Computational Fluid Dynamics and Heat Transfer
Prof. Gautam Biswas
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 11
Finite Volume Method-3

Good afternoon, everyone, today we will cover the last part of general methodology related to Finite Volume Method.

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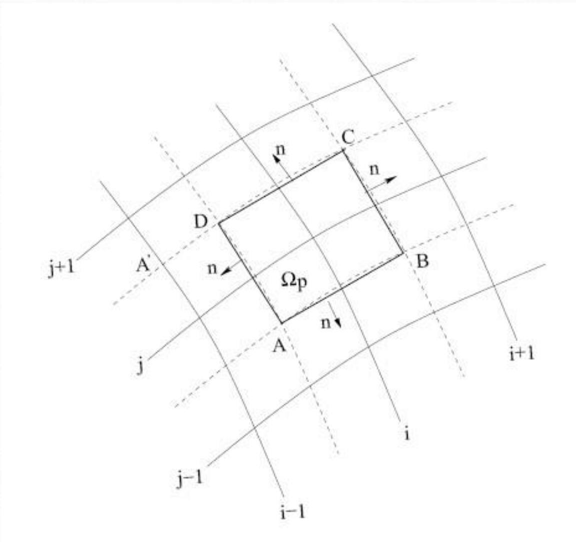
Equations with Second derivatives

The finite volume method requires some modification, when the second derivatives are present in the governing equation. The integration of those terms are to be performed on the finite volume and are to be evaluated over the segments AB, BC,CD and DA:



We will call the domain ABCD as the main integration domain

And specifically, we are going to cover the issues that arise out of presence of second order derivatives in the governing equation. Again, you can see the domain; we have zeta equal to constant and eta equal to constant lines, and we have defined ABCD control volume, which has a center at i, j, and the segments are AB, BC, CD, and DA.

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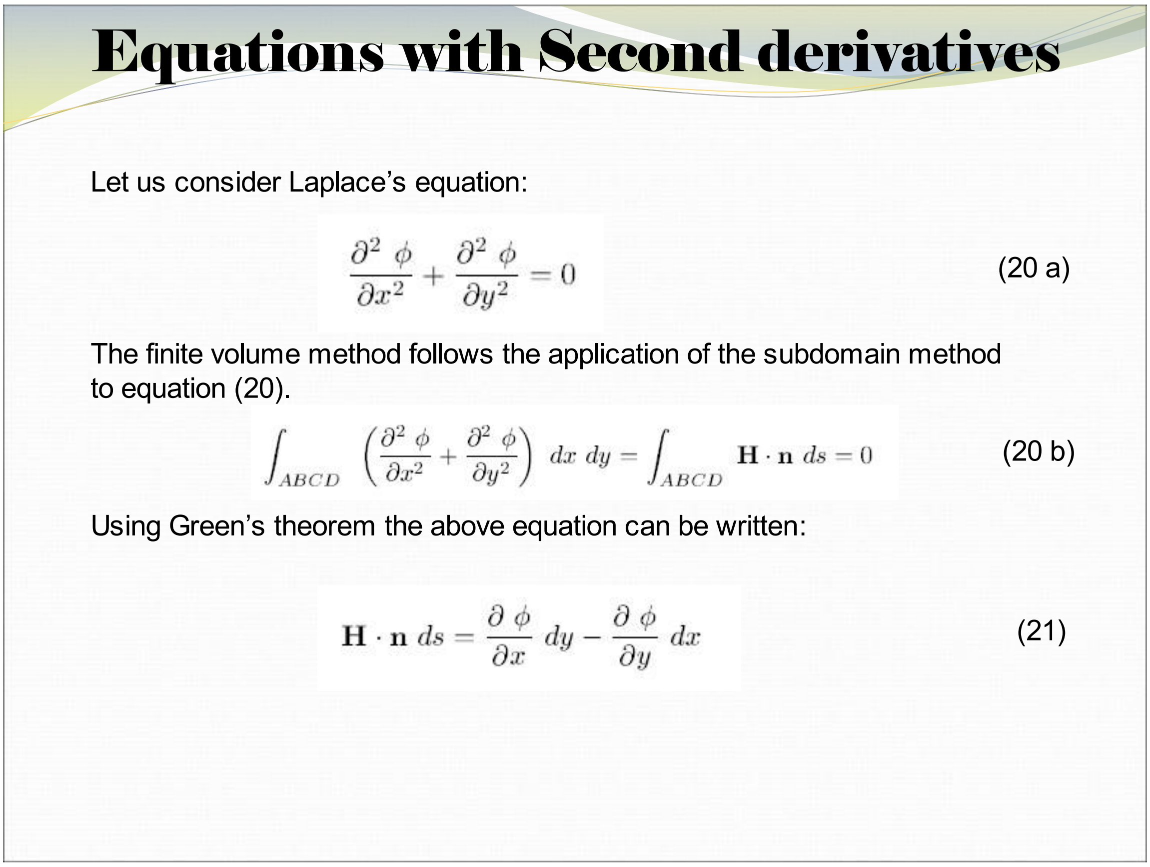

We will consider Laplace’s equation, $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$, and if this equation is integrated over ABCD, then we can write applying Green's theorem, that this is integral ABCD, **H.n** ds equal to 0, where **H.n** ds is $\frac{\partial \phi}{\partial x} dy - \frac{\partial \phi}{\partial y} dx$. This is following Green's theorem and so, we will integrate basically equation 20 b.

$$\int_{ABCD} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) dx\, dy = \int_{ABCD} \mathbf{H} \cdot \mathbf{n}\, ds = 0 \qquad (20b)$$

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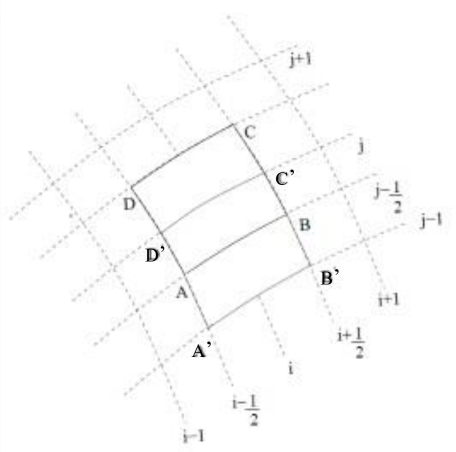
Equations with Second derivatives

The line integral in Eqn. (20 b) can be evaluated approximately over the segments AB, BC,CD and DA by:



$$\left[\frac{\partial\phi}{\partial x}\right]_{i,\ j-1/2}\Delta y_{AB}-\left[\frac{\partial\phi}{\partial y}\right]_{i,\ j-1/2}\Delta x_{AB}$$
$$+\left[\frac{\partial\phi}{\partial x}\right]_{i+1/2,\ j}\Delta y_{BC}-\left[\frac{\partial\phi}{\partial y}\right]_{i+1/2,\ j}\Delta x_{BC}$$
$$+\left[\frac{\partial\phi}{\partial x}\right]_{i,\ j+1/2}\Delta y_{CD}-\left[\frac{\partial\phi}{\partial y}\right]_{i,\ j+1/2}\Delta x_{CD}$$
$$+\left[\frac{\partial\phi}{\partial x}\right]_{i-1/2,\ j}\Delta y_{DA}-\left[\frac{\partial\phi}{\partial y}\right]_{i-1/2,\ j}\Delta x_{DA}=0$$

(22)

And then this integration can be written as we have seen that we if this integral means we have to integrate along the segments AB, BC, CD and DA. So, exactly we have done that.

