the entire element at, i, j and at k this are the vary, I mean this is how dT/dx varies. Major conclusion as we can say from equation 6 is N_i is 1 at i, this is 0 at j and k, N_j is 1 at j and 0 at i and k. N_k is 1 at k and 0 at j and k. This is number 1, and number 2 this Nijk summation of Ns at anywhere, in the domain is always 1, ok. So, that above 3 shape functions, anywhere in the domain between 0 and l it is always unity.

(Refer Slide Time: 15:20)

Background

- It can also be observed that even though the derivatives of the quadratic element are functions of the independent variable x , they will not be continuous at the inter-element nodes.
- The type of interpolation used here is known as **Lagrangian** (as they can be generated by **Lagrangian** interpolation formulae) and it only guarantees the continuity of the function across the inter-element boundaries.
- These type of elements are known as ***C^0 elements***, in which the superscript indicates that only derivatives of zero order are continuous, that is, only the function is continuous.
- The elements that also assure the continuity of derivatives across inter-element boundaries, in addition to the continuity functions, are known as ***C^1 elements*** and such functions are known as **Hermite polynomials**.

So, we can in conclusion say that, even though derivatives of the quadratic element are functions of the independent variable x, they will not be continuous; dT/dx may not be continuous at inter-element nodes. The type of interpolation used here for the temperature is known as Lagrangian interpolation. Since this can be generated by, Lagrangian interpolation formulae, and this guarantees continuity of the function, that is a dependent variable across the inter element boundaries.

So, these types of elements are known as C^o elements, in which the superscript indicates that only derivatives of zero order are continuous, that is, only the function is continuous. The elements that also assure the continuity of the derivatives across the inter-element boundaries, in addition to continuity of the functions, are known as C^1 elements.

So, C^1 elements ensure continuity of the dependent variable and its derivative. C^o elements ensure continuity of the dependent variable only. So, we are very known the, properties of C^o elements and C^1 elements also we have mentioned here, now the functions, which are valid for C^1 elements are called as Hermite polynomials.

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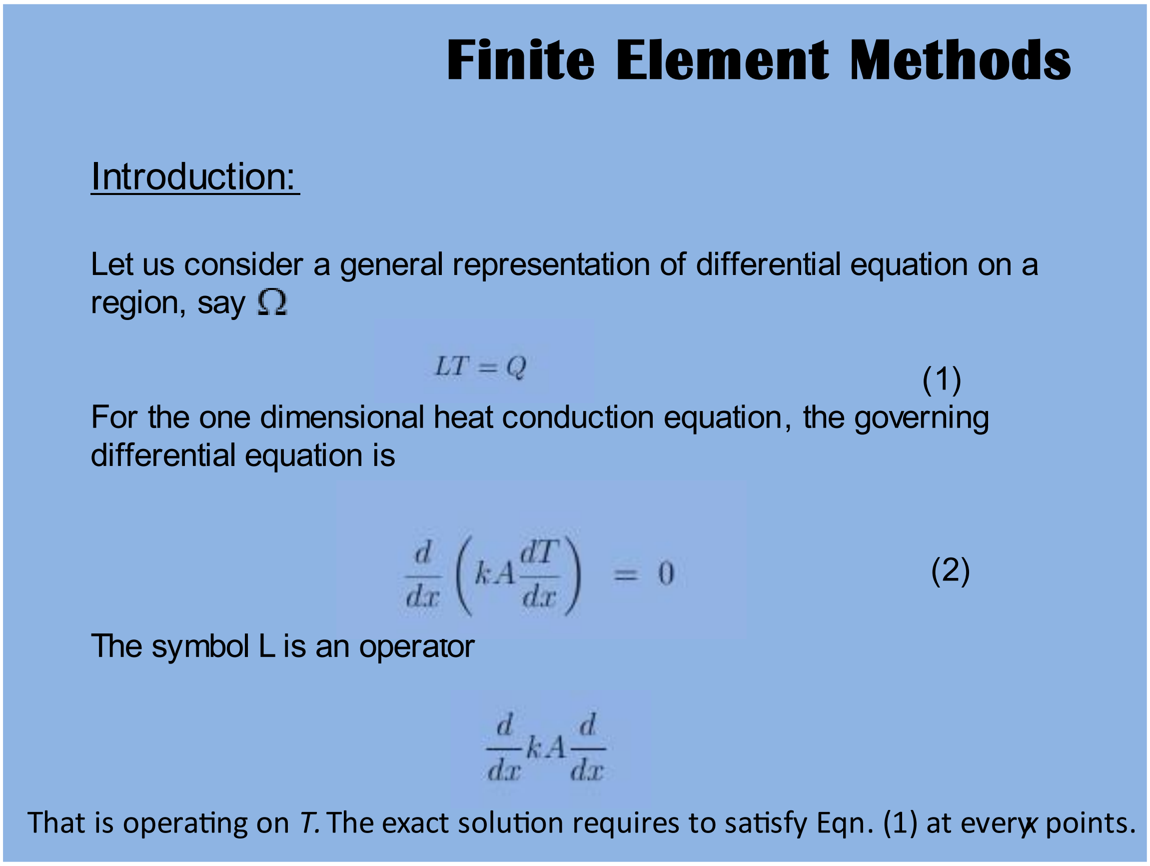

