Basics of Finite Element Analysis - Part II
Prof. Nachiketa Tiwari
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture – 39
2-D Heat transfer problems (Part-I)

Hello, welcome to the Basics of Finite Element Analysis Part II, this is the seventh week of this course. And today is the third day. What we will do today is solve a problem on heat transfer and. So, essentially we will be solving the Poisson equation in context of heat transfer. And as we solve this problem, we will also learn trick or a technique were by we can reduce the size of the domain, for a symmetric problems by factors of two four or even eight or even larger.

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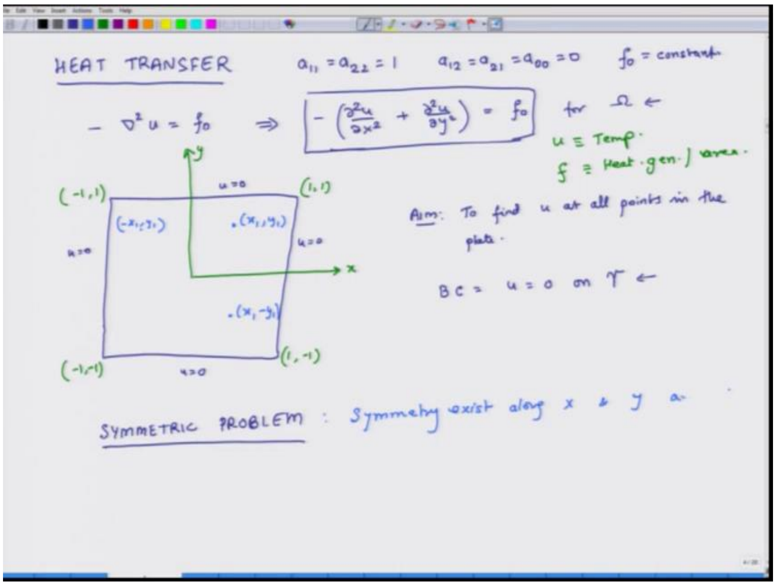

So, first let us look at the problem. So, this is the problem for heat transfer. Here, we are assuming that a 1 1 is equal to a 2 2 is equal to 1. And a 1 2 is equal to a 2 1 is equal to a 0 0 is equal to 0. So, in that case, my governing differential equation, becomes minus del square u equals f naught, and we also assume that f naught is equal to constant. Just to make our problem simple. Even though, we do not lose any complexity in these things. So, in or this can be written as minus del to u over del x square plus second derivative with respect to y and this equals f naught.

So, this is the governing differential equation. And this governing differential equation is valid at what locations. It is valid for the domain of the problem. So, whatever is the domain? So, in this case, we say that the domain of the problem is rectangle. So, we can say that it is like a plate. So, it is a square plate, and for the purposes of the simplicity. So, here this is my x axis is my y axis. So, for the purposes of simplicity, the size of this plate is 2 by 2. So, the coordinates are 1, 1. 1, minus 1, minus 1, 1 minus, 1 minus 1 and it is minus 1, 1.

So, the size of the plate is 2 by 2. So, here u represents temperature. And f represents heat generated per unit area. So, this differential equation, this is valid at all the points at any point in the domain. It is not valid on the edge on the boundary. Also, if we have to known the boundary conditions of the problems otherwise we cannot solve the problem. We say that, the boundary condition is such that on all 4 sides, u is equal to 0 on this side, u is equal to 0 on this side, u is equal to 0 on this side, and u is equal to 0 in this side.

So, what I have done, I have basically on all four sides let us say I put it on in ice, but the. So, on all the four edges, I have ice and that is what I doing. So, our aim is to find t or not temperature, t u at all points in the plate. So, the boundary condition, which I said is gamma. So, note that here, I have not put the superscript e. Because, this is the domain of the entire plate this is the boundary of the entire plate. When I am talking about at elemental level, then I will put a superscript e. So, this boundary condition is valid for the entire plate for all, the boundary and this equation is valid for all points in the system.

