Good afternoon, everybody, today we will continue with Finite Volume Methods.

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Finite Volume Method-2

As it has been observed, the Finite Volume method uses an integral form of the equation to be solved. The computational domain is divided into elementary volumes and the integration is performed within these elementary volumes. The method enables one to handle complex geometry without having the equations written in curvilinear coordinates. The method also preserves the conservative property. The elementary control volumes are described by the coordinates of the vertices of the quadrilaterals (for 2-D) or hexahedrals (for 3-D).

If you recall yesterday's lecture we will start from there. The finite volume method if you recall, we mentioned that finite volume method uses an integral form of the equation to be solved. The computational domain is divided into elementary control volumes and the integration is performed within these elementary volumes.

The method enables one to handle complex geometry without having the equations written in curvilinear coordinates. The method also preserves the conservative property; the

elementary control volumes are described by the coordinates of the vertices of the quadrilaterals or 2D or hexahedral for 3D.

That means, when we get complex geometry grids are to be generated using body fitted coordinates and then the grids are to be generated in such A way for 2D. The elements or the cells are quadrilateral and for 3D these elements are hexahedral sometimes they are also called brick elements.

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Equation with first derivatives

General first-order equation:

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \qquad (12)$$

For $E = \rho$, $F = \rho\,u$ **and** $G = \rho\,v$,

The above equation is the two-dimensional continuity equation, and for $E = \rho u$, $F = \rho u^2$, $G = \rho uv$, it is the inviscid momentum equation in the x-direction, and so on. In a similar manner, for x-direction viscous momentum equation,

$$E = \rho\,u, \quad F = \left(\rho u^2 + p - \mu\frac{\partial u}{\partial x} \right), G = \left(\rho\,u\,v - \mu\frac{\partial u}{\partial y} \right) \qquad (13)$$

Now, general first order equation we can write as

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0$$

$$(12)$$

For $E = \rho$, $F = \rho u$ and $G = \rho v$. The equation will be; obviously, continuity equation and if $E = \rho u$, $F = \rho u^2$ and $G = \rho uv$, then this will be inviscid momentum equation in x direction.

Hence, substitute E by ρu, F by $\rho u^2 + p - \frac{\mu \partial u}{\partial x}$ and G by $\rho uv - \frac{\mu \partial u}{\partial y}$, then the equation will be equation 12 will be then full x momentum equation or x direction Navier Stokes equation.

(Refer Slide Time: 03:48)

Equation with first derivatives

We consider the area integral of general first order equation over Ω_p

$$\iint_{ABCD} \left(\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) dx \, dy = 0 \qquad (14)$$

Green's theorem:

$$\iint_{ABCD} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \, dy = \oint_c f dx + g dy \qquad (15)$$

And now we will try to integrate this equation (12) over a domain may be the elementary control volume given by quadrilateral given by ABCD and integration will be done in this manner equation 14.

$$\iint_{ABCD} \left(\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) dx \, dy = 0$$

$$(14)$$

Also, we can apply Green's theorem. Green's theorem says that for this domain integral of

$$\int\int_{ABCD} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\, dy = \oint_c f\, dx + g\, dy$$

(15)

We will you can very easily understand that $\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}$ in case of continuity equation it will

be substituted $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ and accordingly we have to substitute F and G in this line integral.
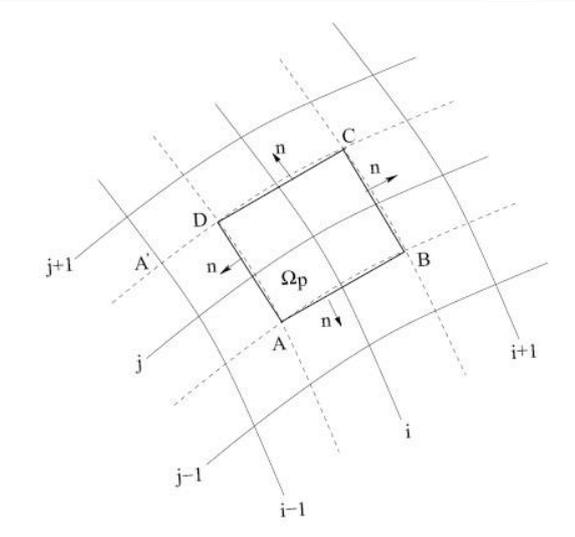
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Figure-5: Finite Volume Grid - Mesh over the domian

Where A is the area of the quadrilateral ABCD in Figure-5. The average value of E over the quadrilateral is represented by $E_{i,j}$

So, if we as I said if we generate a body conforming grade which you know will be a constant eta zeta lines or you know then we can identify a specific control volume say as we have defined A, B, C, D. Now entire domain we can identify such control volumes you know which will have additive property and that is how the whole domain will be covered.