Now, with this background let us start our lesson on Finite Element Methods. Let us consider a general representation of a differential equation on a region say, upper case omega (Ω) given by LT = Q. We are started renumbering equations again, background material we had number from 1 to 9 or 10, this I mean part again we are remembering, LT = Q for the one-dimensional heat conduction equation.

$$LT = Q \qquad (1)$$

We can write basically, $\frac{d}{dx}\left(kA\left(\frac{dT}{dx}\right)\right)$ of k is the thermal conductivity, A is the normal area through, which heat transfer takes place into $\frac{dT}{dx}=0$. Now, we can say, if 1 and 2 are identical then basically, symbol L is an operator, which means, the $\frac{d}{dx}kA\frac{d}{dx}$. That is operating on T, the exact solution requires to satisfy equation 1 at all x points.

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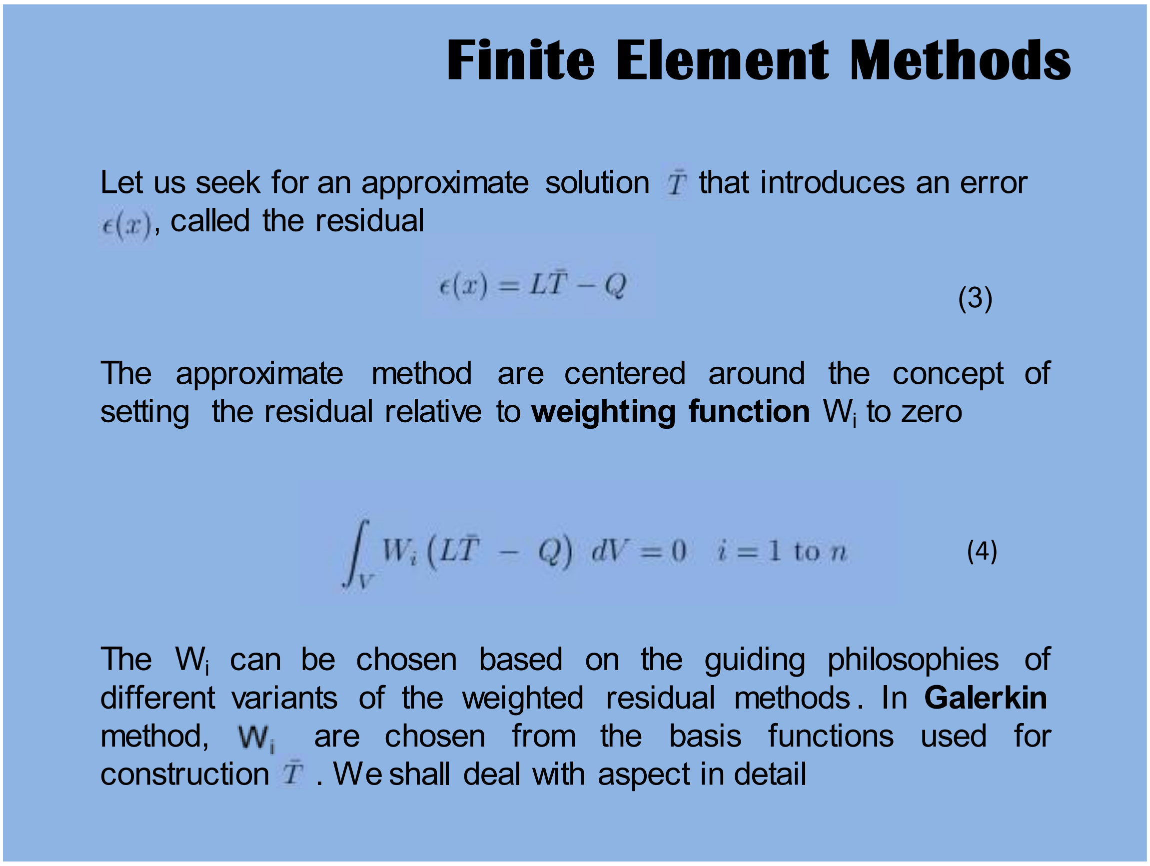

