Basics of Finite Element Analysis – Part II
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Lecture – 30
Interpolation Functions for Linear Triangular Elements
(Part - II)

Welcome to Basics of Finite Element Analysis Part II. This is the fifth week of this course and today is the last day of this week. What we will do is we will conclude our discussion on interpolation functions for linear triangles.

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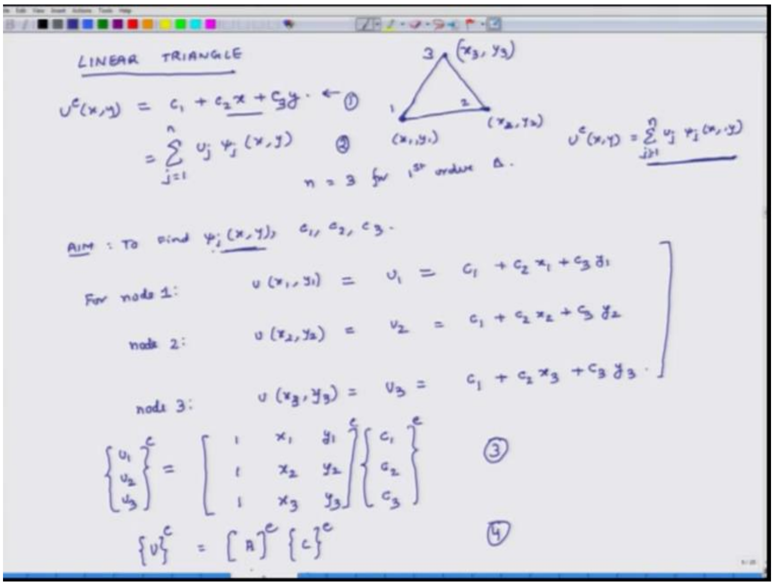

And in the last class we had started discussing singular and non singular matrices. So, what we are trying to do in this particular endure is that, we are trying to express c’s in terms of u’s. What are u’s? Physically, what do u’s represent? They represent the values of unknown variable at nodes. So, u 1 represents the value of u at node 1, u 2 represents the value of u at node 2 and so on and so forth.

So, what we are trying to do is this c’s, they do not have any physical meaning. So, we are trying to express c’s in terms of u’s. We do not know us also. We know x 1, y 1. But we are trying to express c’s in terms of coordinates of the nodes and the displacement or the variables at those specific nodes. So, that is what we are trying to do. And in that endure we had said we had develop this expression four, where u equals 3 by 3 matrix a times a c vector.

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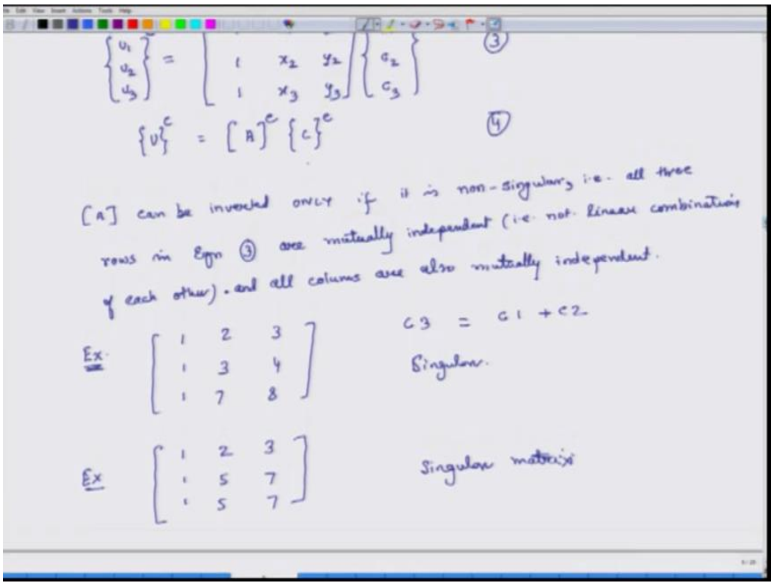

And we had said that this can be inverted only. So, I can express c in terms of u only if I can invert a and I can invert an only if the matrix a is non singular.

Now, what does singular matrix represents? This is another important thing to understand. What does this matrix represent? Matrix A represent is nothing but representation of the coordinates of the nodes. X 1 y 1 is node of coordinate of first node, x 2 y 2 is coordinate of second node and so on and so forth. So, matrix A represents coordinates of the nodes of the element.