If we write down governing equation for one control volume that is valid for all the control volumes and finally, now we will be able to get the matrix equation. So, the A is the area

of this quadrilateral A, B, C, D given here average value of E you recall $\frac{\partial E}{\partial t}$ was the first term and we said E is ρ.

So, this is average value of E or ρ over the entire cell A, B, C, D. We will call and this also we mentioned that this small control volume on the cell will be identified in such a way that, its center will be its identification and center will be identified by i, j. So, if the we call this cell as i, j cell, then we have also written here if E is a property over the cell that will be written as E$_{i,j}$ and i and j are the identification coordinate of center of this cell.

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Equation with first derivatives

Applying Green's theorem, (14) becomes

$$\frac{d}{dt}\iint E\, dV + \oint_{BACD} \mathbf{H} \cdot \mathbf{n}\, ds = 0 \qquad (16)$$

where $\mathbf{H} = (F, G)$ and n is the outward unit normal of segment ds (see Figure). For a segment $d\vec{s} = dx\,\hat{i} + dy\,\hat{j}$. On a counter-clockwise contour, the outward unit normal

$$\vec{n} = \frac{dy\,\hat{i} - dx\,\hat{j}}{ds},$$ where $ds = \sqrt{dx^2 + dy^2}$. For the continuity equation, $F = \rho u$

and $G = \rho v$. In Cartesian coordinates,

$$\mathbf{H} \cdot \mathbf{n}\, ds = F\, dy - G\, dx$$

Equation (16) is just a statement of conservation. For the particular choice, $E = \rho$, $F = \rho u$, $G = \rho v$, in Eqn. (16) coincides with an integral statement of the conservation of mass.

Now we mentioned it earlier applying we will apply Green's theorem; Now Green's theorem means this is basically we will integrate these quantities and E is dependent on t. So, $\frac{\partial E}{\partial t}$ this will be separated out and then $\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}$ we will apply Green's theorem on this part.

So, if we do that we can write down

$$\frac{d}{dt}\iint E\,dV + \oiint_{BACD} \mathbf{H}.\mathbf{n}\,ds = 0$$

(16)

Where H is the identification of the field comprising of F and G. **n** is an outward unit normal of segment ds if we look at the figure.

For a segment ds $= dx\hat{\imath} + dy\hat{\jmath}$ on a counter clockwise contour, the outward unit normal can be calculated from this ds as $dy\hat{\imath} - dx\hat{\jmath}$ divided by ds where this is a modulus value of ds and while the ds or this modulus value is basically root over $\sqrt{dx^2 + dy^2}$. Now as we said for continuity equation then our F will be ρu and G will be ρv.

So, n dot dx will be written as

$$\mathbf{H}.\mathbf{n}\,ds = F\,dy - G\,dx$$

We also said earlier that continuity equation is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$. So, like following that F and G, you will be substituted by v and u and we can write **H.n**ds = Fdy - Gdx. So, this we can write.

Now applying Green's theorem on equation 14 where again let me repeat E means ρ, F means ρu and G means ρv we said that the and then we will apply Green's theorem. And then we can write that you know the time dependent part we separate out, but the volume is not controlling volume is not dependent on time.

So, we write from equation 16 we can write H.**n** ds is Fdy-Gdx basically ρudy $- \rho v$dx. So, equation 16 is a statement of conservation and for this particular choice as I said E = ρ, F = ρu and G = ρv. Then this becomes the integrated form of continuity equation and we will have to perform the integration now.

Equation with first derivatives

Using Green's theorem the equation (14) becomes:

$$\frac{d}{dt} \int \int E \, dV + \oint_{ABCD} \mathbf{H} \cdot \mathbf{n} \, ds = 0 \qquad (16)$$

The finite volume method is a discretization of the governing equation in integral form, in contrast to the finite difference method, which is unusually applied to the governing equation in differential form.

$$\frac{d}{dt} (A \, E_{i,\, j}) + (F \, \Delta y - G \, \Delta x)_{AB} + (F \, \Delta y - G \, \Delta x)_{BC}$$
$$+ (F \, \Delta y - G \, \Delta x)_{CD} + (F \, \Delta y - G \, \Delta x)_{DA} = 0 \qquad (17)$$

So, again let me explain the equation (16) the finite volume method is a discretization of the governing equation in integral form. In contrast to the finite difference method which is usually applied to governing equation in differential form here we have to perform the integration.