So, let us seek for an approximate solution $\bar{T}$ that introduces an error $\epsilon(x)$, and this ϵ is called the residual. So,

$$\epsilon(x) = L\bar{T} - Q \tag{3}$$

And as we said that, this is $\bar{T}$ is a trial solution, is not exact solution as yet, it is approximate solution. The approximate methods are centered around concept of setting the residual derivative to waiting function W_i to 0.

That means over this domain, if $L\bar{T} - Q$, L is the operator $\frac{d}{dx} kA \frac{d}{dx}$ and that is integrated over a domain say, given by V, so integral dV with the weight function; weight functions weighting functions should be such that, it will ask for this quantity to be 0. W_i can be chosen based on the guiding philosophies of different variants of the weighted residual methods. In Galerkin method, W_i are chosen from the basis function used for construction of $\bar{T}$.

So, we have already been acquainted with the shape function, all basis function. And it is told that for this method W_i, will act in such a way that this integral will produce 0; that means, it is minimize the residual and elevate $\bar{T}$ to T; that means, $\bar{T}$ will evolves as the final solution T.

And this weighting function has to be properly chosen so that you know, what we ask for happens finally. And this a Galerkin method says, that W_i the weighting function and the basis function are same. Just to mention here, Galerkin is a very well-respected name in mathematical science. He is he was a Russian mathematician, who died in 1945, I guess.

And he made enormous contribution, in mathematical science. We know contributions about Lebanese contribution about contribution of Lebanese contributions of Newton. And you know the phenomenal contribution of Gauss even today, all the branches of mathematics use Gauss's discoveries.

So, in the name of similar stars in mathematical science, we can also mention about Boris Galerkin. As I said that, he was a in Soviet Union and contributed inner Mascali and you can see, that today, this inter paradigm of weighted residual method of finite elements stand on Galerkin's work.

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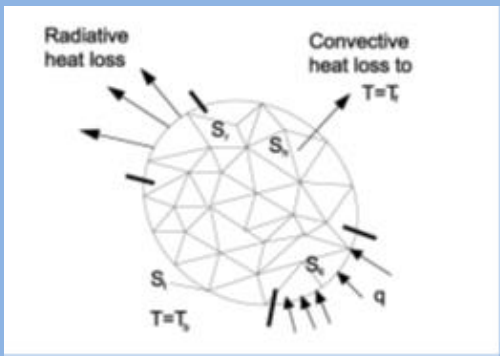
Finite Element Methods

Formulation of a Problem

Consider a steady 2D heat conduction problem in an arbitrary shaped two dimension Domain which is subjected to various types of boundary conditions as shown in the Figure.