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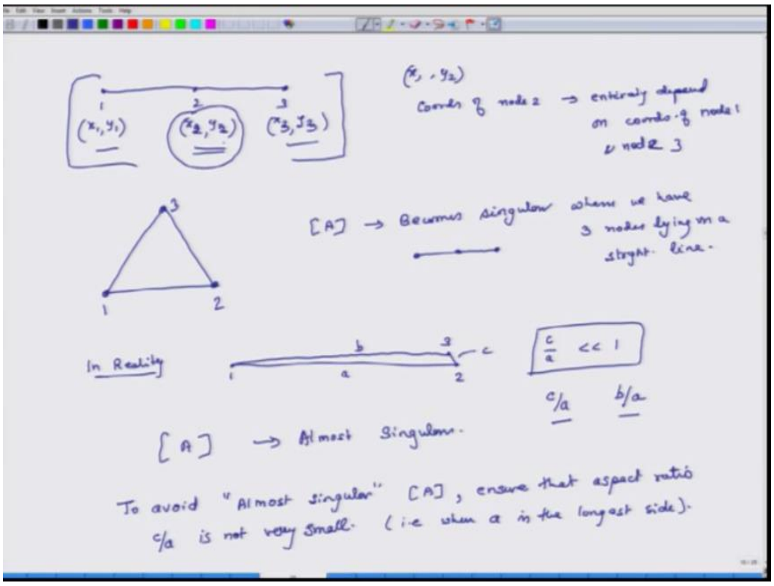

Now, if you have three nodes 1 2 3. So, this is x 1 y 1, x 2 y 2, x 3. So, this is x 3 and this is x 2 y 2. Here x 2 y 2, the coordinates of this are totally dependent. They are not independent of the nodes on, of the coordinates of node 1 and node 2.

So, coordinates of node 2 entirely depend on coordinates of node 1 and node 2 which means if I put these three nodes in my e matrix, my e matrix will be singular because this node is not independent of node three and node 2. If I have a situation like this; so this is node 1 and node three. In this case if I have a triangle then, coordinate of each of the nodes is totally independent of other two nodes, but the thing cannot be said when nodes lie on a straight line. So, physically a becomes singular when we have a straight line, when we have three nodes lying on a straight line.

If there are three nodes which lie on a straight line, then we will have a matrix which will become similar and when it becomes similar, we cannot invert it and if we cannot invert it, we cannot express these cs in terms of u's. So, where ever mesh is being generated, we have to make sure that the nodes never accidentally lie on a straight line. Now in reality, what could also happen is, I can have a triangle, which looks likes this; 1, 2, 3. I could have a triangle which looks like this.

Here, so it has three sides; a, b and c. Here, side c over a is extremely small compared to one. In this case, the coordinates when c in this case, coordinate of node three and node 2 will be very close to each other, which means the a matrix. Mathematically, it will not be singular because they do not lie on the straight line, but it will be all most singular. Essentially what that means is that, the determinant will become an extremely small value. And when we are doing some computations, we have round of errors and truncation errors. Then, even small changes in that small number, in the determinant, will cause large errors. So, what all this discussion tells us is that when we discretize a domain we should not have elements with extreme aspect ratios.

The same thing holds true for rectangular elements and so on and so forth. So, to avoid I will call it almost singular, a matrix ensure that aspect ratio c by a, is not very small. Do not very small. So, for a triangle you will have to calculate three aspect ratios, c by a, actually two aspect ratios, and b by a. And none of these ratios should be extremely small. In this case, c by a is extremely small. In some triangles, v by a could be very small. So, you should. So, is not very small should be very small.

That is, when a is the longest side. Because if you accidentally choose c as a n a s c then, it will be extremely large. So, a general role could be that either it should not be a extremely small or it is should not be extremely large. If it is like twenty thousand it is a bad element. If it is like 0.0001; this ratio then again it is a bad element. It can create a lot of numerical problems in your solution. It may give you a solution, but that solution may not be trust worthy. So, this is very important consideration.

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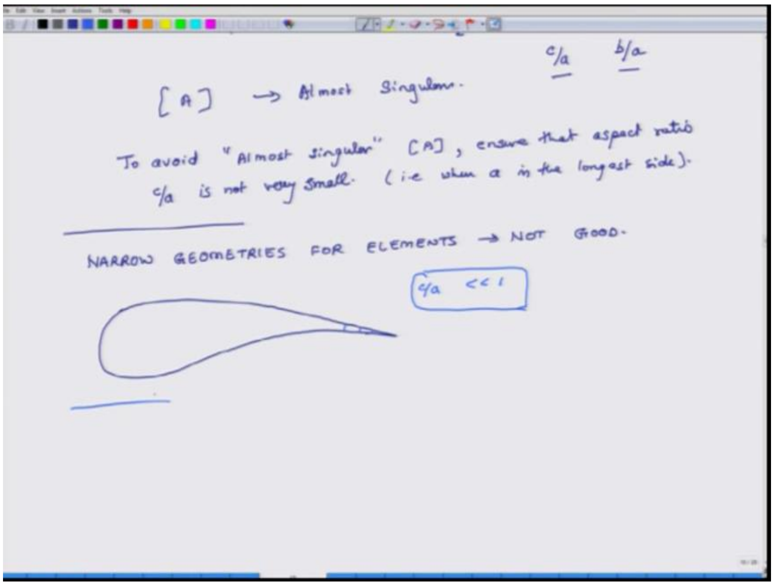