The quantities are $\frac{\partial\,\phi}{\partial x}dy-\frac{\partial\,\phi}{\partial y}dx$. So, $\left[\frac{\partial\phi}{\partial x}\right]_{i,j-\frac{1}{2}}$. So, this is basically we are integrating over ABCD. $\left[\frac{\partial\phi}{\partial x}\right]_{i,j-\frac{1}{2}}$ has been defined at this point, which is basically between A and B and coordinate of this point is i,j-1/2.

So, $\left[\frac{\partial\phi}{\partial x}\right]_{i,j-\frac{1}{2}}$ into Δy_{AB}; that means, A and B point their vertical distance Δy_{AB} - $\left[\frac{\partial\phi}{\partial y}\right]_{i,j-\frac{1}{2}}$ Δx_{AB}, Δx_{AB} means horizontal distance between A and B point. So, this is the first line, that means, integration of $\left[\frac{\partial\phi}{\partial x}\right]_{i,j-\frac{1}{2}}$ Δy_{AB} - $\left[\frac{\partial\phi}{\partial y}\right]_{i,j-\frac{1}{2}}$ Δx_{AB}.

Next, we have to go along BC segment, that means, we have to define the quantities $\left[\frac{\partial\phi}{\partial x}\right]$ and $\left[\frac{\partial\phi}{\partial y}\right]$here. So, $\left[\frac{\partial\phi}{\partial x}\right]_{i+\frac{1}{2},j}$at i+1/2, j and so, $\left[\frac{\partial\phi}{\partial x}\right]_{i+\frac{1}{2},j}$ into again Δy and this Δy is the vertical distance between point B and C. And then again, $\left[\frac{\partial\phi}{\partial y}\right]$ at this point, which is i+1/2,j into the horizontal distance between point C and point B the horizontal component.

So, basically, we are multiplying $\left[\frac{\partial\phi}{\partial y}\right]$ at this point (i+1/2,j) into Δx_{BC}. So, this part is the second line. Similarly, third line will be $\left[\frac{\partial\phi}{\partial x}\right]$ has to be evaluated at this is at this point, which is basically i,j+1/2, $\left[\frac{\partial\phi}{\partial y}\right]_{i,j+1/2}$, and they will be multiplied by again Δy_{CD} and Δx_{CD}.

Then at this point, that means, $\left[\frac{\partial\phi}{\partial x}\right]_{i-1/2,j}$ and $\left[\frac{\partial\phi}{\partial y}\right]_{i-1/2,j}$ and $\left[\frac{\partial\phi}{\partial x}\right]$ will be multiplied by Δy_{DA}, Δy is basically the vertical distance between point A and D. And so, Δy_{DA} and then Δx_{DA} the horizontal distance between point D and point A. Δx_{DA}. So, $\left[\frac{\partial\phi}{\partial y}\right]$ at i-1/2, j into Δx_{DA}.

So, this is the basically we if we perform the integral along the segments AB, BC, CD and DA. If we can evaluate this, we are through our integration is over and basically, we have evaluated this integral and that is what is our target. So, we will see how we can do it step by step.

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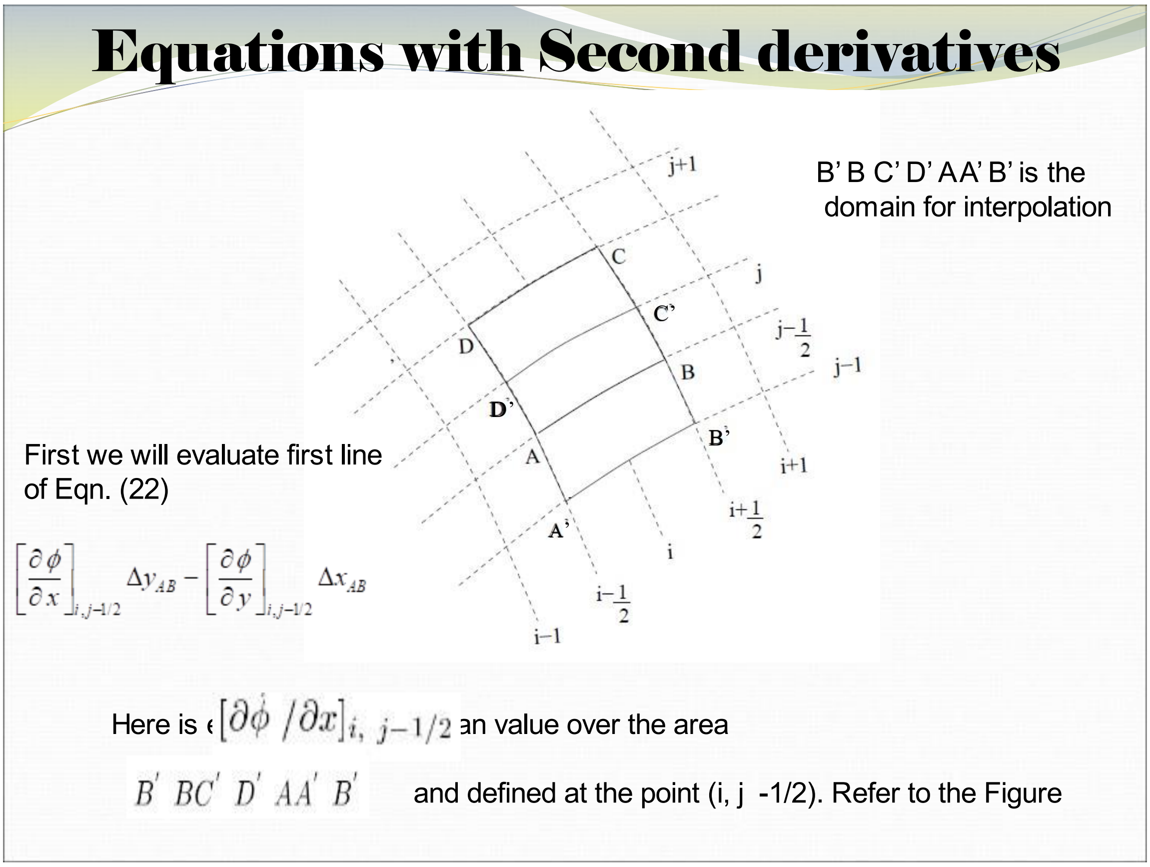

So, in order to first we evaluate the first line of equation 22, which is given by $\left[\frac{\partial\phi}{\partial x}\right]_{i,j-1/2}\Delta y_{AB} - \left[\frac{\partial\phi}{\partial y}\right]_{i,j-1/2}\Delta x_{AB}$, So, at this point we have to find out the gradients $\left[\frac{\partial\phi}{\partial x}\right]$ and $\left[\frac{\partial\phi}{\partial x}\right]$ we have to find out here. We will multiply it with a verticals distance between point A and B and then we will find out $\left[\frac{\partial\phi}{\partial y}\right]$ here and multiply with the horizontal distance between A and B.