Considering a uniform heat generation rate per unit volume (Q) in the entire solution Domain, the governing equation and the boundary conditions can be written as:

Anyway, let us come back to our discussion again. So, consider steady 2D heat conduction problem, in an arbitrary shaped two-dimensional domain, which is subjected to various type of boundary conditions. So, this is a very arbitrary you know, shaped body and this body we are considered we will find out temperature distribution on this body, which generates energy also considering a uniform heat generation rate.

So, it has uniform heat generation rate and the boundaries as you can see, a part of it is boundary you know, different parts are having different condition. A part of it is boundary for example, this part temperature is specified, Dirichlet condition $T = T_b$, may be another part you know, the heat flux is specified S_q part of the boundary.

So, basically $\frac{dT}{dn}$ is specified. Similarly, another part of the boundary, it is basically convective boundary; that means, energy is lost to the local ambience, which is at a lower temperature. And this part of the boundary it I mean, it is subjected to radiative heat transfer, radiative heat loss.

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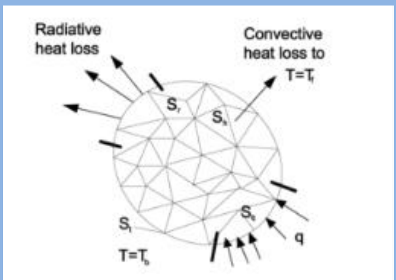
Finite Element Methods

Formulation:

Considering a uniform heat generation rate per unit volume (Q) in the entire solution domain, the equation for heat transfer is

$$k\nabla^2 T + Q = 0 \quad (5)$$

The boundary conditions are:



$$\begin{aligned} T &= T_b \ \text{ on } \ S_t \\ -k\frac{\partial T}{\partial n} &= q \ \text{ on } \ S_q \\ -k\frac{\partial T}{\partial n} &= h\left(T - T_f\right) \ \text{ on } \ S_h \ \text{ and} \\ -k\frac{\partial T}{\partial n} &= \sigma\,\epsilon_s\left(T^4 - T_f^4\right) \ \text{ on } \ S_r \end{aligned} \quad (6)$$

If we you can, you know, we will explain it little more mathematically here. So, governing equation is $k\nabla^2 T + Q = 0$, Q is the volumetric heat generation rate thermal conductivity. Boundary conditions as I mentioned earlier $T = T_b$ one, one part, then minus $-k\frac{dT}{dn} = q$ let me see, plus specified on another segment.

On another segment, basically it is convective condition that is the gradient of temperature equal to h into the boundary temperature minus the local ambient temperature ($h(T - T_f)$). And in another part, basically radiative condition.

$$-k\frac{\partial T}{\partial n} = h\left(T - T_f\right) \ \ on \ \ S_r$$

$$-k\frac{\partial T}{\partial n} = \sigma\epsilon_s\left(T^4 - T_f^4\right) \ \ on \ \ S_r \qquad (6)$$

So, this is a σ is Stefan-Boltzmann constant, this is emissivity, ϵ_s, that means, the temperature at the edge to the power 4 minus the local ambient temperature to the power 4 that is, how the radiative heat transfer is calculated. So, we have tried to prescribe an allpossible forms of boundary conditions, on different segments of the enclosure; different segments of the boundary.

