Basics of Finite Element Analysis - Part II
Prof. Nachiketa Tiwari
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 46
Plane Elasticity Problems: Development of weak form

Hello, welcome to Basics of Finite Element Analysis Part II. Today is the fourth lecture of the current week. Yesterday we have discussing the plane elasticity problem and we had described all the details of the same problem.

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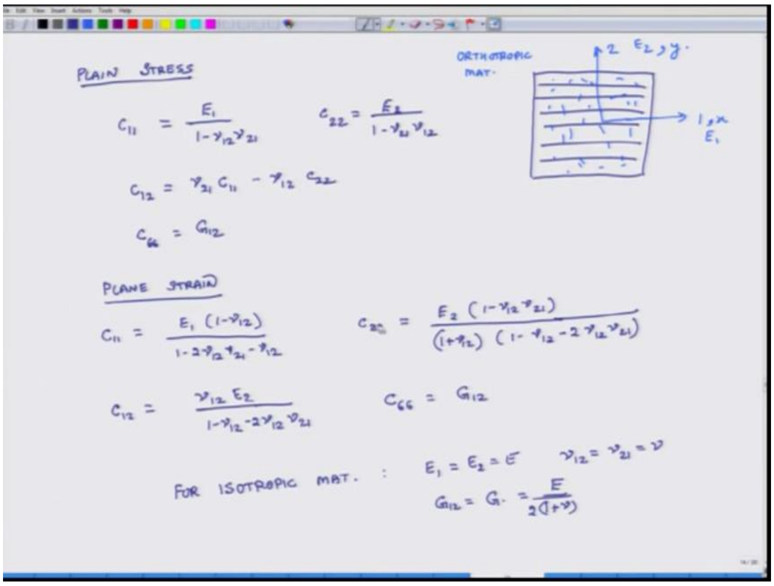

There was one thing yesterday when we were discussing that I had described this constant as C 1 2. So, this was a typographical error and instead of C 1 2 it should have been written as C 2 2. So, this is the correct term. So, what we have discussed till so far is the governing equation of motion is stress strain relations, strain displacement relations and then the definitions for C is the C matrix C 1 1, C 2 2, C 1 2 for plane stress and plane stress problems both for orthotropic and isotropic materials. There was one more thing which we should discuss before we start discussing the finite element formulation and that is Tractions.

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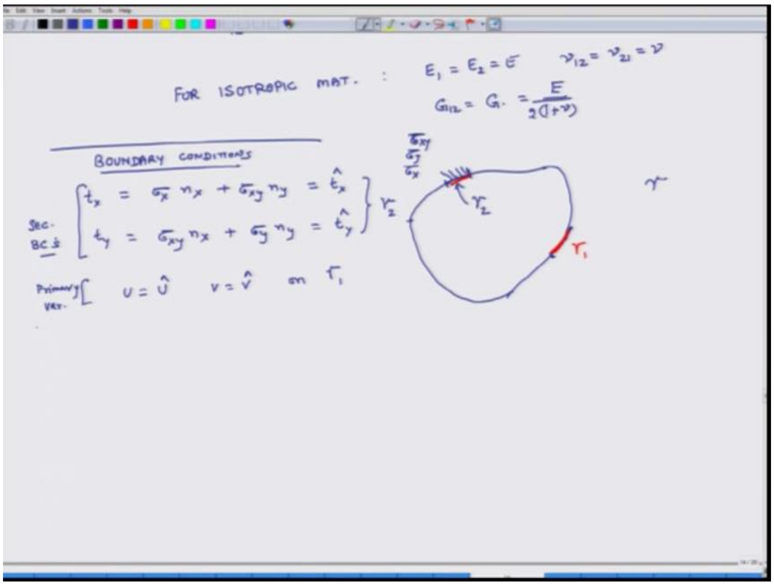

So, what is Tractions? So, traction is there is any force any stress which is being applied on the boundary of the body. It will have an x component and it will have a y component. So, these components are called tractions. So, t x, this discussion is happening in context of boundary conditions.

So, t x is equal to sigma x times n x. So, at any point suppose this is a surface and I am applying some sigma x here. So, at this point or in this suppose over this zone I am applying sigma x and I am also applying sigma y and I am also applying shear stress. I can apply all three of those then and if I know these conditions. So, I am applying some forces in this direction then, sigma x times n x plus sigma x y times and y is called traction. In the x direction and if it is specified I will say that its value is t x hat.