Now, in order to do that, we have to find this quantity $\left[\frac{\partial\phi}{\partial x}\right]$ at i, j-1/2 and $\left[\frac{\partial\phi}{\partial y}\right]$ at i, j-1/2. Now, how we can do that is the question. So, we can do that by considering the mean value of for example, if we calculate $\left[\frac{\partial\phi}{\partial x}\right]$ at this point by calculating mean value of $\left[\frac{\partial\phi}{\partial x}\right]$ over this domain.

This domain is not ABCD it is rather if we take this point as the midpoint then the cell which is being defined is A', B', C', D'. This is a pseudo definition rather we will not this is not being used to evaluate the Laplace's equation over the domain of interest.

Domain of interest is still ABCD, but since we have to find out $\frac{\partial \phi}{\partial x}$ at this point or ∂ϕ/∂y at this point what we will do? We calculate mean value of $\frac{\partial \phi}{\partial x}$ over this domain. If had this been the center of a cell then this would have been A’, B’, C’, D’ would have been the element surrounding that cell and then we have to find out mean value of $\frac{\partial \phi}{\partial x}$ on this area, A’, B’, C’, D’ and that will be eventually defined at i, j -1/2.

So, that is what we have written that this quantity is evaluated at the mean value over this area, this area we have defined as B’,B,C’,D’,A,A’,B ’. And if we calculate the mean value of $\frac{\partial \phi}{\partial x}$ on this domain it will be eventually as I said it will be defined at the point i, j-1/2.

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Equations with Second derivatives

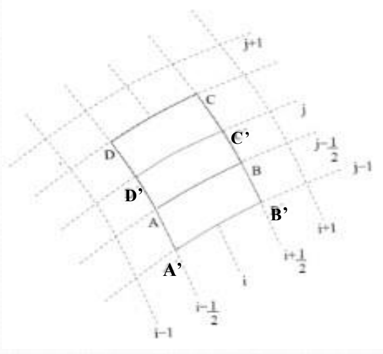
Thus we can write:

$$\left[\frac{\partial \phi}{\partial x}\right]_{i,j-1/2} = \left(\frac{1}{S_{A'B'C'D'}}\right) \int\int \left(\frac{\partial \phi}{\partial x}\right) dx\, dy = \left(\frac{1}{S_{A'B'C'D'}}\right) \oint \phi dy \quad (23)$$

$$\left[\frac{\partial \phi}{\partial y}\right]_{i,j-1/2} = \left(\frac{1}{S_{A'B'C'D'}}\right) \int\int \left(\frac{\partial \phi}{\partial y}\right) dx\, dy = -\left(\frac{1}{S_{A'B'C'D'}}\right) \oint \phi dx \quad (24)$$

Using Green’s theorem:

$$\oint_{A'B'C'D'} \phi\, dy \approx \phi_{i,j-1} \Delta y_{A'B'} + \phi_B\, \Delta y_{B'C'} + \phi_{i,j} \Delta y_{C'D'} + \phi_A \Delta y_{D'A'}$$



So, we will do that integration. So, we have written $\frac{\partial \phi}{\partial x}$ at i, j -1/2 this is equal to as we said it is a mean quantity of $\frac{\partial \phi}{\partial x}$ on this domain A’, B’, C’, D’ which means basically area integral of $\frac{\partial \phi}{\partial x} dxdy$ divided by the mean area, mean area of A’B’C’D’.

And then again. if we apply Green's theorem on this integral, we can write 1 by again area into line integral of ϕdy, which is equation 23. And $\partial\phi/\partial y$ at the same point i, j-1/2 it is calculated as double integral of $\partial\phi/\partial y$ dxdy. So, area integral of $\partial\phi/\partial y$ on the area A'B'C'D', here also we can see A', B', C', D' divided by the area.

$$\left[\frac{\partial \phi}{\partial x}\right]_{i,j-1/2} = \left(\frac{1}{S_{A'B'C'D'}}\right) \int\int \left(\frac{\partial \phi}{\partial x}\right) dx\, dy = \left(\frac{1}{S_{A'B'C'D'}}\right) \oint \phi dy \qquad (23)$$

$$\left[\frac{\partial \phi}{\partial y}\right]_{i,j-1/2} = \left(\frac{1}{S_{A'B'C'D'}}\right) \int\int \left(\frac{\partial \phi}{\partial y}\right) dx\, dy = -\left(\frac{1}{S_{A'B'C'D'}}\right) \oint \phi dx \qquad (24)$$

So, this is negative quantity because the first line of the integral we have seen $\partial\phi/\partial x\ \Delta y$ - $\partial\phi/\partial y\ \Delta x$. So, this quantity again line integral of ϕ dx divided by area. So, we have first evaluated line integral of ϕ dy line integral of ϕ dy over A', B', C', D'. And now we can clearly see that ϕ dy we have to integrate.

So, ϕ at here it is this point is basically i, j-1 point $\phi_{i,j-1}\ \Delta y_{A'B'}$. That means, vertical distance between A' and B' then ϕ_B into again $\Delta y_{B'C'}$ that is vertical distance between B 'and C'. Then ϕ at this point, which is basically $\phi_{i,j}$ we can see $\phi_{i,j}$ into again $\Delta y_{C'D'}$ then ϕ_A into $\Delta y_{D'A'}$.

So, that completes the integration and that means, equation 23 this line integral is evaluated if we divide it by the area, we are through we get $\partial\phi/\partial x$ at i, j-1/2 this point, that was what of interest. Then we will integrate again on ABCD if we find values of $\partial\phi/\partial x$, $\partial\phi/\partial y$ at these points between A and B, between B and C and so forth. Now, equation 24 is integral ϕ dx over again this area A', B', C', D'.

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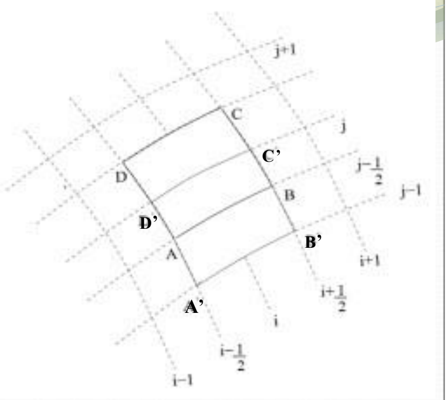
Equations with Second derivatives

$$\oint_{A'B'C'D'} \phi\, dx \approx \phi_{i,j-1}\,\Delta x_{A'B'} + \phi_B\,\Delta x_{B'C'} + \phi_{i,j}\,\Delta x_{C'D'} + \phi_A\,\Delta x_{D'A'}$$



If the mesh is not too distorted,

$$\Delta y_{A'B'} \approx -\Delta y_{C'D'} \approx \Delta y_{AB} \text{ and } \Delta y_{B'C'} \approx -\Delta y_{D'A'} \approx \Delta y_{j-1,j}$$

$$\Delta x_{A'B'} \approx -\Delta x_{C'D'} \approx \Delta x_{AB} \text{ and } \Delta x_{B'C'} \approx -\Delta x_{D'A'} \approx \Delta x_{j-1,j}$$

$$S_{AB} \equiv S_{A'B'C'D'} = \left|\overline{A'B'} \times \overline{B'C'}\right|$$

$$S_{AB} = \left|\left(\Delta x_{A'B'}\hat{i} + \Delta y_{A'B'}\hat{j}\right)\times\left(\Delta x_{B'C'}\hat{i} + \Delta y_{B'C'}\hat{j}\right)\right|$$

$$= \Delta x_{A'B'}\,\Delta y_{B'C'} - \Delta y_{A'B'}\,\Delta x_{B'C'}$$

$$S_{AB} = \Delta x_{AB}\,\Delta y_{j-1,j} - \Delta y_{AB}\,\Delta x_{j-1,j}$$

$$S_{AB} = \Delta x_{AB}\ \Delta y_{j-1,\,j} - \Delta y_{AB}\ \Delta x_{j-1,\,j} \qquad (25)$$

So, that is we can clearly see that ϕ here at i, j-1 into Δx that is horizontal distance between A' and B', $\Delta x_{A'B'}$ ϕ_B $\Delta x_{B'C'}$ $+\phi_{i,j}\Delta x_{C'D'}$ $+\phi_A\Delta x_{D'A'}$.