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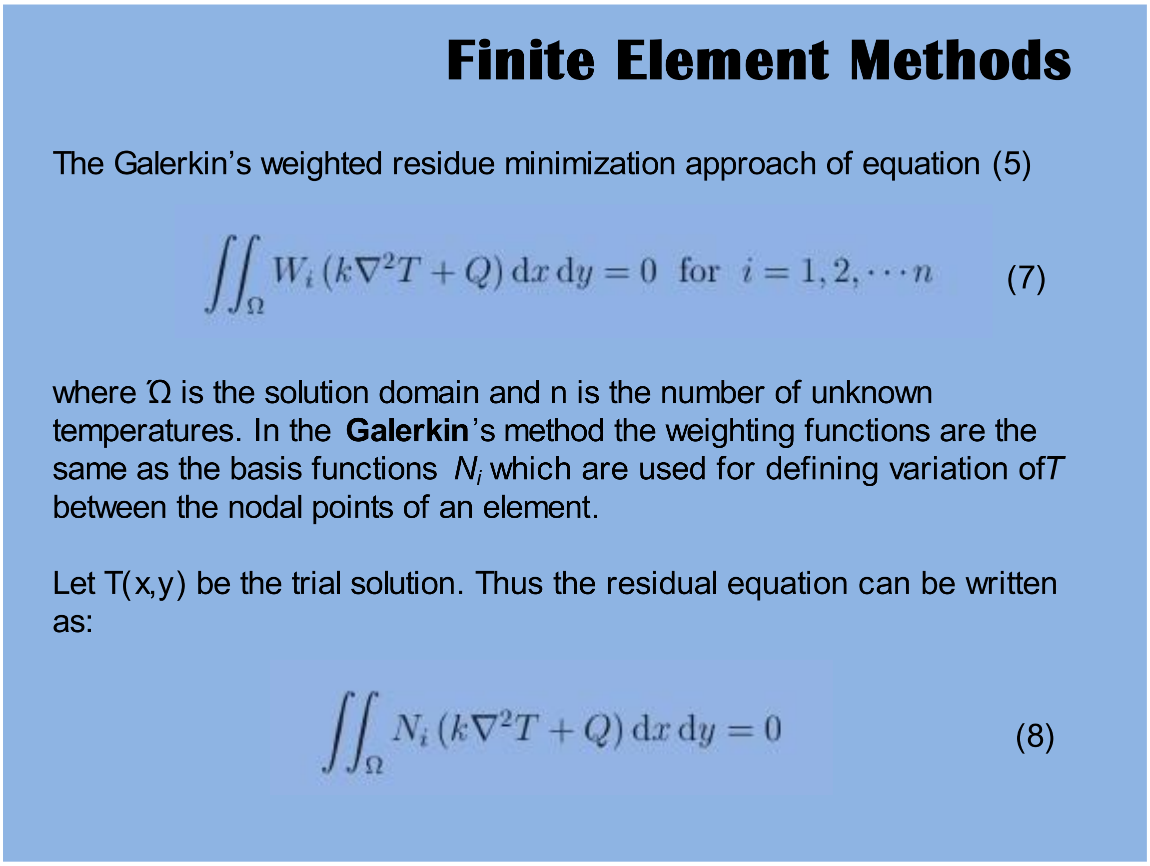

Now, Galerkin's weighted residual minimization approach, as we have already mentioned that, when we bring in weighting function, which is in Galerkin's approach same as basis function or shape function, $(k\nabla^2 T + Q)dxdy = 0$, and this W_i is distributed over the domain, domain $i = n$.

And this integration is done on the domain of you know, specified domains. So, where omega is a solution domain and n is the number of unknown temperatures. In the Galerkin's method the weighting functions are the same as basis function N_i, which are used for defining variation of T between the nodal points of an element.

So, N_i is defined, if we take an element the weight temperature varies at the nodal point of that element, element can be you know in 2D it can be triangle, it can be quadrilateral, in 3D, it can be hexahedral. So, you know, at the nodal point the this is functions are defined in such a way, that it will give the temperature distribution, within the element in a comprehensive manner.

So, let T(x,y) be the trial solution. Thus, the residual equation can be written as;

$$\iint_{\Omega} N_i \left(k\nabla^2 T + Q\right) \mathrm{d}x \, \mathrm{d}y = 0 \qquad (8)$$

area integral over the domain $N_i(k\nabla^2 T + Q)dxdy$. So, simply the difference between 7 and 8 is that, that weighting function we are substituted by the shape function or the basis function.

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Finite Element Methods

In Eqn. (8) the requirement on N_i is that it must be at least twice differentiable so that a non-trivial $\nabla^2 T$ value can be obtained at all locations.

However, this requirement can be weakened by integrating the ∇^2 operator by parts. In addition to the convenience of using lower order interpolation functions, weak formulation also introduces the boundary conditions of the problem in a suitable fashion

Equation 8 is a requirement on N_i is that it must be at least twice differentiable so that non-trivial $\nabla^2 T$ values can be obtained at all locations. I will go back to the earlier slide again. So, N_i should be such, that it is twice differentiable that is a mathematical property.