So, if I know sigma x, sigma x y, sigma y or whatever then I multiply sigma x by n x sigma x y by n y I get sigma x this t x hat. Similarly t y equals sigma x by n x plus sigma y n y and if I have, no its value on the surface then I say that its value is t y hat. So, let us say these values are known on some portion of the boundary. Let us call that portion of the boundary Gamma 2. Let say this is Gamma 2, this is the boundary. So, the overall boundary could be like this, entire boundary is Gamma a portion of that boundary on which traction are known I call that Gamma 2 right. Similarly so these are secondary boundary conditions. What kind of forces I am applying on an external boundary? I have to know those, if I am applying those forces I have to know those otherwise I cannot solve for the problem and the other thing is there could be some other portion of the boundary. Let us say this one. So, here tractions are known, here displacements could be known. So, let us call this boundary is Gamma 1 and on this boundary U is equal to U hat and V equal to V hat on Gamma 1. So, these are primary variables and these are secondary variables. So, we know that. So, the hat indicates it is already known and it is specified. So, this is what the boundary conditions are.

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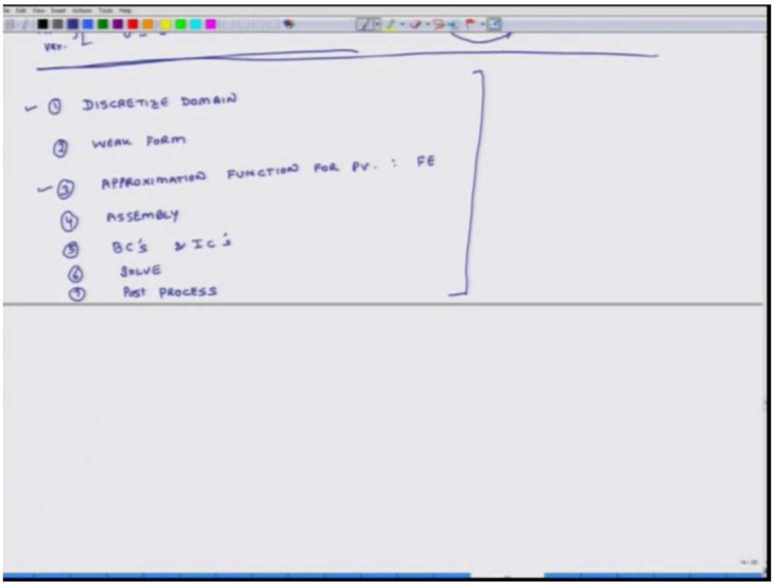

So, now we will start developing a finite element solution for this problem and what is the process? What is the first seven when we start doing finite element solutions? We discretize the domain.

Second step is we develop a weak form. So, we developed a weighted residual statement and then from that weak form of this statement. Once we have the weak form of the statement then what do we do? We have already discretized the domain for each domain we have developed a week from then we introduced approximation functions. So, in the first step domain has been broken into smaller parts. In the second step we have put weak form, in the third step the first step we broke the domain or we approximated the domain right. When we discretized it we are approximating it and then the third step we are approximating the function for primary variables. So, this is first approximation, this is second approximation. So, once we do this approximation for primary variables, we get. What we get? We get FE equations; finite element equations. For each element we get a set of equations. Then what we do? Assembly, then we applies boundary conditions. Now in this case, is it just boundary conditions? So, here I am also including time affects right.

So, if it is a dynamic problem I have to apply boundary conditions as well as initial conditions. The sixth step is we solve and seventh is we post process the data to compute whatever entities we are interested in strains and stresses whatever moments. So, this is a overall road map and this road map has been consistently throughout this as well as the previous course, this has not changed and in any finite element analysis problem this is the road map. You discretize, you develop weak from well in some cases you may not develop a week from, but you will have to do within strong form, but somehow you develop. Then you apply approximation functions for primary variables, you get finite element equations. You assemble them, apply boundary conditions and initial conditions solve it and for post processing you do post processing operations. So, now, we will start doing that. We will start with our development of weak form.