Now, one way to solve this is, I can break this up how, what do we do? First step is we desensitize the domain into a lot of small elements. Then we develop a week form. Then we put interpolation function with we develop assemble, the element level finitely assemble element level equation, f e equations assemble, then apply the boundary conditions and solve it. So, in that case, I will get a solution no problem, but we also classes of problems we can also explain the fact if it is symmetric. So, what we see here is that in this case. So, this is a symmetric problem. What does it means, what it means is that when we get solution of it, just by looking at this problem. We can say that the solution at location x y will be shelled.

So, what does symmetry mean, that if I have any point, Let us say x 1 y 1. Then the solution at x 1 minus y 1 will be same. If that is the case, then there is symmetry across the x i x axis. If the solution is also same at minus x 1 y 1, then there is symmetry across the y axis. So, in this case, just by looking at the problem, I can say that this a symmetry problem and it is going to be symmetric across the x axis and also across the y axis. Why do I say that? So, I can say that here, symmetry exists along x and y y. So, this problem is symmetric let us, just consider the x axis we can use the same argument for y axis also I will explain.

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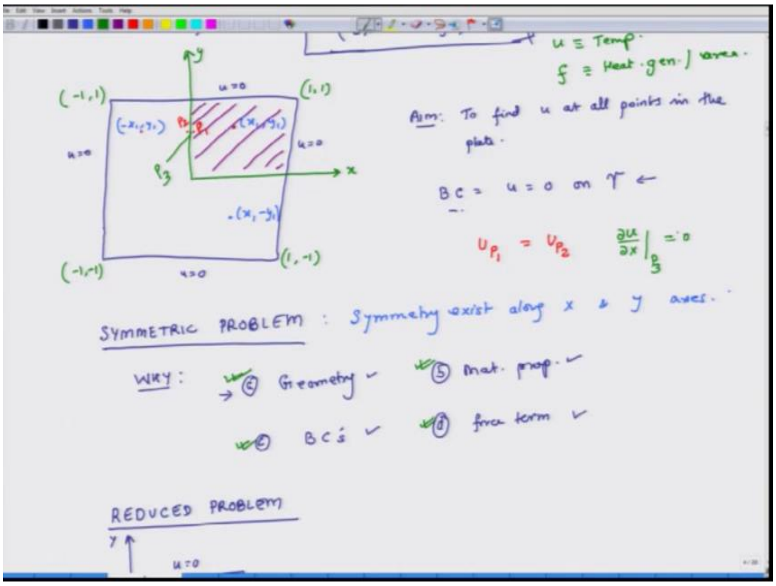

Why is this symmetric along the x axis? It is symmetric along x axis because of a combination of four reasons see the problem of the problem will have a solution, and its solution will depend on what, it will depend on a geometry right. The solution is going to depend on geometry. It will also depend on material properties right. Third thing what else it is going to depend on, it will depend on boundary conditions right, And the fourth thing it will depend on, it depends on four things the four thing look at the equation x relates to geometry right. x and y they relate to geometry, there is b c and then also we have said that a 1 1 a 2 2, they are related to material properties. So, we have considered geometry, we have considered material properties we have consider boundary condition. There is one more thing in the equation which we have not considered. What is it? f naught right u is the solution. So, we that is something we have to predict, but f naught. So, if the force term and in this case the force term is heat generated per unit area.

So, it will depend on four parameters, it will depend on geometry, it will depend on material property, it will depend boundary conditions, and will depend on four term. Now if I think that the problem is symmetric along x axis, then the geometry has to be symmetric along x axis right. So, I have positioned my coordinate system in such a way that it is indeed symmetric it is a square plate it is symmetry in the along the x axis right in this case. So, I fulfill this condition. Second thing is the material properties have to be symmetric along x axis what; that means, is that if they is any point x 1 y 1. Then at another point on the other side of the x axis at the same distance, the properties should be same. In this case, the material properties are a 1 1 a 2 2 is equal to 1, which means these properties are same in the whole field.

