Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 45
Temporal approximation for parabolic problems : Part-I

by
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Hello, welcome to basics of finite element analysis, in the last class we had shown as to how to convert a partial differential equation in X in time into an ordinary differential equation through this spatial approximation process. Today what we will do is we will convert that ordinary differential equation into a set of algebraic equations by integrating that particular equation in time. So this approximation process is known as temporal approximation.

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So our, so temporal approximation, so here our goal is convert ODE into algebraic equations.

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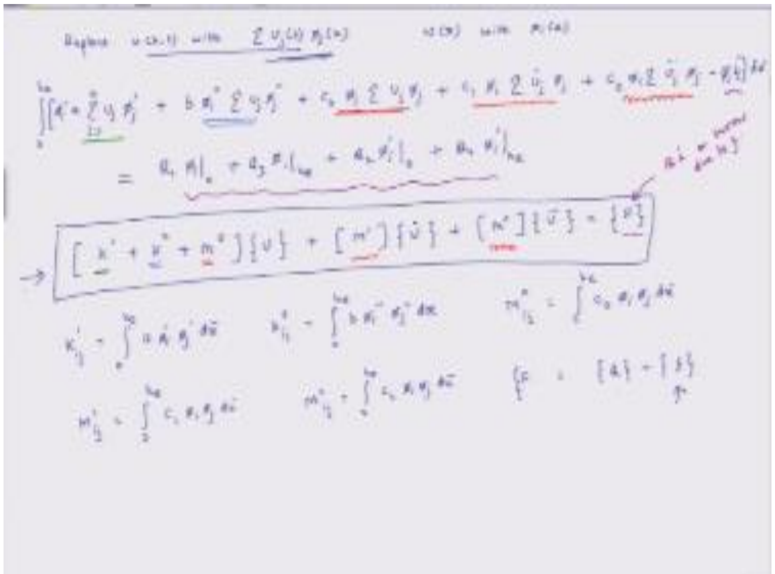

So what we will do is we will write down rewrite this equation.

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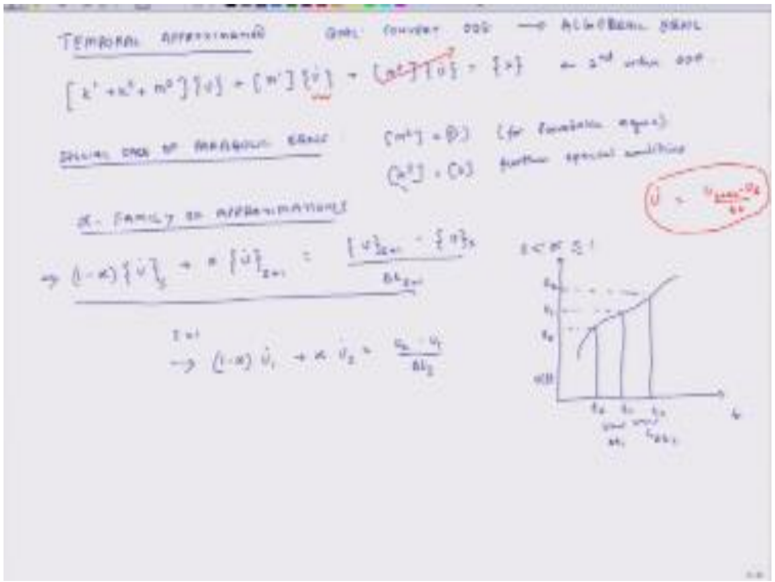

So our k1 + k2 + m0 times displacement of primary variable vector pus m1 times u plus m2 times second derivative of u equals a force vector it is f. So right now this is an ODE, second order ordinary differential equation and our aim is to somehow integrate this in time. So that we once we, once we integrate in time we will get algebraic equations, okay. So we will do this process for a special case and we will do it for special case of parabolic equations.

So in parabolic equation the condition is that the second derivative of u it does not play a role. So that means that m2 is equal to 0, this is for parabolic equations, okay. The other thing we will do is we will now go one more special case that it is the question is not only parabolic but k2 is also zero. Further the special conditions we are assuming this, okay. If that is the case but the solution procedure we will describe does not necessarily depend on this second requirement but why I am making this assumption is that makes physical sense.

Because for heat conduction transient heat conduction problems in one dimension k2 is zero. So just to connect it with some physical reality but the overall solution method for parabolic equation what we will discuss today is valid whether k2 is zero or it is not zero it does not matter. So our aim is to convert this ordinary differential equation into algebraic differential equation.