However, this requirement can be weakened, by integrating this del square or grad square operator by parts. In addition to the convenience of using lower order interpolation functions, so if we do that, we will need your order interpolation functions, also this is called weak formulation, because this helps us in implementing the boundary conditions naturally.

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Finite Element Methods

The integration by parts can be carried out together with the divergence theorem as follows:

$$\iint N_i\,(k\nabla^2 T)\,\mathrm{d}x\,\mathrm{d}y = \iint N_i \nabla\cdot(k\nabla T)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \iint \nabla\cdot(N_i k \nabla T)\,\mathrm{d}x\,\mathrm{d}y - \iint (\nabla N_i \cdot k\nabla T)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \oint_\Gamma N_i\, k \frac{\partial T}{\partial n}\,\mathrm{d}l - \iint_\Omega (k\,\nabla N_i \cdot \nabla T)\,\mathrm{d}x\,\mathrm{d}y \quad (9)$$

where is the boundary of the domain Ω. Thus the final form of residue equation:

$$\iint_\Omega k\left(\frac{\partial N_i}{\partial x}\cdot\frac{\partial T}{\partial x} + \frac{\partial N_i}{\partial y}\cdot\frac{\partial T}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y - \oint_\Gamma N_i\, k\frac{\partial T}{\partial n}\,\mathrm{d}l$$
$$- \iint_\Omega N_i\, Q\,\mathrm{d}x\,\mathrm{d}y = 0 \quad (10)$$

The form of the equation become such that, one term of the equation exactly becomes the boundary condition. So, we will now, integrate $N_i(k\nabla^2 T + Q)dxdy$ over the domain, Ω and integration by parts as to be carried out together with divergence theorem.

So, we can write this as $N_i(k\nabla^2 T + Q)dxdy$, and then integration by parts we can write; divergence of $N_i k \nabla T$ minus again area integral of $\nabla N_i . k\nabla T$ and you can see that naturally, this is converted into line integral. This $\nabla . N_i k \nabla T$ will be simply $N_i\, k\frac{\partial T}{\partial n} dl$ line integral over the boundary (refer equation 9).

And $\partial T/\partial n$ is the temperature gradient in the normal direction, on the confining boundary and these remain as a surface integral k gradient N_i dot gradient T $dx\, dy$. Now, this k gradient N_i dot gradient T can be further written as, k $\frac{\partial N_i}{\partial x}.\frac{\partial T}{\partial x}+\frac{\partial N_i}{\partial y}.\frac{\partial T}{\partial y}$ we get from here, $dx\, dy$ minus transfer it to the left hand side, then minus $N_i k \frac{\partial T}{\partial n}\, dl$ this is basically, line integral.

And in our original proposition; I will go back again by two slides, we had a Q term that is heat generation term and that Q term now, will come here minus surface integral again N_i Q dx dy so, minus double integral N_i Q dx dy. So, that makes now, equation 10, which is the final form of the governing equation.

$$\iint_\Omega k\left(\frac{\partial N_i}{\partial x}\cdot\frac{\partial T}{\partial x}+\frac{\partial N_i}{\partial y}\cdot\frac{\partial T}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y-\oint_\Gamma N_i\,k\frac{\partial T}{\partial n}\,\mathrm{d}l$$
$$-\iint_\Omega N_i\,Q\,\mathrm{d}x\,\mathrm{d}y=0 \qquad (10)$$

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Finite Element Methods

Substitutions for the boundary integral term in terms of given boundary conditions:

$$\underbrace{\iint_\Omega k\left(\frac{\partial N_i}{\partial x}\cdot\frac{\partial T}{\partial x}+\frac{\partial N_i}{\partial y}\cdot\frac{\partial T}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y}_{(i)}+\underbrace{\int_{S_q}N_i\,q\,\mathrm{d}l}_{(ii)}+\underbrace{\int_{S_h}N_i\,h(T-T_f)\,\mathrm{d}l}_{(iii)}$$
$$+\underbrace{\int_{S_r}N_i\,\sigma\,\epsilon_s(T^4-T_f^4)\,\mathrm{d}l}_{(iv)}-\underbrace{\iint_\Omega N_i\,Q\,\mathrm{d}x\,\mathrm{d}y}_{(v)}=0 \qquad (11)$$

Where the terms have been numbered as (i) to (v) for the future reference

So, substitution for the boundary integral terms in terms of given boundary conditions; so, in this equation now, we have coming by again the earlier slide.