So, then again, this equation 24, integrals ϕ dx we have found out, equation 23 integral ϕ dy we have found out. And we have written here this has to be divided by the area. Now, we have to find out the how to define this area. Now, basically this area we can again define as S_{AB}, S_{AB} means $S_{A'B'C'D'}$ this area. This area will call as S_{AB}. And this is basically the area of a quadrilateral and that is |A' B' × B'C'|. So, this and mod value of that.

So, we have one after another found out this magnitude and then we evaluated this you can do it little meticulously, I am giving you one example how the quantities have been defined like $\Delta y_{A'B'}$ is the vertical distance between the point A ', B '.

This is equal to - $\Delta y_{C'D'}$ this is vertical distance between D'and C' and this is equal to Δy_{AB} again this is vertical distance between point A and B.

So, these are all same. If the grid is not distorted these are likely to be very close to each other. And $\Delta y_{B'C'}$ that is Δy vertical distance between B ' and C ' point that is $\Delta y_{j-1,j}$ very clear j-1, j.

So, this way $\Delta y_{A'B'}$ then $\Delta x_{A'B'}$ these have been the such quantities have been calculated. So, that finally, A ', B 'is what? $\Delta x_{A'B'}\hat{\imath} + \Delta y_{A'B'}\hat{\jmath}$ and what is B′C′? B′C′is $\Delta x_{B'C'}\hat{\imath} + \Delta y_{B'C'}\hat{\jmath}$.

So, this is multiplication of these two quantities and we take a mod value of that. Now, if we you know component wise, we have found out each component. If we substitute them final expression will be

$$S_{AB} = \Delta x_{AB}\, \Delta y_{j-1,j} - \Delta y_{AB}\, \Delta x_{j-1,j} \tag{25}$$

The area of this quadrilateral A′, B′, C′, D′. So, if we refer to equation 23, we will be able to see we have seen how ϕ dy has to be integrated, how ϕ dx has to be integrated and what is a final expression of the area A′, B′, C′, D′.

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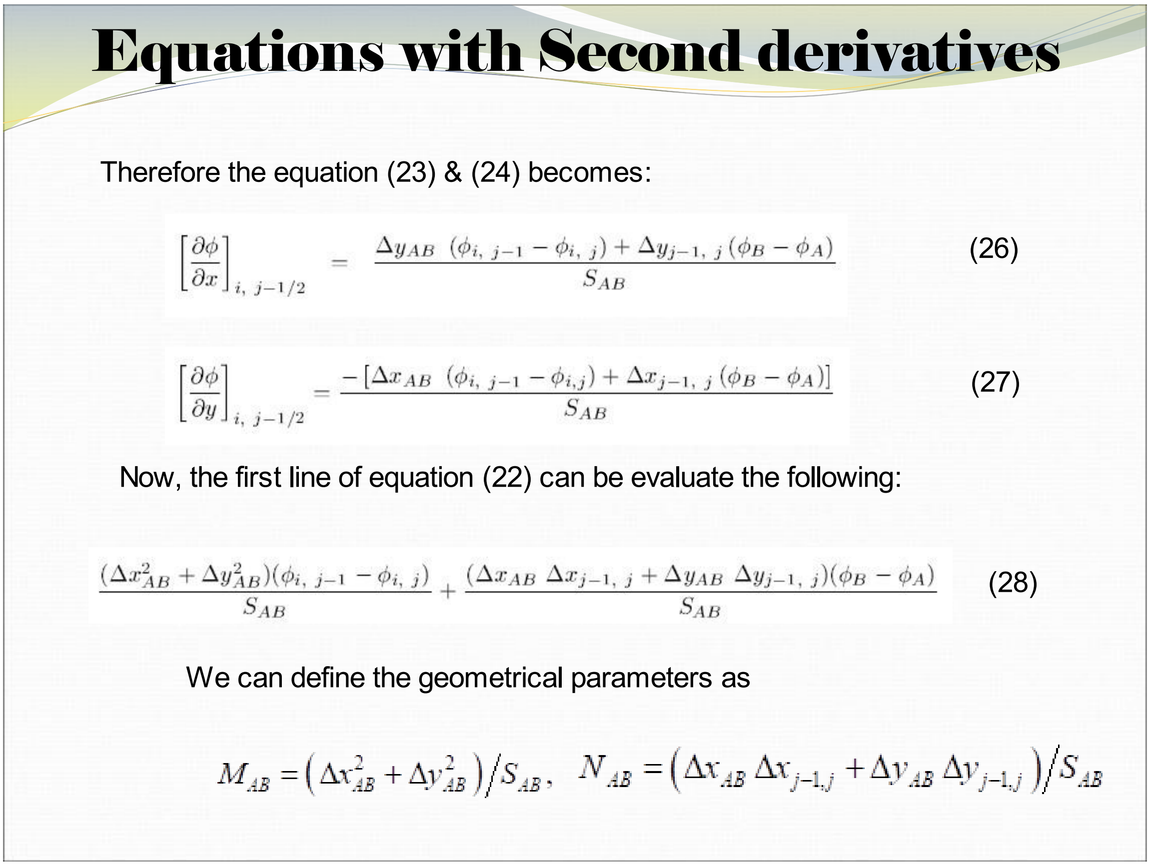

Having found out that we will be able to write from equation 23 and 24 that is what we mentioned was our target ∂ϕ/∂x at i, j-1/2. We can now whatever we have done we this is ϕ dy expression and then you have to divide it by area and if we have found out everything. So, we can get this expression, which is

$$\left[\frac{\partial \phi}{\partial x}\right]_{i,\ j-1/2} = \frac{\Delta y_{AB}\ (\phi_{i,\ j-1} - \phi_{i,\ j}) + \Delta y_{j-1,\ j}\,(\phi_B - \phi_A)}{S_{AB}} \tag{26}$$

Then, ∂ϕ/∂y at i, j-1/2 same point

$$\left[\frac{\partial \phi}{\partial y}\right]_{i,\ j-1/2} = \frac{-\left[\Delta x_{AB}\ (\phi_{i,\ j-1} - \phi_{i,j}) + \Delta x_{j-1,\ j}\,(\phi_B - \phi_A)\right]}{S_{AB}} \tag{27}$$

We found out as I said earlier that ϕ dy, this is ϕ dy this is ϕ d x and this is the area. So, we are dividing integral ϕ dy by area integral ϕ dx by area and we are getting 26 and 27.

Now, if you recall what was first line of equation 22, which was to be evaluated we have evaluated this quantity, we have evaluated this quantity. Now, this quantity ∂ϕ/∂x has to be multiplied by Δy_{AB} that is vertical distance between point A and B and ∂ϕ/∂y we have found out here that has to be multiplied by horizontal distance between A and B, Δx_{AB}.