So the way we do it is that we express derivatives so in this case we do not have, these terms are gone. So we have to somehow convert these u dots derivatives of u in time to differences of u, okay. Now we know that one way to express u dot is equal to suppose time is changing then I can, can call u dot as ut+ Δt – ut/ Δt and if Δt is very small then it approximates to u dot at that particular point.

So we will use similar type of approach to express u dots in terms of differences of u's and of course there will be a time factor because these derivatives are dependent on time. So this type of approximation is what we will use and we will use a particular scheme of approximations to convert derivatives of time into differences of time and this scheme is known as alpha family of approximations, okay.

And first I will write the expression and then I will explain it. So what does alpha family of approximation it says? That 1-α times u. at step number s α times u. at step number s+1, so s stands for step number okay, equals u at step number one s+1 minus u at step number s divided by Δt and which Δt? The (s+1)th time step, okay. And here α is a number and it can vary between 0 and 1.

So let us see what this means. So suppose I am having this is u time. So u is some variable which is a function of time. So I am plotting it and this is my time axis, okay. And let us look at three points, okay. So let us say this is t0, this is where the time is starting, this is the first time step. So this is Δt1 and then the second time step is between t1 and t2, so that is Δt2, okay. And the corresponding values of u's are u1 and u2, excuse me u0.

So it is a little messed up here and I will make it clearer, so this is u2, u1, u0 so in the x, y- axis I am plotting u(t) and the x- axis I am plotting t. And this is how ut is changing with respect to time. So I wanted to explain the meaning of this statement, it is important to understand. So for instance consider s = 1 which is the first time step. So the first time step is Δt1 long, the second time step is Δt2 long, okay.

S is number of the time step number. So if s equal to one then what does it then this expression this expression which I have written it says 1-α times u dot of one plus α of u dot 2 is equal to u2 – u1/ Δt 2, okay. So this is what it means that at when step number is 1 then this relation which I wrote earlier it boils down to this relation. Now let us see what this means further.

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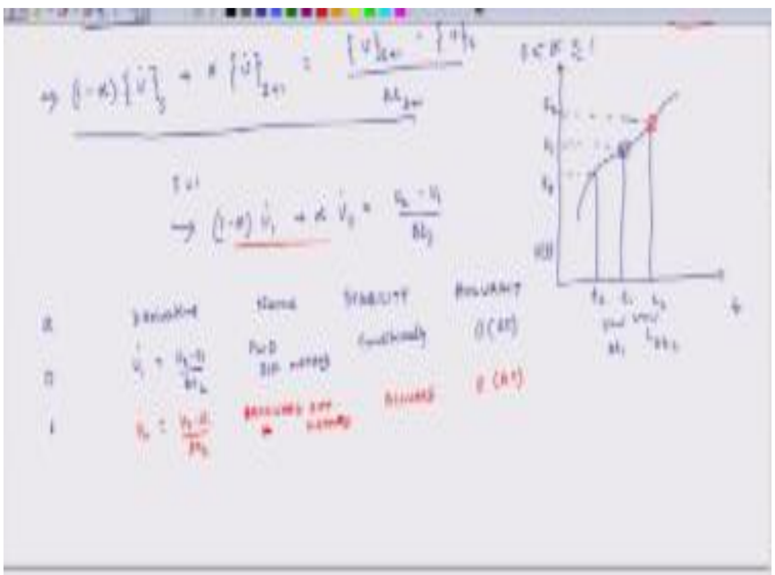

So we will make a small table. So we will have α derivative, name, stability, and accuracy, okay. So first consider when α = 0, α is some number which varies from 0 to 1, okay. So alpha consider alpha is equal to 0 then in that case so if α is equal to 0 then the left side becomes u1 no u2 – u1, right? So it becomes u1 dot is equal u2- u1/ Δt2. Now look at this picture, so what it says is that the derivative at this point the derivative of time, a derivative of u at this point which is encircled is equal to u1, u2-u1/ Δt2, right?

This formula when alpha become zero is known as forward difference method. And it is conditionally stable, I will explain stability later. So it is conditionally stable and accuracy of this method is of the order of Δts. So if you reduce Δt it will become more accurate if you double the if you reduce Δt by a factor of two it becomes twice as accurate, okay. So what this forward difference method or it is also known as Euler's methods it says that the derivative at u1 is equal to the value at u2 minus value at u1 divided by $\Delta t2$, okay. Similarly derivative at u2 will be u3-u2/ $\Delta t3$.