So, this integral plus this is line integral and line integral mean (equation 10); this is falling on confining boundary. Confining boundary means, earlier we have defined that it can be convective boundary, it can be specified heat plus boundary, it can be a boundary through each radiative heat transfer is taking place.

So, in addition to the boundary y temperature is known, that is Dirichlet condition, but that is not a problem, whenever temperature and at the boundary has to be defined will directly substitute that, by the specified boundary; a basic specified temperature at the boundary. But, otherwise, we have to define all those you know, $\partial T/\partial n$ related conditions, which we are calling natural boundary conditions.

In this problem those are either constant specified heat flux condition; that means, $N_i\ q$; q is $-\frac{k\partial T}{\partial n}$ so, wherever that is known or convective heat transfer condition. We have seen $N_i h(T-T_f)dl$ or radiative heat transfer condition N_i σ Stefan-Boltzmann constant epsilon emissivity, this epsilon s is emissivity. This is not the error that we defined; while defining our formulation this is different emissive surface emissivity; $(T^4-T_f^4)dl$ again this will be on the segment, where it is applicable plus the heat generation rate equal to 0. And we have identified each term, by roman I, II, III, IV, V, so, where the terms have been numbered as roman I to V for our future reference.

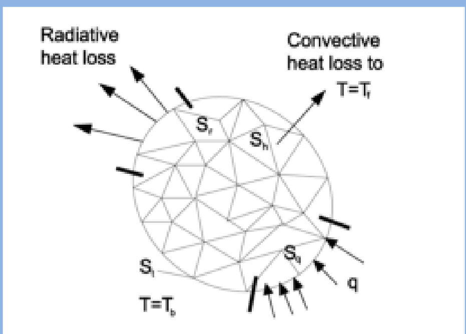
$$\underbrace{\iint_\Omega k\left(\frac{\partial N_i}{\partial x}\cdot\frac{\partial T}{\partial x}+\frac{\partial N_i}{\partial y}\cdot\frac{\partial T}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y}_{(i)}+\underbrace{\int_{S_q} N_i\,q\,\mathrm{d}l}_{(ii)}+\underbrace{\int_{S_h} N_i\,h(T-T_f)\,\mathrm{d}l}_{(iii)}$$

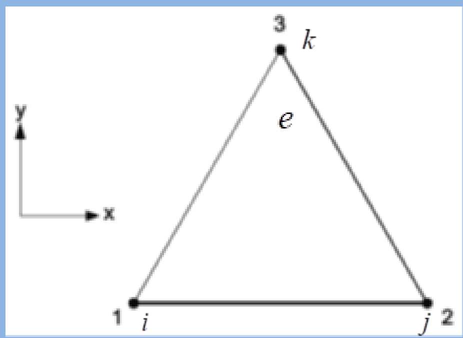
$$+\underbrace{\int_{S_r} N_i\,\sigma\,\epsilon_s(T^4-T_f^4)\,\mathrm{d}l}_{(iv)}-\underbrace{\iint_\Omega N_i\,Q\,\mathrm{d}x\,\mathrm{d}y}_{(v)}=0 \tag{11}$$

(Refer Slide Time: 37:19)

Finite Element Methods

After formulating the residue equation, including the boundary contributions, the next task is to evaluate the area and the line integrals of Eqn. (11). For this purpose, the domain Is divided into small elements, each of which has a certain number of nodes placed on the Boundary and inside for interpolating the field variables. We can use a 3-noded triangular Element (Figure) for our problem of interest.