So, these two quantities we have, so long we have spent our time in finding out these two terms. Now, we have to multiply these two terms with Δy and Δx. So, we will come to equation 26 and 27. So, 26 now, for example, if we find if we multiply by Δy_{AB} and 27, if we multiply by Δx_{AB} we will be able to evaluate the first line and by doing. So, we can see we are getting equation 28, this is basically first line of equation 22.

Now, if we look at these quantities we will see that here all these quantities Δx^2_{AB}, Δy^2_{AB} area AB, Δx_{AB} $\Delta x_{j-1,j}$ Δy_{AB}, $\Delta y_{j-1,j}$ and again area. These are all comprising of all these are comprising of the geometrical parameters. (Indicating equation 28)

$$\frac{(\Delta x^2_{AB} + \Delta y^2_{AB})(\phi_{i,\ j-1} - \phi_{i,\ j})}{S_{AB}} + \frac{(\Delta x_{AB}\ \Delta x_{j-1,\ j} + \Delta y_{AB}\ \Delta y_{j-1,\ j})(\phi_B - \phi_A)}{S_{AB}} \tag{28}$$

And here the field variable that is the dependent variable is here $\phi_{i,j-1}$ - $\phi_{i,j}$ and ϕ_B - ϕ_A. So, we can say that the first term is basically M_{AB} into $\phi_{i,j-1}+\phi_{i,j}$ where M_{AB} is given by this geometrical parameter, I mean it M_{AB} is a term comprising of geometrical parameters. (Refer equation 28)

Similarly, second term ϕ_B - ϕ_A that is being multiplied by this quantity, which is again geometrical parameter and these two geometrical parameters we are giving a new nomenclature M_{AB} and N_{AB}. So, M_{AB} into $\phi_{i,j-1}$ - $\phi_{i,j}$; N_{AB} into ϕ_B - ϕ_A these two I mean additive quantities of this two is equation 28. So, we have evaluated equation 28 is basically sorry, equation 28 is basically evaluation of first line of equation 28.

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Equations with Second derivatives

In a similar manner, one can evaluate the second line of Eqn. (22)

$$+\left[\frac{\partial \phi}{\partial x}\right]_{i+1/2,j} \Delta y_{BC} - \left[\frac{\partial \phi}{\partial y}\right]_{i+1/2,j} \Delta x_{BC}$$

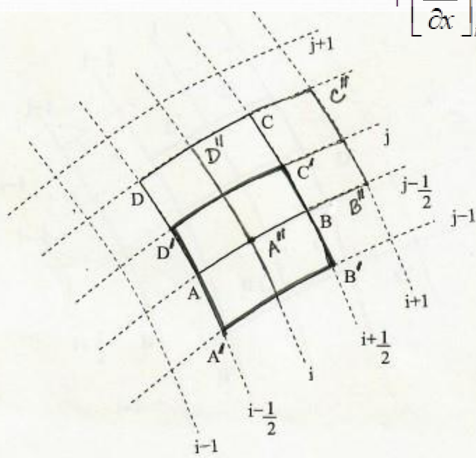

Figure 3.6: Finite Volume for a Curvilinear Grid.

B" C"C D"A" BB" is the domain for interpolation

$$\left[\frac{\partial \phi}{\partial x}\right]_{i+1/2,j} = \frac{1}{S_{A"B"C"D"}} \iint \frac{\partial \phi}{\partial x}\, dx\, dy = \frac{1}{S_{A"B"C"D"}} \oint \phi\, dy \quad (29)$$

$$\left[\frac{\partial \phi}{\partial y}\right]_{i+1/2,j} = \frac{1}{S_{A"B"C"D"}} \iint \frac{\partial \phi}{\partial y}\, dx\, dy = -\frac{1}{S_{A"B"C"D"}} \oint \phi\, dx \quad (30)$$

Now, we evaluate the second line of equation 22, which is again $\partial\phi/\partial x$ at i+1/2, j, means at this point and $\partial\phi/\partial y$ at j +1/2 j, at this point. This I mean $\partial\phi/\partial x$ at this point will be multiplied by Δy_{BC}. That means, vertical distance between B and C and $\partial\phi/\partial y$ having evaluated that will be multiplied by Δx_{BC} that is horizontal distance between C and B.

So, we have to in order to find out this ∂ϕ/∂x quantity and ∂ϕ/∂y quantity here like we did in earlier case. We have to integrate them over this domain which is we can call it domain of interpolation and this we are defining as A'', B'', C''and D''.

So, this domain of interpolation on this we will find the mean value of ∂ϕ/∂x and mean value of ∂ϕ/∂y, which is meant that they are being defined at this point. So, ∂ϕ/∂x at i+1/2,j is basically double integral of ∂ϕ/∂x dx dy divided by area of this new quadrilateral A'', B'', C'', D''.

$$\left[\frac{\partial \phi}{\partial x}\right]_{i+1/2,j} = \frac{1}{S_{A''B''C''D''}} \iint \frac{\partial \phi}{\partial x}\, dx\, dy = \frac{1}{S_{A''B''C''D''}} \oint \phi\, dy \qquad (29)$$

$$\left[\frac{\partial \phi}{\partial y}\right]_{i+1/2,j} = \frac{1}{S_{A''B''C''D''}} \iint \frac{\partial \phi}{\partial y}\, dx\, dy = -\frac{1}{S_{A''B''C''D''}} \oint \phi\, dx \qquad (30)$$

And again, if we apply Green's theorem here this integral means 1 by area line integral of ϕ dy and ∂ϕ/∂y at again the same point i +1/2, j this is 1 by area double integral surface integral of $\frac{\partial \phi}{\partial y}\, dx\, dy$, which is equal to - 1 by area into line integral of ϕ dx.

We did same thing for our earlier case for evaluating first line of equation 22, this is second line and in the second line these are the terms ∂ϕ/∂x and ∂ϕ/∂ y, ∂ϕ/∂x ∂ ϕ/∂ y, how they are to be evaluated we have clearly written it down. Now, after having evaluated this will be through.

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Equations with Second derivatives

In a similar manner, one can evaluate the following:

$$\left[\frac{\partial \phi}{\partial x}\right]_{i+1/2,j} = \frac{1}{S_{A''B''C''D''}} \iint \frac{\partial \phi}{\partial x}\, dx\, dy = \frac{1}{S_{A''B''C''D''}} \oint \phi\, dy \qquad (29)$$

$$\left[\frac{\partial \phi}{\partial y}\right]_{i+1/2,j} = \frac{1}{S_{A''B''C''D''}} \iint \frac{\partial \phi}{\partial y}\, dx\, dy = -\frac{1}{S_{A''B''C''D''}} \oint \phi\, dx \qquad (30)$$

$$\oint_{A''B''C''D''} \phi\, dy \approx \phi_B\ \Delta y_{A''B''} + \phi_{i+1,\,j}\ \Delta y_{B''C''} + \phi_C\ \Delta y_{C''D''} + \phi_{i,\,j}\ \Delta y_{D''A''}$$

and

$$\oint_{A''B''C''D''} \phi\, dx \approx \phi_B\ \Delta x_{A''B''} + \phi_{i+1,\,j}\ \Delta x_{B''C''} + \phi_C\ \Delta x_{C''D''} + \phi_{i,\,j}\ \Delta x_{D''A''}$$

So, $\frac{\partial \phi}{\partial x}$ at i+1/2,j point is 1 by area again we wrote a surface integral of ∂ϕ/∂x, which is basically line integral of ϕ dy. So, ϕ dy line integral over the segments A”B”C”D”A”.