So this is forward difference method. The second method is when α equals one, so here if α equals one then we again look at this equation if $\alpha = 1$ this u1 dot goes to zero. So it says u2 dot equals u2-u1/$\Delta t2$. This is called backward, backward method, backward difference method, this method is always stable and here also the accuracy increases linearly with time, if time becomes half then accuracy becomes double and so on and so forth.

What does this mean? So this is a little different than the first method, here it says that if I have so if I use the forward difference method u2-u1/Δt is considered to be the differential of u at the first point, here it is considered as the differential at the second point, okay. So the same value in first backward difference method I say that it is the derivative at an earlier point of time, in the second situation I say that it is at a later point of time, there is a fundamental difference.

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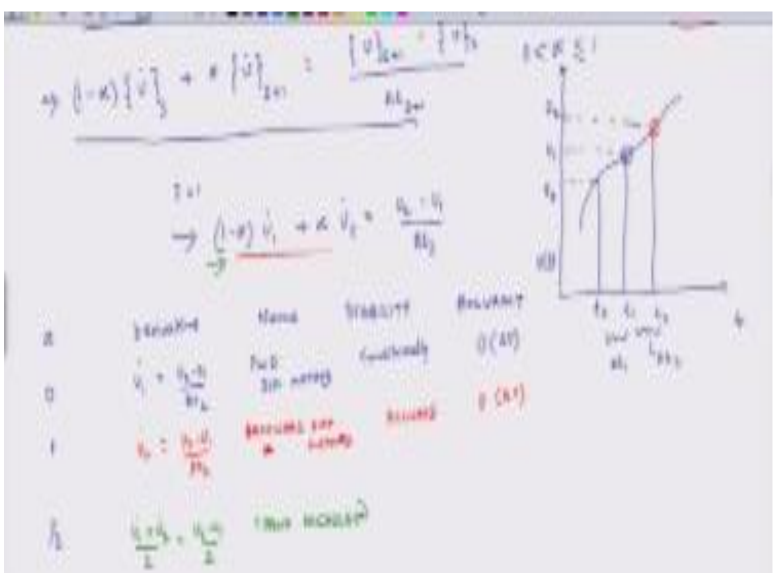

But if I use the second approach that approaches it is always stable and I will explain what is stable later but it is always stable. The first approach is stable in certain situations and we will explain that later. And then I can have different values of alpha between 0 and 1. So another value of alpha I can have is half and here what does it say? It says that u 1 dot plus u2 dot divided by 2 again I am going to plug this alpha in this equation, equals u2 – u1/ 2. This method is known as Crank Nicolson method.

So there are different methods for calculating derivatives, in each derivative I am taking the same differences u2-u2/2 but in one case I call it u1 dot, in another case I call it u2 dot, in the third case I call it u1+u2 dot divided by two.

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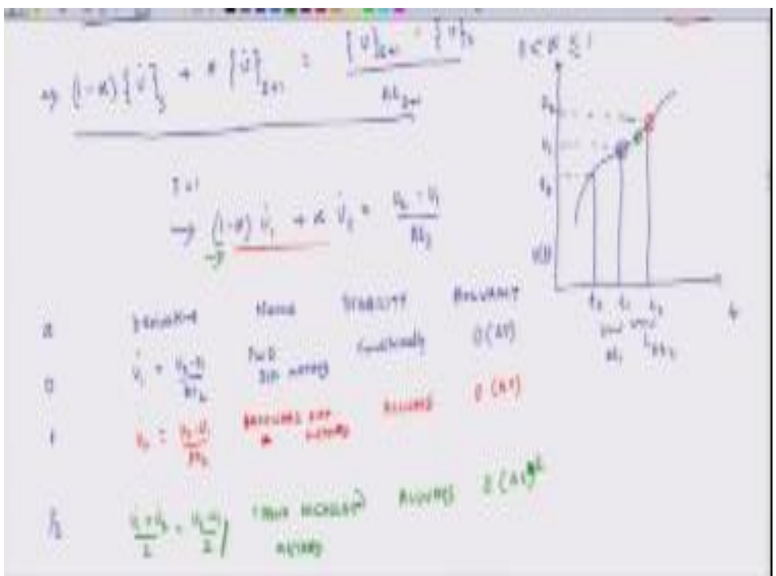

So this is always stable and here the accuracy increases quadratically with respect to time increments, okay. So if I have my Δt my accuracy goes up by a factor of four, this is important to understand. Graphically what it means? That u2 so this is u1 dot this point is u1 dot it says that if I calculate this term u2 – u1/ 2 essentially what I will get is the time derivative of u at this point, the central point between u1 and u2, okay.