So, this we can say d/dt of this integral means total area this volume integral here per unit depth, if this is the volume this is total area. So, area into $E_{i,j}$ we mentioned it earlier that any property which is distributed over this control volume will be identified by the identification of the control volume, identification here is i, j.

So, E will be $E_{i,j}$ and our E is ρ. So, basically it will substitute ρ here later then we will integrate over A, B, C, D. That means, over you know this sides of the quadrilateral and that is what we will integrate H.**n** d s, that means, Fdy-Gdx we got it through application of Green's theorem.

So, that means, this H is now Fdy-Gdx. So,

$$\frac{d}{dt}\left(A\,E_{i,\,j}\right)+\left(F\,\Delta y-G\,\Delta x\right)_{AB}+\left(F\,\Delta y-G\,\Delta x\right)_{BC}$$
$$+\left(F\,\Delta y-G\,\Delta x\right)_{CD}+\left(F\,\Delta y-G\,\Delta x\right)_{DA}=0$$

(17)

We go along the contour we integrate F and G over AB, BC, CD, DA.

Now basically when you are integrating, we are integrating mass flux because F is nothing but ρu, into Δy. That means, Δy into unit depth that is the area through which ρu, that is passing through. That means u into area gives the volume and multiplied by ρ gives the mass.

So, through the control volume we basically calculate the mass flux total mass flux in x-direction total mass flux in y-direction. And then this net mass flux will be zero that will be the governing equation.

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Equation with first derivatives

Where A is the area of the quadrilateral ABCD in Figure-5, and the average value of E over the quadrilateral is represented by $E_{i,j}$ and the remaining terms are approximations for the line integral over segments AB, BC, CD and DA respectively
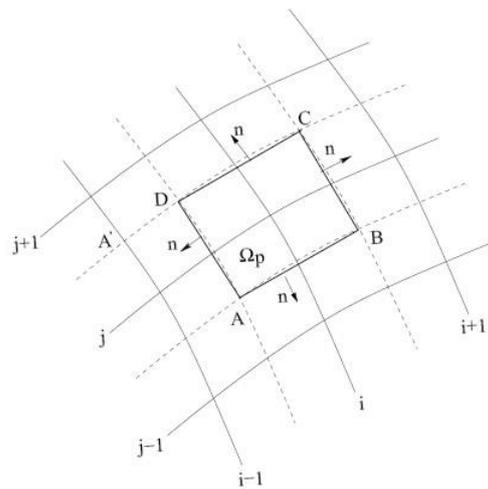
Figure-5: Finite Volume Mesh System

$$\Delta y_{AB}=y_B-y_A,\quad \Delta x_{AB}=x_B-x_A \text{ and}$$
$$F_{AB}=0.5\,(F_{i,\,j-1}+F_{i,\,j}),\quad G_{AB}=0.5\,(G_{i,\,j-1}+G_{i,\,j})$$

So, how to execute this? So, where we have already explained it that A is the area of this quadrilateral and ABCD are the sides, average value of E over the quadrilateral is represented by $E_{i,j}$. And the remaining terms are approximation for the line integral over the segments AB, BC, CD and DA.

Now when we perform the line integral on AB the Δy_{AB} is basically y coordinate value of B and y coordinate value of A so $y_B - y_A$ is the Δy. Similarly, Δx AB is x coordinate of B and x coordinate of A. So, $x_B - x_A$ is Δx_{AB}.

So, delta Δy_{AB}, we have seen Δx_{AB} we have seen because we are taking the projection of this mass in the x-direction. So, projection of mass, which is passing through it in x-direction. So, and in y-direction, so ρu will be multiplied through basically Δy_{AB} into 1 and similarly ρv will be multiplied by Δx_{AB} into 1.

Now what is F_{AB}? If we are multiplying Δy_{AB} into, F_{AB}, F_{AB} will be the value of F here. Now $F_{i,j}$ is defined here and similarly $F_{i,j-1}$ is defined here. So, these are all constant i lines these are all constant j lines we have. So, j-1, j, j+1, j+2 this way it will be counted in j direction and i-1, i, i+1, i+2 these are the constant lines. So, which will basically be coming out of the grid that has been generated.