It is just around the this contour over we have to go and that is how we have done ϕ_B into $\Delta y_{A''B''}$ $\phi_{i+1,j}$ $\Delta y_{B''C''}$ ϕ_C $\Delta y_{C''D''}$ + $\phi_{i,j}$ into $\Delta y_{D''A''}$.

And the in order to find out ∂ϕ/∂y at same point again we have to integrate ϕ dx over A”, B”; B”C”; C”D”; D”A”. And if we do that ϕ dx is ϕ_B into $\Delta x_{A''B''}$ $\phi_{i+1,j}$ $\Delta x_{B''C''}$, ϕ_C $\Delta x_{C''D''}$, $\phi_{i,j}$ $\Delta x_{D''A''}$. So, this is these are the integrals and divided by similar way if we now can calculate the area of this quadrilateral A”B”C”D”. (refer equation 30)

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Equations with Second derivatives

If the mesh is not too distorted,

$$\Delta y_{A''B''} = -\Delta y_{C''D''} = \Delta y_{i,\ i+1} \quad \text{and} \quad \Delta y_{B''C''} = -\Delta y_{D''A''} = \Delta y_{BC}$$
$$\Delta x_{A''B''} = -\Delta x_{C''D''} = \Delta x_{i,\ i+1} \quad \text{and} \quad \Delta x_{B''C''} = -\Delta x_{D''A''} = \Delta x_{BC}$$

$$S_{BC} \equiv S_{A''B''C''D''} = \left|\Delta x_{A''B''}\hat{i} + \Delta y_{A''B''}\hat{j}\right| \times \left|\Delta x_{B''C''}\hat{i} + \Delta y_{B''C''}\hat{j}\right|$$

The equivalent expressions for ar $[\partial\phi/\partial x]_{i,\ j+1/2}$, $[\partial\phi/\partial y]_{i,\ j+1/2}$ other similar terms in equation (22), finally yields:

$$M_{AB}\ (\phi_{i,\ j-1} - \phi_{i,\ j}) + N_{AB}\ (\phi_B - \phi_A) + M_{BC}\ (\phi_{i+1,\ j} - \phi_{i,\ j}) + N_{BC}\ (\phi_C - \phi_B)$$
$$+ M_{CD}\ (\phi_{i,\ j+1} - \phi_{i,\ j}) + N_{CD}\ (\phi_D - \phi_C) + M_{DA}\ (\phi_{i-1,\ j} - \phi_{i,\ j}) + N_{DA}\ (\phi_A - \phi_D) = 0$$

(31)

Which we have done here again we have done some parametric evaluation $\Delta y_{A''B''}$ equal to $-\Delta y_{C''D''}$ equal to $\Delta y_{i,\ i+1}$. So, this way we have calculated these parameters and as we know that this area means basically, $|A'', B'' \times B'', C''|$.

And A″, B″ means its x component into i, y component into j. Similarly, B″, C″ means its x component i plus y component into j. So, here we can see $\Delta x_{A''B''}\hat{\imath} + \Delta y_{A''B''}\hat{\jmath}$ and $|\Delta x_{B''C''}\hat{\imath}$ plus $\Delta y_{B''C''}\hat{\jmath}|$.

$$S_{BC} \equiv S_{A''B''C''D''} = \left|\Delta x_{A''B''}\hat{i} + \Delta y_{A''B''}\hat{j}\right| \times \left|\Delta x_{B''C''}\hat{i} + \Delta y_{B''C''}\hat{j}\right|$$

So, if we evaluate this and then we divide these integral values integral ϕ dy and integral ϕ dx will be able to get these two quantities $\frac{\partial\phi}{\partial x}$ at i +1/2, j; $\partial\phi/\partial y$ at i +1/2, j.

Again, we have to have been done that complete evaluation of the second line of equation 22, we have to multiply this with $\Delta y_{B''C''}$ and we have to multiply this with $\Delta x_{B''C''}$ and then again, we will get an expression.

Similarly, we will evaluate the third line, third line means $\frac{\partial \phi}{\partial x}$ at i, j+1/2 and ∂ϕ/∂y at i, j+1/2. So, basically then we have to find out $\frac{\partial \phi}{\partial x}$ at this point. That means, i, j +1/2 and $\frac{\partial \phi}{\partial y}$ at this point i, j +1/2. Here and then again, we have to do the multiplication and for finding out here again we have to define a new area. And we have to integrate those quantities on those area and finding values here then we have to find that by Δy_{CD} and Δx_{CD}.

So, this way if we complete the entire integration this point into Δx Δy component value of this point into Δx Δy component value at this point into Δx Δy value of this point into Δx Δy. Then we have evaluated the entire you know integration, but in order to do that all this $\frac{\partial \phi}{\partial x}$ and ∂ϕ/∂y quantity are being defined here, here, here, here indicating to some points (Refer Slide Time: 35:39).

So, for defining here we have interpolated it on this domain we have interpolated on this domain. And similarly, here we have not shown, we have to interpolate if we want to find out on you know this domain. So, this way we have to find out all the quantities and then we if we assemble them, we will see just from the first line if you recall we got equation 28.

Basically geometric parameters M_{AB} multiplied by $\phi_{i,j-1} - \phi_{i,j}$. Again, N_{AB} $\phi_B - \phi_A$. So, every line will generate some such geometric parameter equivalence and multiplied by the difference in field variables. Exactly we can see M_{AB} multiplied by $\phi_{i,j-1} - \phi_{i,j}$ N_{AB} $\phi_B - \phi_A$, M_{BC} from second equation we will get these two terms M_{BC} and N_{BC}.

$$M_{AB}\ (\phi_{i,\ j-1} - \phi_{i,\ j}) + N_{AB}\ (\phi_B - \phi_A) + M_{BC}\ (\phi_{i+1,\ j} - \phi_{i,\ j}) + N_{BC}\ (\phi_C - \phi_B)$$
$$+ M_{CD}\ (\phi_{i,\ j+1} - \phi_{i,\ j}) + N_{CD}\ (\phi_D - \phi_C) + M_{DA}\ (\phi_{i-1,\ j} - \phi_{i,\ j}) + N_{DA}\ (\phi_A - \phi_D) = 0 \quad (31)$$

It is multiplied by $\phi_{i-1,j} - \phi_{i,j}$ this is multiplied by $\phi_C - \phi_B$. So, we will you know from the third line we will get these two terms and from the fourth line we will get these two terms. So, equation 22 is now completely evaluated as equation 31, you can see as I mentioned earlier M_{AB}, N_{AB}, M_{BC}, N_{BC}, M_{CD} etcetera are all geometric parameters.