So physically this green dot makes more sense because it is balanced between the two ends, then there are other approaches also where alpha is equal to two over three that approach is known as Galerkin method, Galerkin method of approximation and there that system is always also stable and then there is, and there also the accuracy increases with square of time, so that is the, so overall we have.

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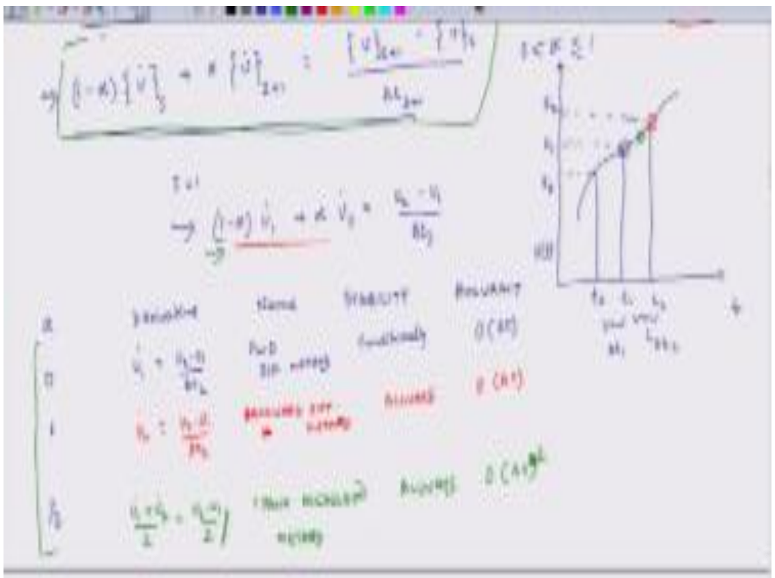

So, but all these methods for calculating derivatives they belong to this alpha family of approximations and the general relation is given here what we have shown here is, that when alpha when s is equal to one then what does this mean physically that is what we have explained.

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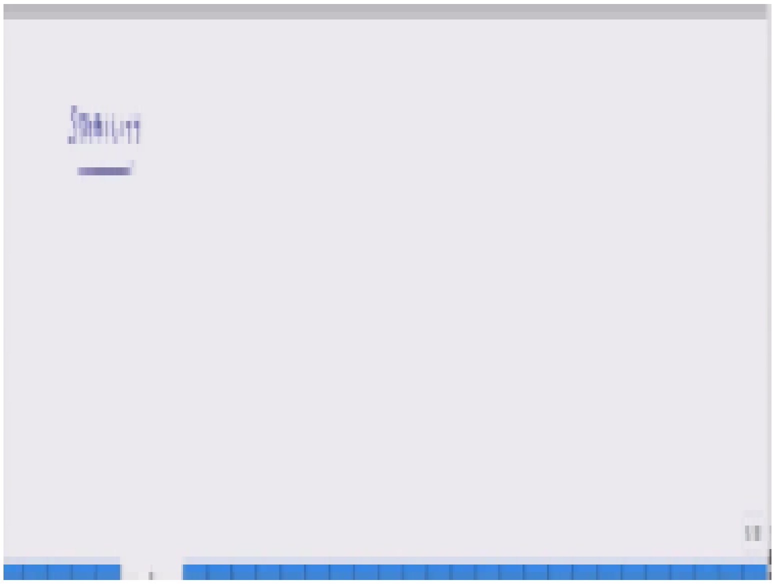

Now I have explained two terms accuracy and stability. So I will briefly explain these so stability and accuracy, what does accuracy mean? So what happens is that when we are doing finite element analysis we are approximating the time derivative.