So, F what will be the value of F here? That is $F_{i,j}$ this is you know from the coordinate it is clear i and j crossing here $F_{i,j}$ and F here $F_{i,j-1}$ divided by 2. So, $(F_{i,j-1} + F_{i,j})/2$ is F_{AB} ,G what will be the G_{AB} see G is again $G_{i,j}$ is here and $G_{i,j-1}$ is defined here.

So, average of these 2 is G here, G at this point which is we can call it G_{AB}. So, G_{AB} is half of $G_{i,j-1} + G_{i,j}$. And then this will be multiplied by Δx_{AB} will be the mass flux in y-direction this is G is ρv and F is nothing, but ρu.

So, its entire flux in x direction u is basically x direction velocity, v is y direction velocity. So, net flux in x mass flux in x direction, net mass flux in y direction that way we can calculate, now if we go for the.

Equation with first derivatives

Where A is the area of the quadrilateral ABCD in Figure-5, and the average value of E over the quadrilateral is represented by $E_{i,j}$ and the remaining terms are approximations for the line integral over segments AB, BC, CD and DA respectively
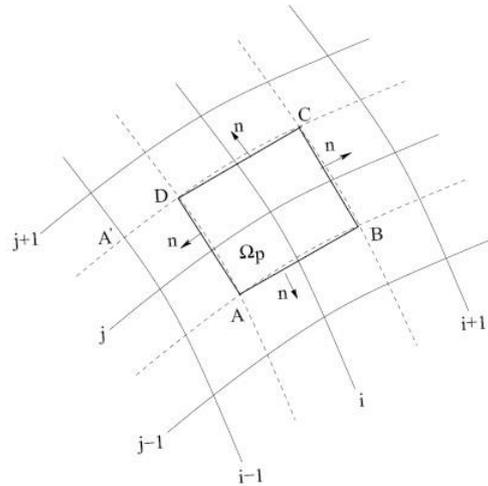
Integration over BC: $+0.5\left(F_{i,j}+F_{i+1,j}\right)\Delta y_{BC}-0.5\left(G_{i,j}+G_{i+1,j}\right)\Delta x_{BC}$

Integration over CD: $+0.5\left(F_{i,j}+F_{i,j+1}\right)\Delta y_{CD}-0.5\left(G_{i,j}+G_{i,j+1}\right)\Delta x_{CD}$

So, the integration over basically F and G over AB we have performed. Now if we perform integration over BC. Now again integration over BC will be first we have to find out Δy_{BC}, Δy_{BC} will be y coordinate of C and minus y coordinate of B

And this will be multiplied with F value of F defined here or whatever is the representative value of F on BC this is basically $(F_{i,j}+F_{i+1,j})/2$.

So, you can see $(F_{i,j}+F_{i+1,j})/2$ into the projection of y coordinate in x-direction this is Δy_{BC} y_C-y_B. Similarly, x_B-x_C will be delta x_{BC}, now Δx_{BC} will be multiplied by the value of G or which is assigned on BC. Now value of G we are going for cell centered method.

So, value of G on BC is basically whatever is the value of G here, that means, $(G_{i,j}+G_{i+1,j})/2$ that is the value of G here. So, that value of $G\Delta x_{BC}$ will be the flux again in x-direction and as we have mentioned that $(F_{i,j}+F_{i+1,j})/2$. Whatever is F value here, that means, $\rho u,$ multiplied by Δy_{BC} that is the x-direction mass flux.

So, we are we have calculated these 2, then we go for integration over CD and this integration again basically the y_C-y_D will be Δy_{CD} and x_C-x_D will be Δx_{CD}. So, that is from geometry and then the variable, that means, $F_{i,j}$ here and $F_{i,j+1}$ is the value here. So, we have to find out F value on CD, that means, say value at this location and that is half of $F_{i,j}$ +F_{ij+1} into y_{CD} that is again. Basically, we are getting the projection on x plane; because this is the x direction mass flux x plane means the plane here which is generated by y z, z is 1. So, generated on y.

So, this is again as I as we are discussing this will be y_{CD} multiplied by F value here and Δx_{CD} multiplied by the G value here. So, G value will also be $(G_{i,j}+G_{i,j+1})/2$ is here. So, this way we can keep calculating we will also calculate on DA side.