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Equations with Second derivatives

Where the geometrical parameters are:

$$M_{AB} = (\Delta x_{AB}^2 + \Delta y_{AB}^2) \,/\, S_{AB}, \qquad N_{AB} = (\Delta x_{AB}\ \Delta x_{j-1,\,j} + \Delta y_{AB}\ \Delta y_{j-1,\,j}) \,/\, S_{AB}$$

$$M_{BC} = (\Delta x_{BC}^2 + \Delta y_{BC}^2) \,/\, S_{BC}, \qquad N_{BC} = (\Delta x_{BC}\ \Delta x_{i+1,\,i} + \Delta y_{BC}\ \Delta y_{i+1,\,i}) \,/\, S_{BC}$$

$$M_{CD} = (\Delta x_{CD}^2 + \Delta y_{CD}^2) \,/\, S_{CD}, \qquad N_{CD} = (\Delta x_{CD}\ \Delta x_{j+1,\,j} + \Delta y_{CD}\ \Delta y_{j+1,\,j}) \,/\, S_{CD}$$

and

$$M_{DA} = (\Delta x_{DA}^2 + \Delta y_{DA}^2) \,/\, S_{DA}, \qquad N_{DA} = (\Delta x_{DA}\ \Delta x_{i-1,\,i} + \Delta y_{DA}\ \Delta y_{i-1,\,i}) \,/\, S_{DA}$$

In equation (31) a $\phi_A,\ \phi_B,\ \phi_C,\ \phi_D$ average of the four surrounding nodal values. Thus

$$\phi_A = 0.25\ (\phi_{i,\,j} + \phi_{i-1,\,j} + \phi_{i-1,\,j-1} + \phi_{i,\,j-1})$$

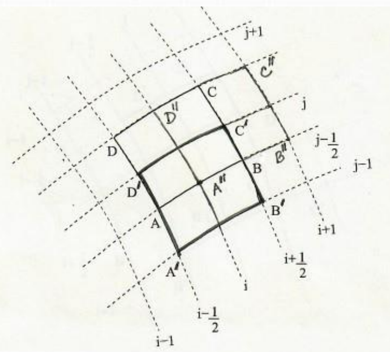

Figure 3.6: Finite Volume for a Curvilinear Grid.

So, these are basically geometrical parameters we have written M_{AB}, N_{AB}, M_{BC}, N_{BC}, M_{CD}, N_{CD}, M_{DA} and N_{DA} and how they are geometrical parameters. We have understood that these are different projections of you know line segments, on horizontal line or on vertical line. Basically, vertical distance between if we define a line segment by two points vertical distance between those two points and horizontal distance between those two points.

Then, the area of the quadrilateral and such parameters. So, we these are the geometrical parameters and equation 31 if you see. We have we can see that $\phi_{i,j-1}$ $\phi_{i,j}$ $\phi_{i,j+1}$ $\phi_{i,j}$ all these i, j subscripted variables these are basically nodal points well defined nodal points where the grid lines are intersecting. But ϕ_C, ϕ_B, ϕ_A these are not nodal a point these are between you know two nodal points lie between two points.

So, they have to be evaluated in terms of nodal points like ϕ_A see this ϕ_A it is basically it is not i, j or j-1 i, I, j-1 or i, j + 1 or i + 1, j + 1. This is you know falling here, which is basically i+1/2 and j-1/2, right. So, this is intersection between j-1/2 and i-1/2.

So, if we want to find out this point in terms of four neighboring coordinates where you know things are well defined it is crossing i, j-1 line. So, this point then basically we can say i, j then this is i-1/2, j and this i-1, j sorry and this is i-1, j-1. So, this is i-1, j-1 point this is i, j-1 point this is basically i, j point this is i-1, j point.

So, we have written all these points and the mean value of that you know divided by 4 is the coordinate value of ϕ_A. So, that is how ϕ_A, ϕ_B, ϕ_C and ϕ_D they have to be evaluated because ABCD was defined such a way that its center point is i j, but it vertices are not falling on intersection lines of the grids i, i + 1, i - 1 j, j + 1, j - 1.

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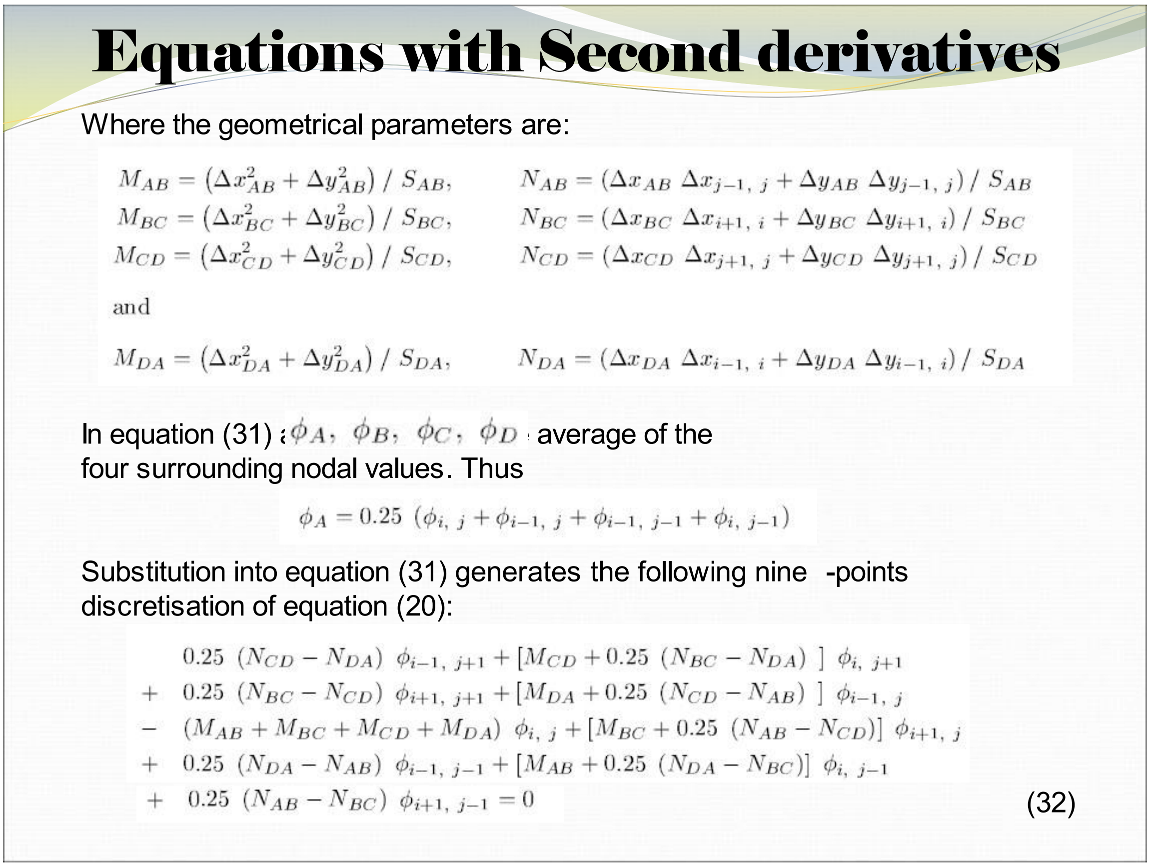

So, if we do that ϕ_A, ϕ_B, ϕ_C and ϕ_D it just the way ϕ_A has been calculated if we do not follow the same calculation ϕ_B, ϕ_C and ϕ_D and substitute in the equation that we got as equation 31 we will be able to get equation 32. So, equation 32 is basically evaluation of the line integral of basically, $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ along ABCD which is the control volume.