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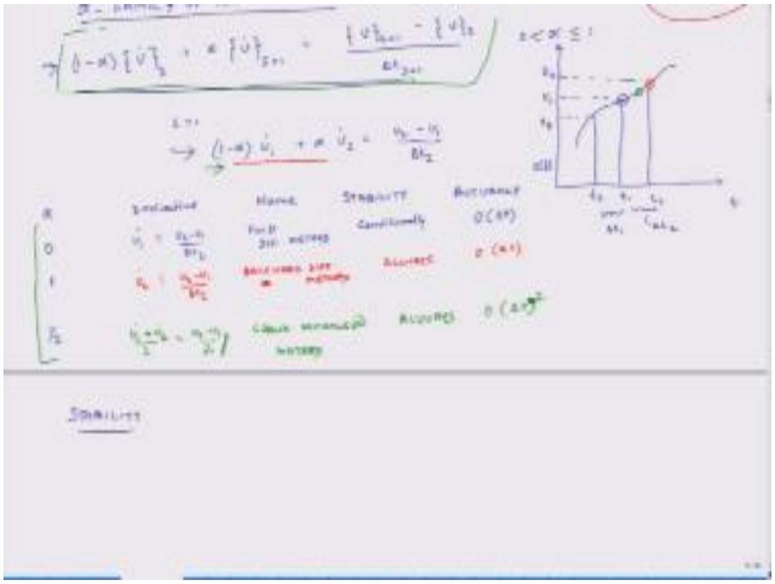

So remember these u2, u1 are not position coordinates they are coordinates in time. So we are approximating derivatives in time by differences in time, okay. Theoretically the difference in time has to be infinitesimally small but for practical reasons we cannot do that otherwise we will have to solve this problem in for infinite time steps we are in time-dependent problem not only discretizing the system in X but also discretizing in time, and this discretizion in time also causes some error. It also causes some error now depending on the method we use to calculate derivatives, forward difference method, backward difference method, Crank Nicolson or Galerkin method.

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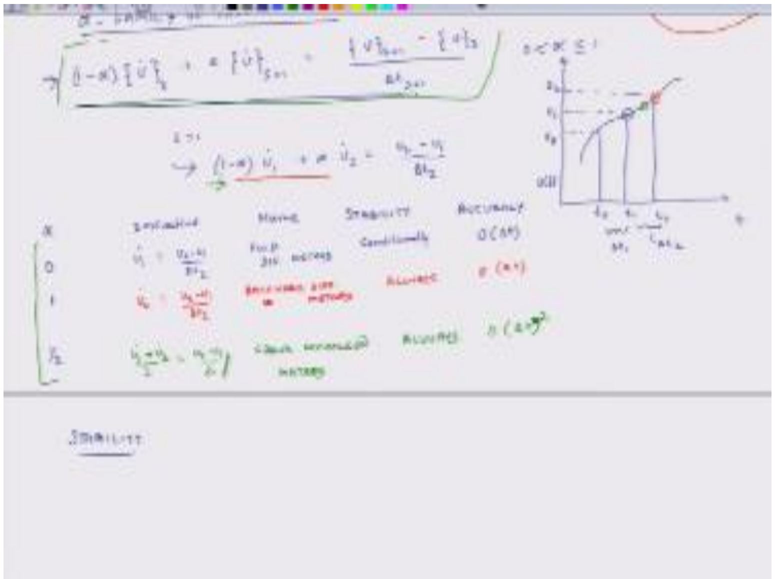

Some of in some situations suppose I in first step let us say I get an error of one let us say 0.1 percent. In second step this error will influence the solution for the next time step also, okay. So maybe it will become 0.2 percent, in the third step this error will again get added up because every time we are we are adding up errors. So if the system is stable what will happen is that in general maybe in first step time step you get 0.1 percent error in next time step the error is on negative 0.1 percent. So it adds up to zero or it becomes very small.

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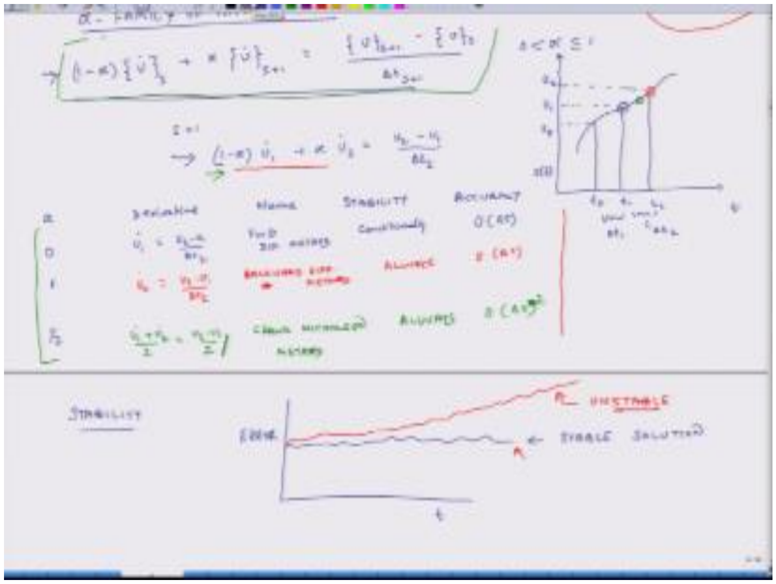