So, properties are symmetric, third condition boundary conditions should be symmetric. What do I see u is 0 on the top edge, and u is 0 on the bottom edge? So, boundary conditions are also symmetric, and the fourth condition is force term, force term is f naught. And the value of the f naught is same, it is not a functions of x and y it is a constant. So, again it is same at all the places. So, that is also symmetric only if all these four conditions are satisfied, we can exploit the feature of symmetry. If even one of these is not exploited, then we cannot say that the problem is symmetric in nature. So, this is very important to understand all these things have to be simultaneously true, and then I can explain the feature of symmetry.

So, what we see here, is that the problem is symmetric not only along the x axis, but it is also symmetric along y axis. So, in that case, what I will do is. I can only discretize 1 fourth of the domain. And solve for only 1 fourth of the domain. I do not have to solve for the whole domain what; that means, is that if I require 4 n elements for the entire domain right. Then if I am only solving for one fourth of the problem, I need only n element. So, my computation will be faster because the number of equation will be go down by the factor of four. So, lot of times you may see in the your problems presence of symmetry, if it is symmetric along one plane you can divide your problem complexity goes down by the half, if it is symmetric along two planes it goes on by factor of four it can be symmetric about three planes or it could be asymmetric also. So, we will exploit this feature of symmetry.

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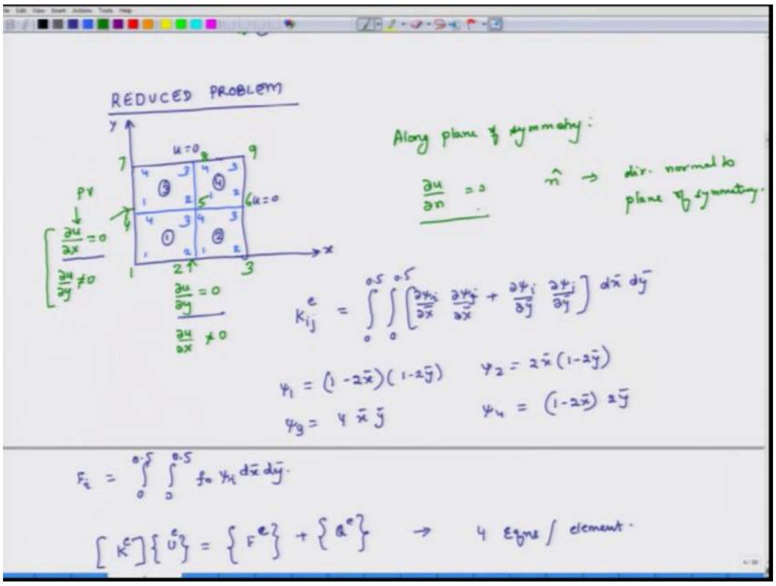

So, then my reduce problem is this. So, please note we have not shifted the coordinate system. Because once we have put the coordinate system at the center, the location of the coordinate system also tells us whether it is symmetric or not. So, now, we are taking the first quarter of the problem, and the coordinate system has note shifted. And this is the reduced problem. Now what is the boundary condition on the top edge, u is equal to 0 on the right edge, u is equal to 0. Now the other thing we had to note is we have to know the boundary condition on these two edges. Here, in the original problem we knew boundary condition on all four edges unless we know the boundary condition on these two edges. We cannot solve the problem. So, what will be the boundary condition on this edge, what is the boundary condition on this edge? I will give an answer you have to first understand the physics of the problem.

Suppose, there is this point, x x 1 y 1 and 1 the other side it is, minus x 1 y 1 the value of u at x 1 y 1 and minus x 1 y 1, will be same because of symmetry. So, when you take these 2 points, which are very close to each other let us saying this is p 1, and this is p 2. The value of point p 1 and p 2 across the line of symmetry will be same right. Which means that the value of u at point p 1 will be same as u at point p 2? It is the same thing we had said it. So, many times right because of symmetry which means, what and if these 2 points are extremely close to each other, if they are extremely close to each other. So, let us say the midpoint is this p 3 which is at the center.