So, in general, narrow geometries, for elements not good. Narrow geometries are not good. Suppose you have an air foil type of a structure and you want to analyze it, you can discretize it. This could be one element. In this case, c by a will be very small. Then how do you avoid this problem. You avoid this problem by making smaller elements. So, that you are a becomes small. Then c by a is automatically starts between reasonable. So, this is important to understand, because sometimes you may have to take very narrow geometries. But when we are talking about narrow geometries, we are talking in contexts of this ratio. So, this is the ratio which is important.

So, with that understanding we now proceed, invert this matrix u is equal to a times c. So, the first thing we will do is, we will develop an expression for a.

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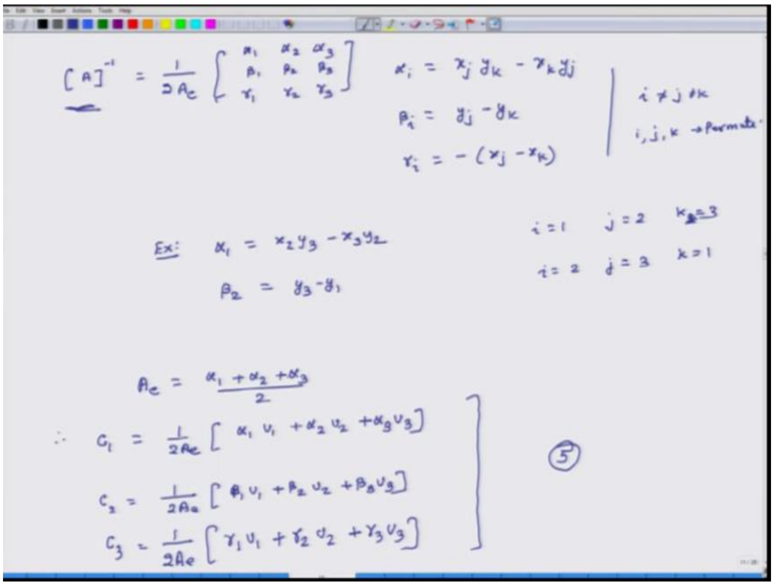

So, a inverse equals 1 over 2 a e and I will through in some. So, a inverse is the inverse of that a matrix and it is equal to some number 1 over 2 a e times this matrix which is alpha 1, alpha 2, alpha 3, beta 1, beta 2, beta 3, gamma 1, gamma 2, gamma 3. What are the alphas and betas? Alpha i equals x j y k minus x k y j. beta i equals y j minus y k and gamma i equals minus x j minus x k. Here i is not equal to j, is not equal to k and i j k permute.

What does that mean? Example, alpha 1; if I have to do alpha 1, if I have to calculate alpha 1 then, i is equal to one j is equal to two and k is equal to 3; it is not a summation. Then alpha 1 equals x 2 y 3 minus y x 3 y 2. Suppose, I have to calculate beta 2, the beta

2 is equal to. So, here i is equal to 3; no sorry 2; j is equal to 3, k is equal to 1, 2, 3, 1. So, beta 2 equals y 3 minus y 1. So, understood how we calculate? So, using this formula I can calculate alpha 1, alpha 2, alpha 3 and all other betas. And a e equals alpha 1 plus alpha 2 plus alpha 3 divided by 2. So, therefore, c 1 equals 1 over 2 a e alpha 1 u 1 plus alpha 2 u 2 plus alpha 3 u 3. c 2 equals 1 over 2 a e beta 1 u 1 plus beta 2 u 2 plus beta 3 u 3 and c 3 equals 1 over 2 a e gamma 1 u 1 plus gamma 2 u 2 plus gamma 3 u 3.

So, we understand how these relations came for c's. Basically, I have inverted a matrix and I have plugged that a matrix inverse in this relation three or relation four. So, from that I can express c's in terms of u's; a is alpha 1 plus alpha 2 plus alpha 3 divided by 2. So now, we have accomplished our first goal that is, to express cs what was the goal? We have to explain c in terms of us which we have done. Now we have to find expression for psi j. So, now, that is actually pretty straight forward.