So what that means is that if I am seeing this and it is error then the error fluctuates between some limits as time is increasing, this type of a solution is called stable solution where the overall error level does not keep on increasing with time, but if the system is if the solution methodologies unstable then the error may grow with time. So as I move forward and further down in time my solution might become totally inaccurate, they may become totally wrong, here my solution is having some error but that error is controlled, okay.

So this solution methodology not methodology this solution is unstable, this is an unstable solution. So what this table tells us is that the backward difference method is, so what this says is that the backward difference method and the Crank Nicolson method provide this kind of a solution stable solution, the forward difference method provides stable solution only in some specific conditions not always, what are those conditions I will quickly explain later, but if we do not meet those conditions.

Then the forward difference method it provides an unstable solution. So then in that case we do not have that solution may not be reliable. So that is what is the meaning of stability. So what is the condition for stability for forward difference method?

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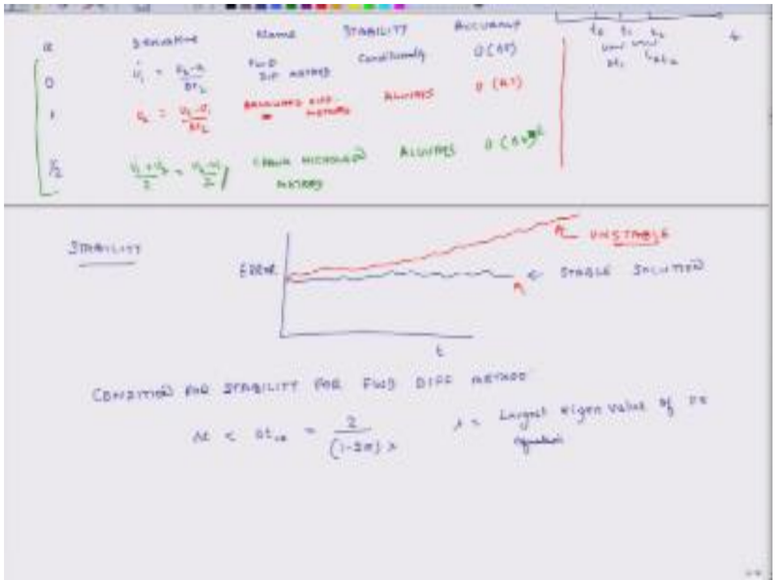

The condition for stability is that the time step should be smaller than a critical time step number Δt critical and this value is 2/1-2α times λ, okay. So in this case alpha is 0, so it will be basically 2/λ and what is λ? λ is largest Eigen value of f_e equation, okay. So first, so how do we calculate time critical to the critical time? We have defined the largest Eigen value from this finite element analysis and how do we find the largest Eigen value?

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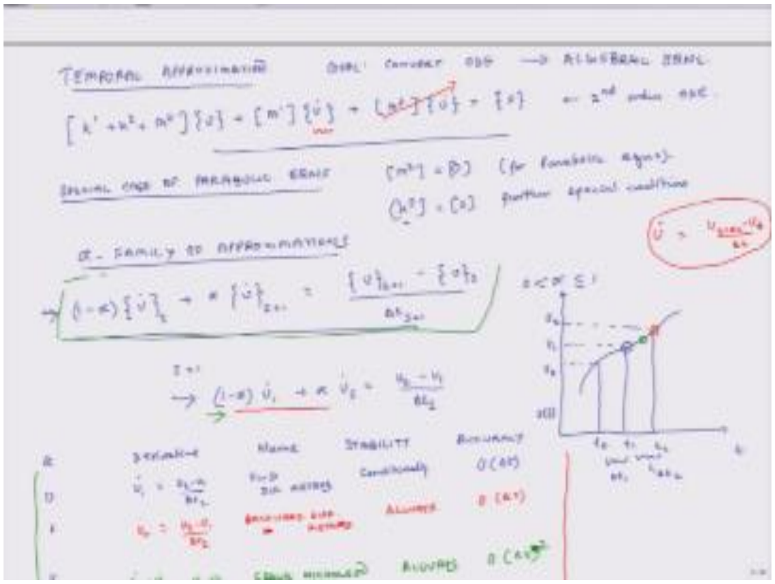

Essentially what we do is that in this equation how do you find Eigen values? The forcing function is 0. So we make this side 0 and then and we have already explained how to calculate Eigen values we find the Eigen values whatever is the largest Eigen value.