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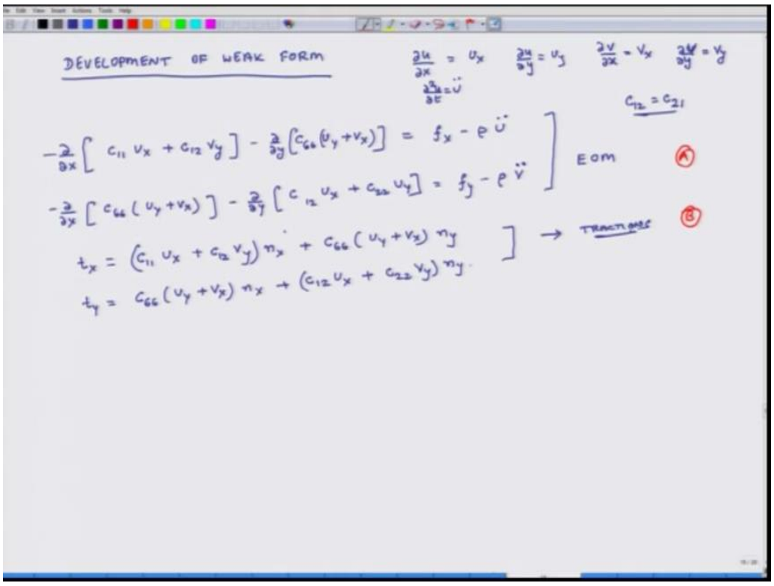

So, we will write down the equations. Now as we write down these equations for purposes of gravity I will write Del u of partial derivative of u with respect to x as U x partial of u with y is U y partial of v of x as V x and partial of v of y s V y.

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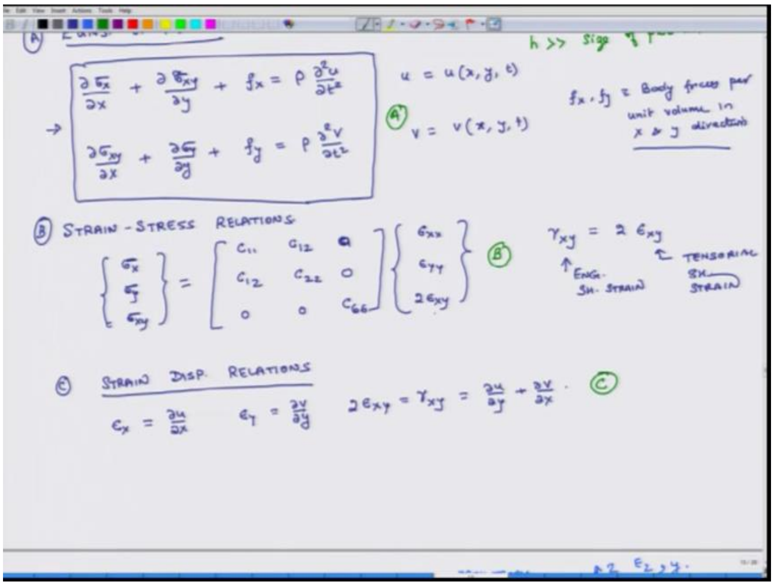

So, my equations of motion let us call this equation A. These are equations B and these are equations C. What we want to do is, we want to rewrite equation A only in terms of Us and Vs. So, first step we do is we put the definitions of sigma x sigma y and tau x y from equation B into A. Once we put B into A, we will have A only in terms of strains, agreed? Once we put B into A we get only equations involving strains. Then we put C into the new A. So, then those strain related equations become displacement based equations. So, it is fairly mechanical process. So, you can do it in your homes. That is one part of the thing we will not worry about in this course, but you please do this and convince yourself that whatever equations we are going to use further, they are correct.

So, if I plug C and B into A I get equations in U and V only. So, what are those equations? So, how many questions we will get? 2 equations. They were 2 equations of motion; we will still get 2 equations. So, the first equation is minus Del over Del x C 1 1 U x plus C 1 2 V y minus Del over Del y C 6 6 U y plus V x is equal to f x minus rho U dot, dot. So, what does U dot, dot mean? It is the second derivative of U with respect to time. Similarly from the second equilibrium equation, I get C 6 6 U y plus V x minus Del over Del y C 1 2 U x plus C 2 2 U y. So, please note that C 1 2 is equal to C 2 1. This is a property of orthotropic materials. So, that is why we do not have C 2 1 here. Here we do not have C 2 1 because C 1 2 and C 2 1 are same and this is equal to f y minus rho V acceleration in y direction and then we had relations for tractions right.

So, we will also rewrite relations for tractions as t x is equal to C 1 1 U x plus C 1 2 V y n x plus C 6 6 U y plus V x n y and traction in the y direction equals C 6 6 U y plus V x n x plus C 1 2 U x plus C 2 2 V y n y. So, these are equations of motion and these are for tractions. The second set of equations we will use later when we do the weak formulation. We have not yet started the weak formulation, but what we have done till so far is now that we have a displacement based formulation.