Then what is the value of del u over del x at point p 3. What will be the value? 0. It will be 0. So, what symmetry tell us is, direct mathematical consequence of the symmetry tell us is that on this edge, the derivative del u over del x this is my primary variable. The derivative of primary variable along the plane of symmetry, in the direction normal to the plane of symmetry will be 0. It does not mean that del u over del y will be 0 right on this line. So, it will be only del u over del x will be 0, what about here on this line. del u over del y will be 0, but del u over del x will not be 0, that tangential component will not be 0, the normal component to the boundary will be 0.

So, in general I can say that along plane of symmetry del u over del n where n is equal to 0, where n corresponds norm direction normal to actual normal to the plane of symmetry. So, with that understanding we will do. So, these are the extra boundary conditions, I will enforce del u over del x and del u over del y 0 on this plane. So, with that understanding what I do is I break this up. And let us say I break it up into four elements for purposes of this class. And I my global node numbers are 1, 2, 3, 4, 5, 6, 7, 8, 9. And my local node numbers are 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, and 4. And my element numbers are element 1, element 2, element 3, element 4, agreed.

So, if I do the math k i j for e-th element. If I calculate it in local coordinate system you have seen it earlier then it will be. So, what is the size of each element, overall size was 2 by 2, now the size of the overall domain is 1 by 1. So, each element is half by half. So, if I am doing this in local coordinate system. So, I will integrate between 0 to 0.5, 0 to 0.5, agreed. So, this is d x bar, d y bar. And because I am using local coordinate systems I say x bar and y bar. And what are psi i and psi j. So, for each element how many psi will be there. For each element how many approximation functions, will have 4 approximation functions, because a 4 nodded linear rectangle. So, psi 1 is equal to 1 minus 2 x, we have derived this relation earlier. So, I will not go back, psi 2 is equal to 2 x bars into 1 minus

2 y psi 3 equals 4 x bar y bar and psi 4 equals 1 minus 2 x bar 2 y bar. So, using these we can develop expression for each k matrix and also for the f vector d x d y.

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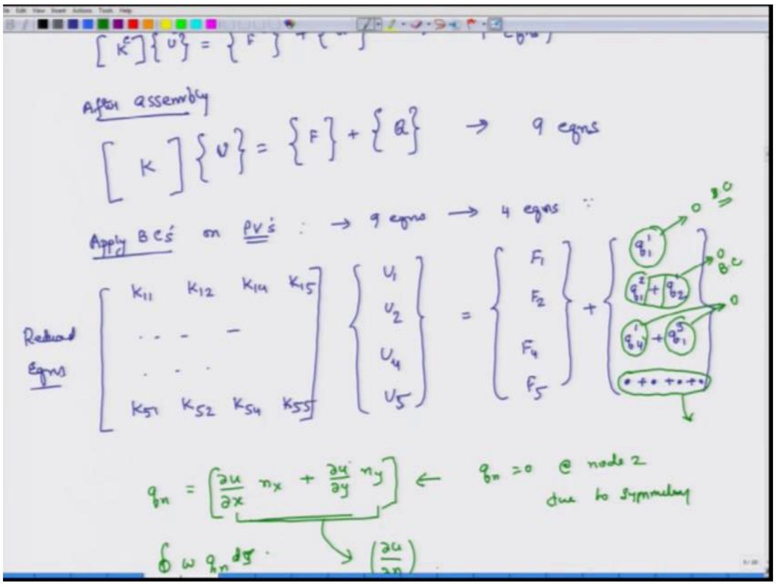

So, eventually I will get the k matrix time's u. So, this is for e-th element right and finally. So, this will be how many equations for each element. So, it will be 4 equations per element 4 equations for each element. And my after assembly, I will get an overall global k matrix times u vector equals, a global f vector, plus a global q vector, and how many equations these be, 9 equation why because in the global domain we have 1, 2, 3, 4, 5, 6, 7, 8, 9 nodes right. So, we have 4 equations per element. So, at element level will get 16 equations. And those 16 elements 16 equations will be reduced to 9. Because of elimination of 7 degrees of freedom, what are those 7 degrees of freedom? They are associated with node 2, node 4, node 5, node 6 and node 8.