And here we can see that, basically it is comprising of multiplication of geometrical parameters and the ϕ value at different nodes i, j is a node of interest i+1 j, i-1 j, i j+1, i j-1, i+1 j+1 then i-1 j, i-1 j-1 then i+1 j-1.

So, all the you know nodes we have to see appearing we have to get participated in this calculation and that is how equation 32 has been obtained and in the equation 32, if we carefully now look at it every ϕ term is having some coefficient.

Now, $\phi_{i,j}$ is having coefficient $M_{AB} + M_{BC} + M_{CD} + M_{DA}$. So, and this equation is equal to 0, that means, this $\phi_{i,j}$ with it is coefficients if shifted to the other side of the equation and then we can explicitly get expression of $\phi_{i,j}$ by dividing the remaining terms by this term $M_{AB} + M_{BC} + M_{CD} + M_{DA}$.

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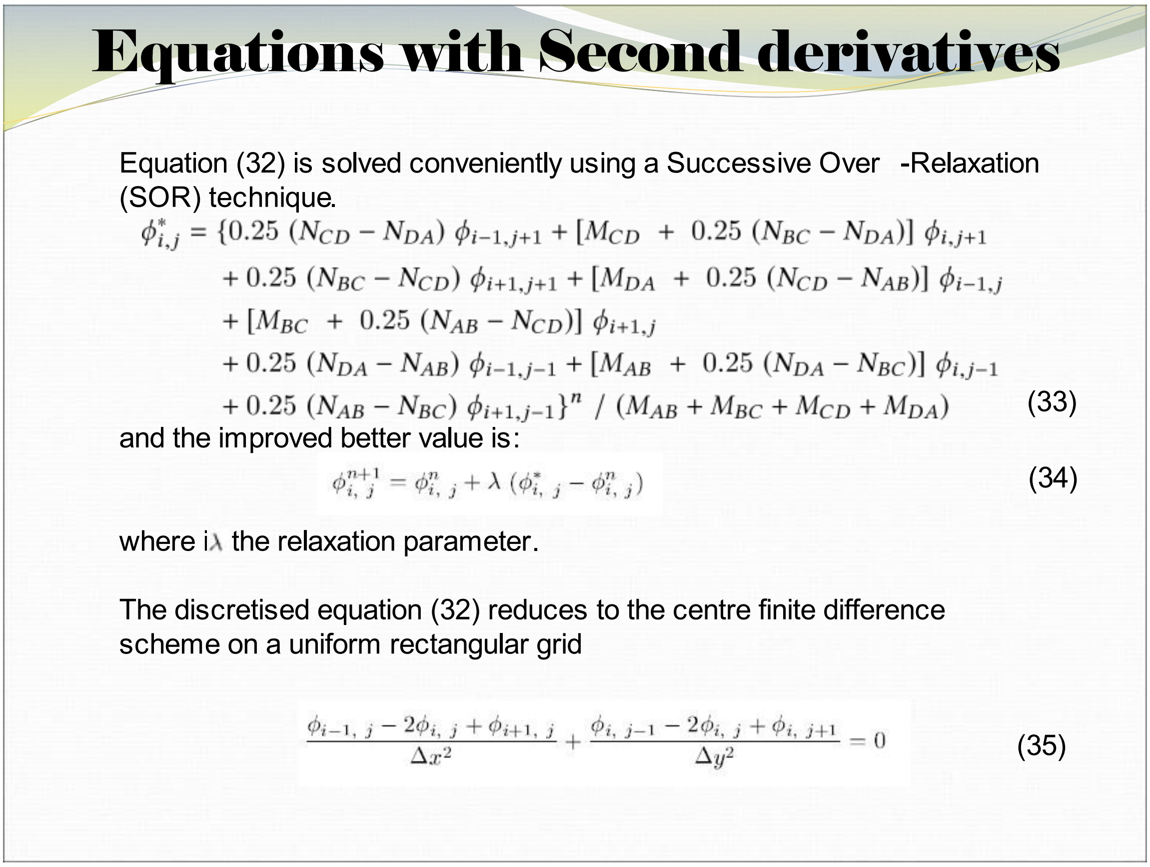

So, that is what we have done we are calling this as $\phi^*_{i,j}$ this is all the remaining terms on the other side divided by $M_{AB} + M_{BC} + M_{CD} + M_{DA}$. These four terms this term basically these four quantities bracketed terms together is the coefficient of $\phi_{i,j}$.

And we are transferring this to the right hand side dividing the entire remaining terms with this term and we are getting $\phi_{i,j}$ we are calling it $\phi_{i,j}^{*}$, this is explicit evaluation. And you know if all ϕ's right hand side ϕ's are at the nth level then $\phi_{i,j}$ left hand side ϕ's is at the advanced level.

If you are going from N to N+1 this is you cannot we are not calling it N + 1 you can say it is pseudo N + 1 is this and then this start term can be the initial evaluation in order to accelerate the calculation. We can then call that

$$\phi_{i,\ j}^{n+1} = \phi_{i,\ j}^{n} + \lambda\ (\phi_{i,\ j}^{*} - \phi_{i,\ j}^{n}) \tag{34}$$

Where, λ is over relaxation per is it is an over relaxation parameter.

So, all ϕij, ij-1, i j+ 1, i+1 j, i-1 j all these ϕ values are at nth level, from there we are calculating $\phi_{i,j}^{*}$ and using $\phi_{i,j}^{*}$ and nth level value and having performed the over relaxation exercise we are calculating $\phi_{i,j}$ at n+1th level at the next iteration level.

So, this way we can keep on iterating till we get a constant value or the values of i, j are i, j remain unchanged at every cell or every point, then we have gotten the final solution. So, that is what is the solution strategy of basically Laplacian ϕ and $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$.

While doing this we have learned how to handle the second order derivatives number 1, and number 2 is we can see instead of x = constant line, y = constant lines, which may make basically Cartesian grid instead of that we used curvilinear grids. Basically, body conforming zeta = constant, eta = constant lines and we got this expression, but this expression (33) or expression 32 whatever we consider they are same, we get this 32 or 33.

But now, if you change the grid to basically Cartesian grid as I said that all x = constant line, y = constant lines Cartesian grid. Then this expression you can test it in your notebook, this expression will be transformed into equation 35, which is

$$\frac{\phi_{i-1,\ j} - 2\phi_{i,\ j} + \phi_{i+1,\ j}}{\Delta x^2} + \frac{\phi_{i,\ j-1} - 2\phi_{i,\ j} + \phi_{i,\ j+1}}{\Delta y^2} = 0 \tag{35}$$

So, this is basically central difference in x direction central difference in y direction. So, this is finite difference quotient for $\frac{\partial^2 \phi}{\partial x^2}$ this is finite difference quotient for $\frac{\partial^2 \phi}{\partial y^2}$. And we get this through difference approximation, but finite volume method is basically integral representation, but finally, we can see that they are equivalent.

If the curvilinear grid is confirmed is curvilinear grid is changed to partition grid. So, this is possible for cylindrical polar coordinate and spherical polar coordinate also, the similar exercise can be done. So, what we did in this part of learning finite volume method?

We have just conceptualized the ideas and then implemented it on simple equations, but directly or indirectly our target was to basically integrate the variable its first order derivative, its second order derivative in the elemental control volumes to get the final set of algebraic equations. So, with this we will stop today and in the next class hopefully, we will take up another new topic.

Thank you very much, thank you.