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Equation with first derivatives

With similar expressions for integration along DA , and if A is not a function of time, then:

$$
\begin{aligned}
& A \, dE_{i,j} \, / dt \; + \; 0.5 \, (F_{i, \, j-1} + F_{i, \, j}) \, \Delta y_{AB} \; - \; 0.5 \, (G_{i, \, j-1} + G_{i, \, j}) \, \Delta x_{AB} \\
& + \;\; 0.5 \, (F_{i, \, j} + F_{i+1, \, j}) \, \Delta y_{BC} - 0.5 \, (G_{i, \, j} + G_{i+1, \, j}) \, \Delta x_{BC} \\
& + \;\; 0.5 \, (F_{i, \, j} + F_{i, \, j+1}) \, \Delta y_{CD} - 0.5 \, (G_{i, \, j} + G_{i, \, j+1}) \, \Delta x_{CD} \\
& + \;\; 0.5 \, (F_{i-1, \, j} + F_{i, \, j}) \, \Delta y_{DA} - 0.5 \, (G_{i-1, \, j} + G_{i, \, j}) \, \Delta x_{DA} = 0
\end{aligned}
$$

(18)

For the irregular grid -mesh (I, j) , the finite volume Eqn (18) provides a discretization using the dependent variables defined on Cartesian coordinates without introducing the variables on the curvilinear coordinates.

So, similar expression for integration along DA and then the cycle will be complete it will be integration over ABCD, this elemental control volume will be over. So, we have written now all the terms and the first term as we said will be the area integral. That means, total area into $E_{i,j}$ whatever is the dependent variable it is $\frac{\rho dA\rho}{dt}$.

So, here we will write that way and then remaining terms time derivative is separately calculate and spatial derivatives are calculated using Green's theorem. As we mentioned that we apply Green's theorem basically you know this is the Green's theorem, however, F is ρu and G is ρv.

So, we get this final expression. So, for the irregular grid mesh the finite volume provides. So, this is a discretized equation a discretization using the dependent variables defined on Cartesian coordinate. So, we can see all the dependent ρu, ρv they are all dependent on we have written the expressions on Cartesian coordinates and.

So, ρu, ρv as if you know whatever is their value on Cartesian coordinate, we are writing them, but we are evaluating them on curvilinear coordinate by calculating net projection of mass in x-direction net projection of mass in y direction. And obviously, net mass flux will be 0 and we are setting that equal to 0 plus these are all spatial derivatives plus the temporal derivative.

$$\begin{aligned}
A\ dE_{i,j}\ /dt\ &+\ 0.5\left(F_{i,\ j-1} + F_{i,\ j}\right)\Delta y_{AB}\ -\ 0.5\left(G_{i,\ j-1} + G_{i,\ j}\right)\Delta x_{AB} \\
+\ &0.5\left(F_{i,\ j} + F_{i+1,\ j}\right)\Delta y_{BC} - 0.5\left(G_{i,\ j} + G_{i+1,\ j}\right)\Delta x_{BC} \\
+\ &0.5\left(F_{i,\ j} + F_{i,\ j+1}\right)\Delta y_{CD} - 0.5\left(G_{i,\ j} + G_{i,\ j+1}\right)\Delta x_{CD} \\
+\ &0.5\left(F_{i-1,\ j} + F_{i,\ j}\right)\Delta y_{DA} - 0.5\left(G_{i-1,\ j} + G_{i,\ j}\right)\Delta x_{DA} = 0
\end{aligned} \tag{18}$$

So, we do not transform u, v, w etc. along the curvilinear coordinate. So, we are writing it here for the irregular grid mesh the finite volume equation 18, provides a discretized discretization using the dependent variables defined on cartesian coordinates without introducing the variables on the curvilinear coordinates, curvilinear coordinates you know after coordinate transformation the variable transformation has to be also done.

And it is a quite complex process by we are not going through that route we are going through a different route and this is the beauty of finite volume method. So, and we have taken the first order derivatives today and we will take second order derivative in the next lecture. To you know understand really that it is finally, even though it is integral method finally, we are getting the equation which is equivalent to finite difference method.

Equation with first derivatives

If the grid-mesh is uniform and coincides with lines of constant x and y, the above equation (18) becomes:

$$\frac{d}{dt} E_{i,\,j} + \frac{F_{i+1,\,j} - F_{i-1,\,j}}{2\,\Delta x} + \frac{G_{i,\,j+1} - G_{i,\,j-1}}{2\,\Delta y} = 0$$

(19)

which coincides with a central difference representation for the spatial terms in equation (12)

Now what we have done that if this grid mesh is uniform and coincides with the lines of constant x it is not constant $\eta\ \zeta$ line, instead if it is constant x lines. That means, all y's are vertical constant y lines or x's are horizontal and then you know basically a Cartesian grid mesh then this equation which is looking so complex you if we substitute that you will see it will become

$$\frac{d}{dt} E_{i,j} + \frac{F_{i+1,j} - F_{i-1,j}}{2\Delta x} + \frac{G_{i,j+1} - G_{i,j-1}}{2\Delta y} = 0 \qquad (19)$$

So, this is the central difference $\frac{F_{i+1,j} - F_{i-1,j}}{2\Delta x}$ in x direction and this is again central difference in y direction $\frac{G_{i,j+1} - G_{i,j-1}}{2\Delta y}$. This is easily possible from this equation you can try by yourself and you can see this is plain and simple finite difference discretization.