So, what we will do is. So, I get 9 equations. And then I apply, boundary conditions right, I apply boundary conditions, but before I apply boundary conditions. And what are the boundary conditions? First related to the primary variables right, the boundary conditions are that u 3 is equal to 0, u 6 is equal to 0, u 9 is equal to 0, u 8 is equal to 0, and u 7 is equal to 0. So, how many terms are 0, 1, 2, 3, 4, 5 right.

If I apply boundary condition on primary variables, then these 9 equations reduce to how many 4 equations. Because u 3 is equal to u 6, is equal to u 9, is equal to 3, 6, 9, 7, 8 u 7 is equal to, u 8, is equal to 0. So, eventually I am left with, a reduced number of equations and the size of those numbers of equation is 4 rights, because these five things got eliminated. And what are the things which are left u 1 u, 2 u, 4 and u 5. And here, we have global k 1 1, k 1 2, k 1 4, k 1 5 and similarly, we can fill up this matrix. The last row will be k 5 1, k 5 2, k 5 4, and k 5 5. So, these are reduced number of equations.

And here, I get f 1, f 2, f 4, f 5 plus. I get some q terms right. So, this will be q and I will not write the global thing, I will write it. So, the first this will be because of. So, I am going to erase this one because it is generating some confusion. So, this will be q 1 1 plus right. f 1 will be q 1 1, q 1 1, then q 2 will be q 2 1 plus q 1 2, third thing is q 3 is 0 what is what else is q 4. So, it is 1 4 plus 3 1, 1 4 plus 3 3 1. And the last one is the fifth and this will have four contributions. So, it will have I will not write this, but there will be four terms, now this q 1 2 plus q 2 1, first let first look at this term, q 1 1. Now we know that qs. So, these are basically related to q n, if you go back to our earlier thing. We had written that as, del q del u over del x and x plus del u over del y and y this thing had to be integrated right. And the term here, was basically integral over the domain w q n d omega d s this is how we use to get at elemental level right.

Now, q n in this case, for q 1, what is the n y, n y is 0. n y is the direction cosine of the normal to corresponding to the this edge 1 3 right. So, this is 0 and also del u over del x at the first node this is 0. Because, of boundary condition related to symmetry. Same thing here, this q n is equal to 0 at node 2, due to symmetry. See this thing, this I can also write it as del u over del n. And we know that on a plane of symmetry del see this is along any plane of symmetry del u over del n is 0. And this is nothing this entire expression is nothing, but del u over del n. So, when I multiply this del u over del n with w and d s I am integrated, because the integral q n is 0 its integral will also be 0.

So, this is also 0 because, of boundary condition symmetry. What about node 4 nodes, 4 is this, again on this at node 4 del u over del n is 0. So, it is not that the sum is 0 individual components this is 0 and this is 0. Same thing here individual components are 0, is not that the sum is 0 individual component themselves are 0. Because of symmetry and finally, we look at the force term and force term relates to this mid node. What is this value?

So, these first three terms, in the q vector they were 0 because, of the boundary condition related to symmetry. What about this sum we had said that, if a node is not if a node is shared on all four sides, on all four sides if it is shared. Then the flux which is going in is same as flux which is coming out. So, when I add up all the individual, if it is shared from all the sides, then because of that condition this is also 0 rights. So, it just happens in this case, that the entire q vector is 0, but that may not be case in all situation in other situations you will have to look at it. So, with that understanding I get a k matrix,times u matrix is equal to times f matrix. So, these are reduced equations, and I solve for u. And with that approach I can solve it very easily and fastly. Because I have reduced the complexity of my problem by a factor of four, I just, I am just solving only one fourth of the problem.

So, this is what I wanted to discuss, and what we will do is will continue this discussion in the next class. I think we have spent significant time of today on this, topic and with that we close the discussion for today and we will meet once again tomorrow

Thank you.