Now, as we said that, we have defined these boundary conditions and this is the domain of interest and in the domain of interest, we are discretizing the domain by the triangular elements. So, the domain is divided into small elements, each of which has a certain number of nodes and some nodes will be falling on the boundary, some nodes will be within the you know domain.

And we will use basically, a 3-noded triangular element. So, this element we have defined, and now you recall the line element that, we studied in our background material, there we identified the nodal points as i, j, k. so, here also we are trying to identify our nodal points as i, j and k these are vertices of the triangle.

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Finite Element Methods

Using a 3- noded triangular element for 2D application, the interpolation of the temperature can be obtained as follows:

$$T = ax + by + c \quad (12)$$

Where the constants a, b, c depend on the nodal coordinates and the nodal temperature value. The values of a, b, c for a typical element is given by:

$$T_i = a\,x_i + b\,y_i + c \quad (13)$$

$$T_j = a\,x_j + b\,y_j + c \quad (14)$$

$$T_k = a\,x_k + b\,y_k + c \quad (15)$$

And the 3-noded triangular element for 2D application, 3D application as I said; obviously, elements will be different will be hexahedral or prisms.

But here it is basically, a triangular element and we can heat in this variation, the $T = ax + by + c$, but constant a, b, c depends on nodal coordinates and the nodal temperature values. The values of a, b, c for a typical element will be given by $T_i = ax_i + by_i + c$, at j; $T_j = ax_j + by_j + c$ and $T_k = ax_k + by_k + c$ equations 13, 14 and 15.

$$T_i = a\,x_i + b\,y_i + c \quad (13)$$

$$T_j = a\,x_j + b\,y_j + c \quad (14)$$

$$T_k = a\,x_k + b\,y_k + c \quad (15)$$

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Finite Element Methods

Using matrix notation, we obtain:

$$\begin{bmatrix} T_i \\ T_j \\ T_k \end{bmatrix} = \begin{bmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (16)$$

Since the equation (12) can be written as:

$$T = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

We get within the element (e):

$$T = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{bmatrix}^{-1} \begin{bmatrix} T_i \\ T_j \\ T_k \end{bmatrix} = \begin{bmatrix} N_i & N_j & N_k \end{bmatrix} \begin{bmatrix} T_i \\ T_j \\ T_k \end{bmatrix} \quad (17)$$

So, now in a comprehensive manner we can write, T_i, T_j, T_k in a column matrix form so, this will give T_i, T_j, T_k and then from here, basically the we if we tried to take out a, b, c as again a column matrix, we will able to get one matrix with x_i y_i 1, x_j y_j 1, x_k y_k 1 this is basically, matrix with 3 rows and 3 columns 3 by 3 matrices into a, b, c. So, we will get back equation 13, 14, 15 from the summarized form as 16.

Since, our major target was to write equation, represent equation 12. So, equation 12 is then a$x + by + c$.

And if you know, T_i, T_j, T_k is given by expression 16 then we can write equation 12 is; x y 1 again I am going back to the earlier slide, ax plus by plus c can be written as x y 1 as a row and a, b, c as a column matrix, which will give me ax plus by plus c. So, T can be written as this. Now, in this; we can now, re-write for the element T, that T is given by x y 1 multiplied by; we have to write a b c column matrix.

What is the value of a, b, c column matrix from 16? We can get it is basically, inverse of these matrix into T_i, T_j, T_k. So, x_i y_i 1, x_j y_j 1, x_k y_k 1 exactly, this matrix inverse into T_i, T_j, T_k. We have just found out what is $a\ b\ c$? So, x y 1 multiplied by inverse of this matrix into T_i, T_j, T_k. Now, we can call these which are multipliers of T_i, T_j, T_k are N_i, N_j, N_k.

So, necessarily this part which is coefficient here, as the column matrix of T_i, T_j, T_k will give us the shape functions or the basic functions for the element, that are given by N_i, N_j, N_k. So, we will stop here today, and we will find out in the next class, N_i, N_j, N_k individually, and will see, how do they are the shape functions, basis functions help us finally, in solving this problem. So, thank you very much, we will meet in the next class.

Thank you.