Everything is in terms of displacement based variables U and V. So, that is what we have accomplished. So, let us call this equation A, let us call this equations B. Now we will develop the weak form. Now how many equations we have? We have 2 equations. So, to develop a weak form what do we do? We multiply first by a weight function and then integrate it over the domain and that integral is equated to 0 and then we reduce the differentiability requirement.

So, now we have 2 equations. So, both these equations will be multiplied by 2 different weight functions W 1 and W 2. The first equation will be multiplied weight function W 1, the second equation will be multiplied weight function W 2. Then we will compute the residue weighted residue and then we will weaken the differentiability.

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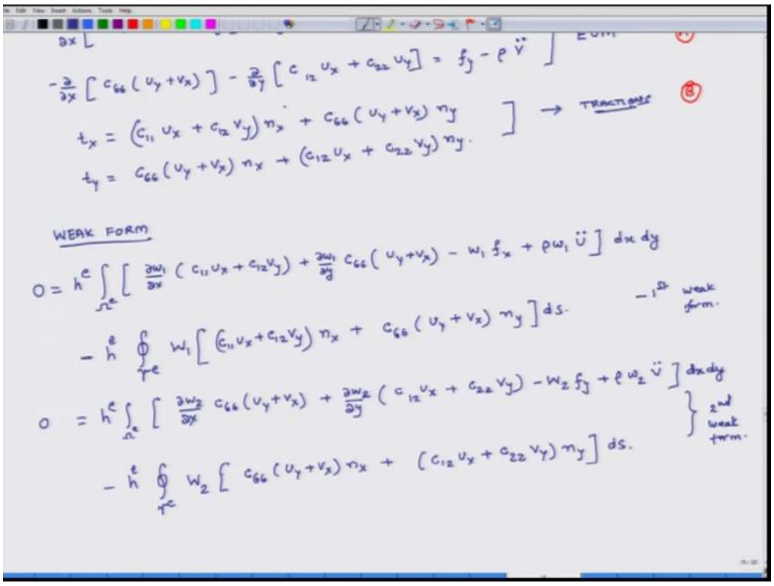

So, once we multiply the first equation by W 1, when we weaken the differentiability we have seen again and again that when we integrate this thing whole thing over the domain and we weaken the differentiability we also get a boundary term. We get a boundary term. So, what I will do is we know all these mathematics and we have done this several times. So, I will directly write the weak form. So, the week from for the first equation is 0 equals h e. So, the only thing is that we have developed the weak form and we have multiplied the weak form by a constant which is the thickness of the element h e. That is because later we will calculate some parameters and if we multiply it by thickness then they are physical significance. So, h e is a constant the entire equation we are multiplying it by.

So, this is first weak form, this is from the first equation. Similarly from the second equation, so in the second equation we are going to multiply it by which weight function? W 2 it is a different weight function. C 2 2 V y minus W 2 f y plus rho W 2 second derivative of v in time, integrated over the domain and I am yeah I am integrating it over domain and actually to be a little clearer because I am in Cartesian frame of reference then, this d omega is nothing, but d x d y and d x and d y and then I have a boundary term.

So, this is the second weak form, I am sorry this is h e and h e and we have said that it is a plane elasticity problem. So, the entire body is enclosed between two parallel planes and the distance between those two parallel planes is h e. So, the thickness of the body is h e. Now in this weak from if the thickness of the bodies extremely small then, you go for a plane stress assumption and you accordingly choose your Cs and if the thickness of the body is extremely large then, you go for a plane strain assumption and again appropriately choose your Cs, but the problem formulation does not change. So, these are the two equations. This is first equation, this is the second equation. Now we look at this entire function. So, before we move further we look at the boundary terms.