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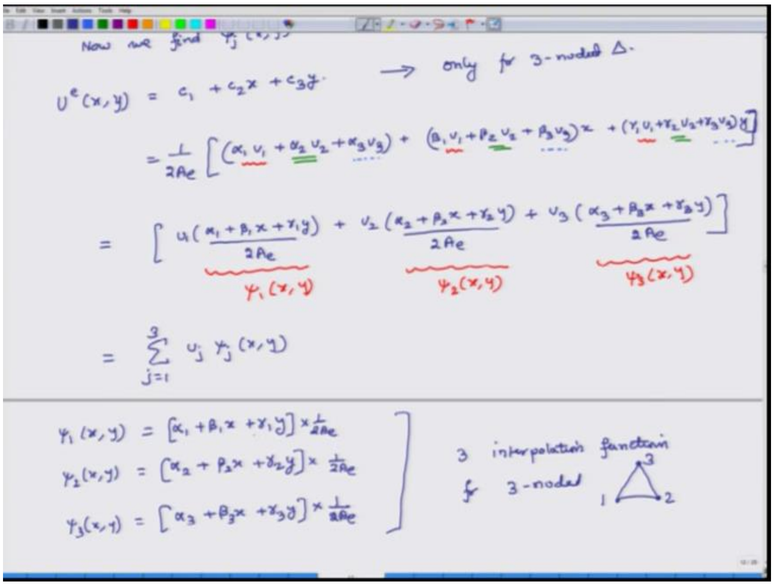

So, now we find psi j has a function of x and y. So, we know that u of e which is a function of x y is what? C 1 plus c 2 x plus c 3 y. This equation is valid for which type of element? Only for a three nodded triangle.

If it is a rectangle, this expression will change. All this mathematics is going to change. If it was a rectangle, we will use the same approach. But this matrix, it will be four by four. This will be a four by four matrix. If it is a six nodded triangle, which is a quadratic triangle, it will be a 6 by 6 matrix and so on and so forth. So, the procedure is going to be the same but all these details will change. So, u is equal to c 1 plus c 2 x plus c 3 y and we have already calculated the values c's. So, this equals 1 over 2 a e and what is c 1? Alpha 1 u 1 plus alpha 2 u 2 plus alpha 3 u 3; so this is alpha 1 u 1 plus alpha 2 u 2 plus alpha 3 u 3. So, that is my c 1 plus c 2 x. So, that is beta 1 u 1 plus beta 2 u 2 plus beta 3 u 3 times x plus gamma 1 u 1 plus gamma 2 u 2 plus gamma 3 u 3 times y.

Now, what we do is. So, we have developed the expression for u. What we do is, we collect terms involving u 1. Then we also collect terms involving u 2 and then, we collect terms involving u 3. So, what we get is, 1 over 2 a e u 1 alpha 1 plus beta 1 x plus gamma 1 y plus u 2 alpha 2 plus beta 2 x plus gamma 2 y plus u 3 alpha 3 plus beta 3 x plus gamma 3 y, agreed? This function I call it is psi 1 x y because it depends on x and y only and it is multiplied by, no it is, and of course. So, actually I will just to be consistent and accurate. I will divide this by 2 a e 2 a e 2 a e. So, this function I call it psi 1 x y. This function I call it psi 2 x y. This function I call it psi 3 x y.

So, we see that psi 1 is multiplied by u 1, psi 2 is multiplied by u 2, psi 3 is multiplied by u 3. So, this can be expressed as, j is equal to one two three u j psi j x y where psi 1 x y equals alpha 1 plus beta 1 x plus gamma 1 y psi 2 x y equals alpha 2 plus beta 2 x plus gamma 2 y. And psi 3 x y equals alpha 3 plus beta 3 x plus gamma 3 y times 1 by 2 a e. So, these are the three interpolation functions for three nodded triangle. Three nodded triangle.

Here what we see is that, psi 1 depends on x and y. o, all these functions they are functions of x and y and the values of alpha 1, beta 1 and gamma 1, they are depend on the coordinates of these three nodes. If you see what are alphas, betas and gammas? These are nothing but they depend. They are functions of coordinates of nodes. So, alpha beta and gammas, they are known constants. So, this is actually. They are known constants and these constants depend on the coordinates of node. So, if I change the from of one element to other element.

The coordinates of the node change and because of that these psi functions also changed. And while developing these psi functions, we have ensured that the value of these functions is same as chronicle delta i j because that condition was implemented we were developing these first set of equations. So, here we had said that u of x and y is u 1 which implied implicitly that psi 2 and psi 3 were zero and so on and so forth.

So, because of that reason these psi's they satisfy our requirements that they are one at individual nodes, and at all other nodes they are zero and their maximum value is only one. So, this concludes the discussion for today and also for this week. What we will do in the next week is, we will do a couple of examples for these linear interpolation functions and then actually solve one or two problems for one or two dimensional single variable problem and also we will learn how to assemble these equation, because now we know how to develop these equations at element level.

And then the next thing is, we have to figure out how to assemble these equations at the assembly level and then impose boundary conditions. With that we conclude our discussion for today.

Thank you, and have a great day. Bye.