So, they are equivalent right this follows the integral philosophy, but you know final equations are equivalent, which coincides with a central difference representation for the spatial terms in equation 12. Equation 12 means this if we say F = ρ G = ρv and if we

discretize it on constant x lines and constant y lines, that means, on Cartesian grid we will get that expression $\frac{d}{dt} E_{i,j} + \frac{F_{i+1,j} - F_{i-1,j}}{2\Delta x} + \frac{G_{i,j+1} - G_{i,j-1}}{2\Delta y}$. But, in the finite volume methodology we will get finally, governing equation which is equation 18 for the element. Now, we have to I mean vary the indices, that means, i has to be varied from i = 1 to imax j = 1 to jmax or it is usually j = 2 to jmax-1 and i = 2 to imax-1.

Because when you go to i = 2 already i = 1 point will come and that will be the boundary point. I have similarly when j = 2 will start because j = 1 point will be the boundary point. When we will define equation at j = 2 already j = 1 will be included in the discretized form.

So, this you know when we take up the practical problems it will be even more clear how to induct the boundary condition. We mentioned it in our last lecture how to include Norman boundary, how to include racial boundary condition. We will do it further, but overall philosophy I think is clear now.

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Finite Volume Method-2

Desirable properties of the Curvilinear Grid Mesh

Skewness: Estimate of non-orthogonality between two Intersecting grid lines. Larger the skewness, larger will be truncation error

Aspect Ratio: This is maximum ratio between any two sides of a cell. Desirable range is between 6 and 20.

Adjacent Cell Ratio: Ratio of areas of two adjacent cells. Should be as close to unity (1) as possible. It estimates grid smoothness

Now I will just mention one thing that in the Cartesian coordinate these are constant x lines are you know all vertical lines are constant y lines are all horizontal lines and we get the grid mesh. For complex geometry we do not do that instead we try to fit in body fitted coordinate which can be called as constant ζ lines and constant η lines, but having generated the grid you know we have to analyze the properties of the grid.

Now, we do not have the scope here, but if you really look this grid, we can you can clearly see that although this is not you know following Cartesian coordinate system, this is conforming curvilinear coordinate system, but grids have been laid in such a way that this quadrilateral is not too distorted and if it is a rectangular or square control volume we call it is definitely its orientation is different, but it is not too distorted, and what does it mean?

That means, the before generating the grid. We have to test before generating the grid we have to know that the curvilinear grid mesh should have these properties. One is skewness I mean what is called skewness? Skewness is estimate of non-orthogonality between two intersecting grid lines larger the skewness larger will be truncation error.

That means, as I said instead of x = constant line y = constant line. If we generate grid by ζ = constant line and η = constant lines then you know depending on the location in the domain this ζ η lines will may not be able to maintain their orthogonality. But this when they deviate this deviation will not be much regeneration technique has to follow that.

Aspect ratio this is maximum ratio between any two sides of a cell. So, if it is quadrilateral right this is ratio between any two sides of a cell. So, this is desirable this stays between 6 and 20 and then adjacent cell ratio of areas of two adjacent cells. And then we have to have a check that two adjacent cells they will keep on varying it is never you know same throughout the domain.

Because somewhere grid density will be more somewhere grid density will be coarser and all such things will have, I mean we have to use in order to handle a complex geometry. But as I said the two adjacent grids grid grids if when we look focus at two adjacent quadrilateral then their area ratio will be close to 1 it will not be 1 it will deviate, but will not deviate too much from 1.

Finally, you know over the whole geometry maybe if we pick up a grid at the you know westernmost point and another grid at the easternmost point. They may not be able to keep

that ratio, but you know adjacent ratios of two grids will be close to unity. Now if we follow you know these guidelines while generating grids.

Then you know grid properties will be such, that finite volume method will be applicable with more accuracy inherently accuracy will be enhanced because of these properties of the grid.

So, we will stop here today and we will take up some more aspects of finite volume method in the next class.

Thank you very much.