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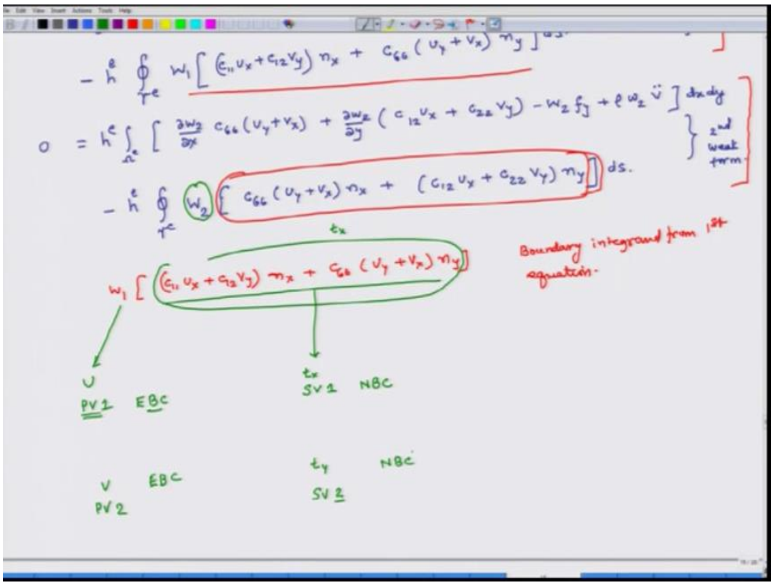

In the first equation the boundary term is W 1, see this is the boundary term W 1 times C 1 1 U x plus C 1 2 V y n x plus C 6 6 U y plus V x n y right. This is boundary integrant from first equation. Which means that, this tells me w 1 is the variation in first primary variable which is U. So, this means that U is a primary variable and this entire term, what is this entire time? We have defined it earlier. This is traction t x agreed. So, this is t x and this is a secondary variable. If I is specify the primary variable, I am specifying the essential boundary conditions. If I specify the secondary variable, I am specifying the national boundary conditions.

So, this entire thing is t x. Similarly W 2 appears in the second equation boundary term which means the other primary variable is V. So, this is the other primary variable and if I specify this variable, I am specifying essential boundary condition and the other secondary variable is this thing. So, that is t y that is the other secondary variable. So, this is primary variable number 1, this is this is primary variable number 2, t x is secondary variable number 1, t y secondary variable number 2 and if I specify it I say that I am prescribing national boundary conditions. So, with this understanding;

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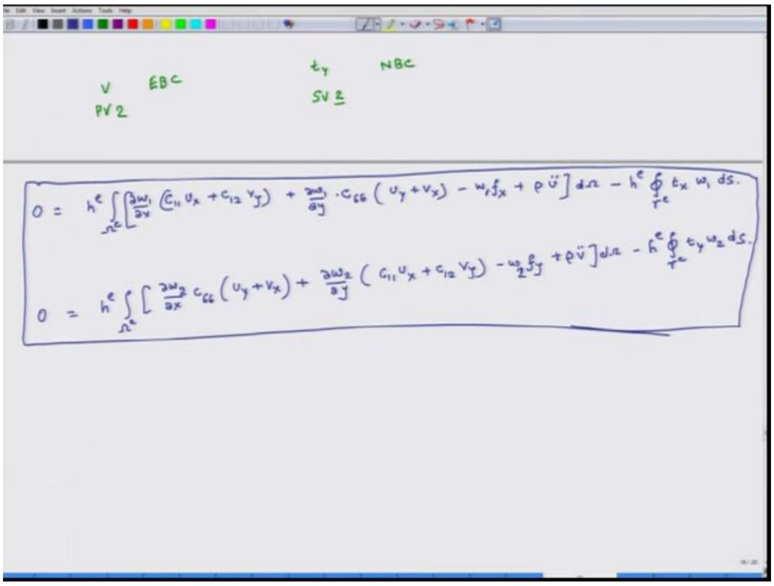

We rewrite the two equations once again in the weak form. So, 0 equals minus h e integral over the domain Del W 1 over Del x C 1 1 U x plus C 1 2 V y plus Del W 1 over Del y times C 6 6 U y plus V x minus W f x plus rho u dot, dot integrated over the domain minus h e boundary integral of t x W 1 d s [FL] and the second equation is 0 equals minus h e omega e Del W 2 over Del x C 6 6 U y plus V x plus W 2 over Del y C 1 1 U x plus C 1 2 V y minus W f y plus rho v double prime d omega minus h e Phi t y W 2 d s.

So, these are the two weak forms for plane elasticity problem and depending on what type of Cs we choose we can either address plane elasticity, a plane stress or a plane strain problem. Actually first thing I have to say is that this negative sign will go away. What else.

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W

Student: (Refer Time: 26:35) of x

This?

Student: W (Refer Time: 26:39)

W 1

Student: (Refer Time: 26:42)

W 1, W 2

So, this closes the discussion for today. Tomorrow we will develop element matrices for this problem and then, we will also learn how to assemble them, solve them and also work out some specific details for this problem in next couple of lectures. So, that covers it for today. We look forward to seeing you for seeing you tomorrow.

Thank you very much.