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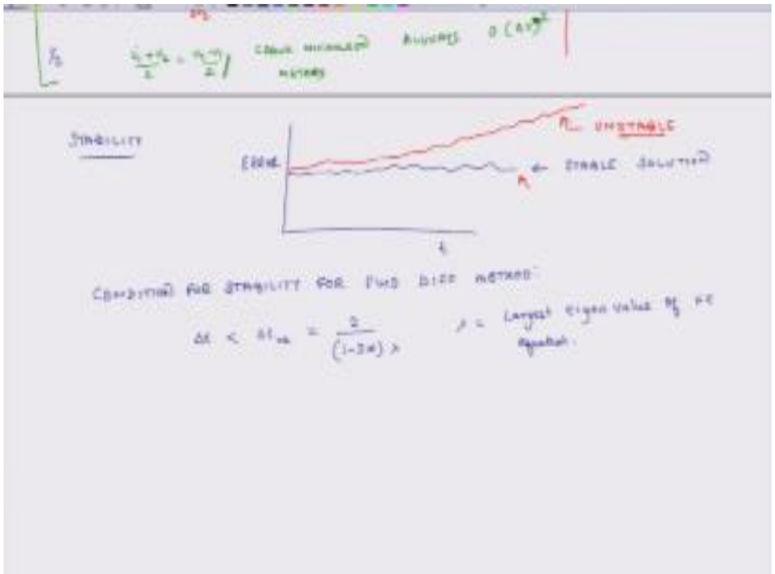

We plug that number here in this equation, calculate the time increment critical time increment and once we have calculated that time increment step we have to make sure that all my time increment steps remain smaller than that, if it is a smaller than that then the solution will be stable. So that is the meaning of stability.

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In context of this particular problem, the other one is accuracy.

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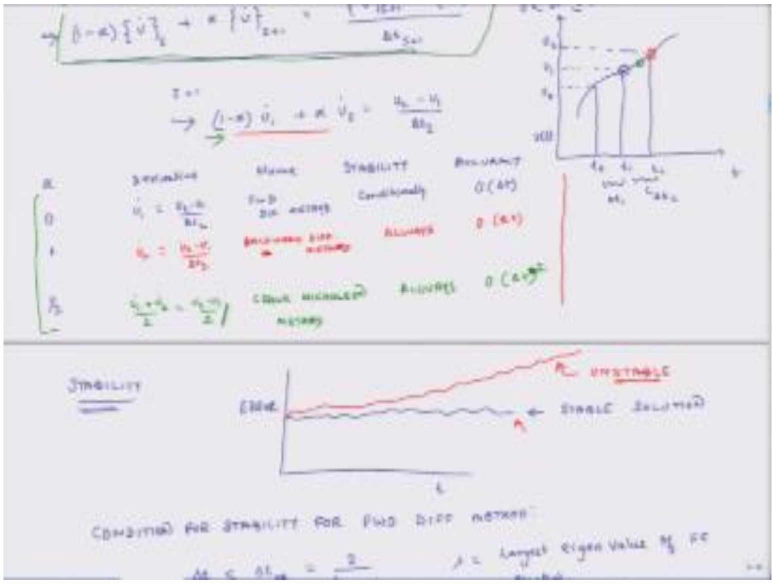

Because this table also tells you how accurate the process is.

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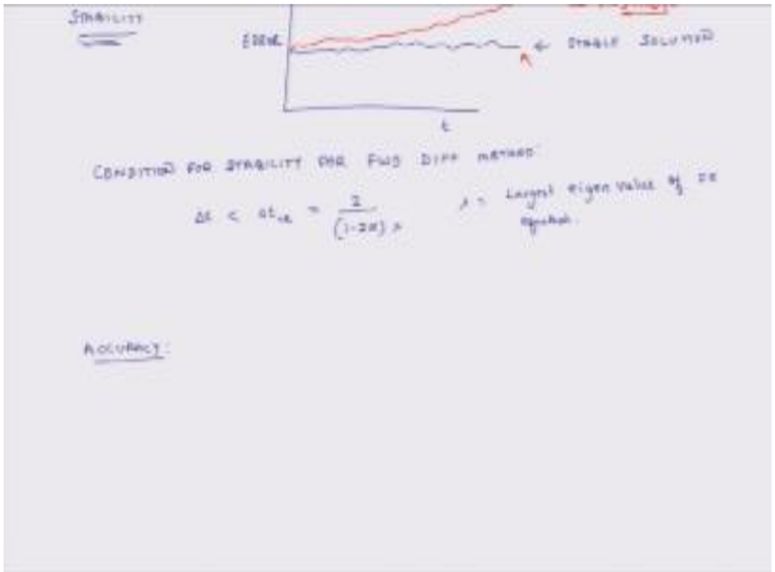

So accuracy means the difference between the finite element answer and the exact answer, okay.

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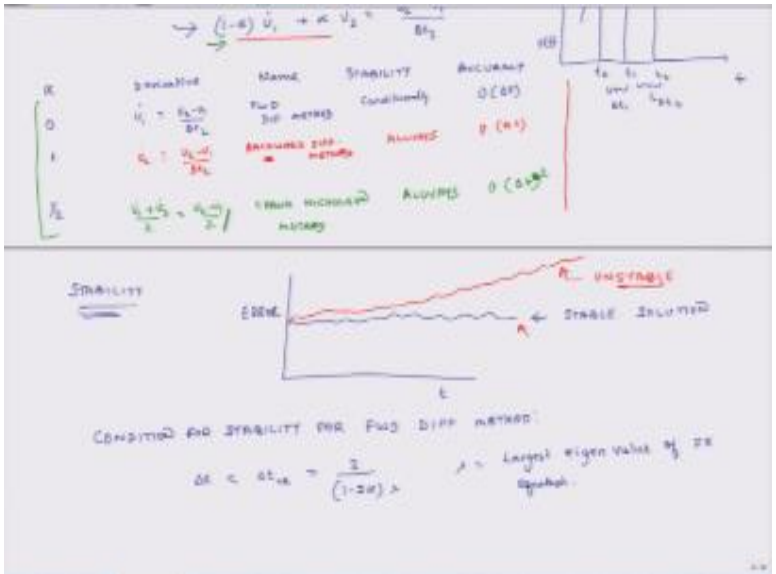

So what this means is that the forward and backward methods accuracy increases linearly with time, if I made my time increment half accuracy doubles, if I make it one fourth accuracy also becomes four times better but if I use this Crank Nicolson method.

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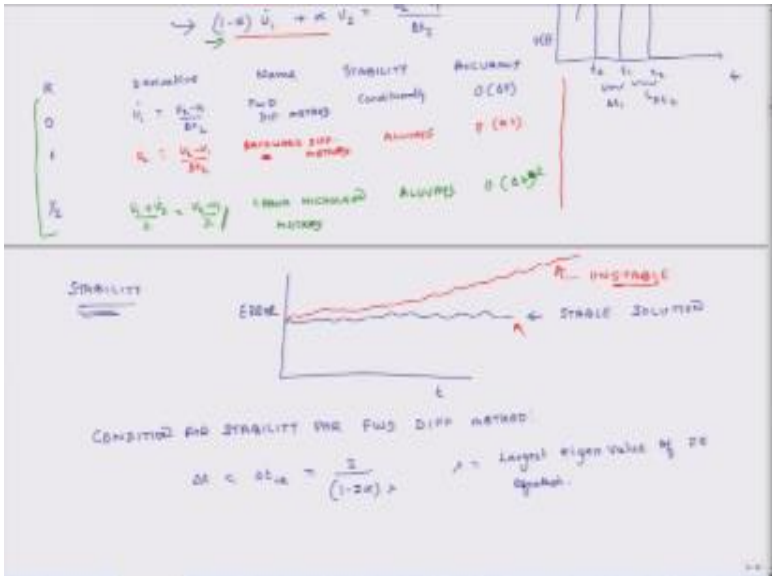

Then it increases at a square piece, so if I half the time increment accuracy becomes better by a factor of four and so on and so forth. So that is the meaning of accuracy and stability and what we will do is we will continue this discussion for hyperbolic equations in our next lecture tomorrow, thank you.

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