Mechanics of Fiber Reinforced Polymer Composite Structures
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Module-07
Elastic Behaviour of Laminates-I
Lecture-
Classical Lamination Theory-Part I

Welcome to the second lecture of this module. We have been discussing the macromechanical analysis of laminate. In the last class laminates, the requirement of laminates has been discussed. Different types of laminates, designation of laminates, layups and stacking sequence, stacking of different laminae with different fiber orientation to achieve required strength and stiffness in a laminated component have also been discussed in brief. The objective of macromechanics of laminate is to understand the response of a laminate subjected to force (axial force, shear force, bending).

In light of the fact that a lamina is made from large number of laminae stacked together, perfectly bonded at the interfaces and the fact that these laminae are orthotropic and heterogeneous, makes analysis of laminate little more involved compared to that in the case of components made from conventional isotropic and homogeneous metallic materials. One of the simplest theories put forward for analysis of such laminates was the classical lamination theory and in this lecture, classical lamination theory will be discussed.

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Classical Lamination Theory (CLT)

Assumptions:

1. Each layer is homogeneous and orthotropic.
2. Laminate is thin and its lateral dimension is much larger compared to its thickness and the laminate is loaded in its plane only i.e the laminae are in the plane stress state ($\sigma_z = \tau_{xz} = \tau_{yz} = 0$).
3. All displacements are small.
4. Displacements are continuous throughout the laminate.
5. In plane displacements vary linearly along the thickness of laminate i.e are linear function of z. $u, v \rightarrow$ *linear function of z*
6. Transverse shear strains γ_{xz} and γ_{yz} are negligible. This along with assumption 5 implies that a straight line perpendicular to the middle surface remains straight and perpendicular after deformation.
7. Strain displacement and stress-strain relations are linear.
8. ϵ_z is negligible compared to ϵ_x and ϵ_y.
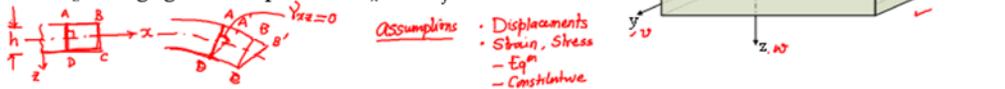
Classical lamination theory is based on some assumptions as follows:

1. **Each layer is homogeneous and orthotropic** – Actually each lamina is heterogeneous but in macromechanical analysis, the lamina is considered to be homogeneous represented by it is average properties even though the average properties are actually functions of the properties of constituent fiber and the matrix. Though the laminae are considered to be homogeneous, but are considered to be orthotropic with direction dependent properties average properties E_1, E_2, v_{12} and G_{12} and orthotropic means there is no in-plane shear extension coupling in the material direction.

2. **Laminate is thin and its lateral dimension is much larger compared to its thickness and the laminate is loaded in its plane only i.e the laminae are in the plane stress state ($\sigma_z = \tau_{xz} = \tau_{yz} = 0$).** Referring to the figure, then the out of plane stresses, $\sigma_z = \tau_{xz} = \tau_{yz} = 0$; this is true for thin laminate.

3. **All displacements are small** – meaning that the strain displacement relationship is linear and the nonlinearity which might occur because of the large displacement is eliminated.

4. **Displacements are continuous throughout the laminate**.

5. **In plane displacements vary linearly along the thickness of laminate i.e are linear function of z.**

6. **Transverse shear strains γ_{xz} and γ_{yz} are negligible**. This along with assumption 5 implies that a straight line perpendicular to the middle surface remains straight and perpendicular after deformation (as shown in Fig.) – it is kind plane sections remain plane even after deformation. As shown, in the x-z plane, considering a small rectangle ABCD (*refer lower*

left corner in slide 1), the line AD is straight and perpendicular to the mid surface before deformation. Suppose after deformation in the deformed shape, A, B, C and D, even after deformation, line AD remains straight and perpendicular to the middle surface. That means this is a pure rotation there is no shear stain. If there is shear strain in the x-z plane then the rectangle ABCD would not have remain as rectangle and it would have taken the shape of a parallelogram as shown. That means the point A would have moved to *A'*, B would have moved to *B'*, the measure of shear strain γ_{xz} is the deviation from the right angle as shown. Because $\gamma_{xz} = 0$, the transverse strain is negligibly small or zero and the line AD which was straight and perpendicular to the mid surface still remains straight and perpendicular to the mid surface after deformationSimilarly we can take a section in the y-z plane and show that $\gamma_{yz} = 0$ means a line which is initially straight and perpendicular to the middle surface will be so after deformation.

7. **Strain displacement and stress-strain relations are linear** means it obeys Hook's law.

8. **ϵ_z is negligible compared to ϵ_x and ϵ_y**. This means the laminate is thin; therefore when it is deformed change in length of this line AD is negligibly small compared to the thickness.

Assumptions are made on the nature of displacement, like, in-plane displacements are actually linear functions of z. Assumptions are made on the strain displacement relationship as linear. So, we have assumed displacement, we have assumed stress strain. Assumptions are also made on the stress strain relationship as linear following Hooke's law. Based on these assumptions, the overall the laminate characterization, that means the laminate constitutive relationship will be developed using classical lamination theory as will be discussed now.

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Classical Lamination Theory (CLT)

Displacements

$u_o, v_o, w_o \longrightarrow x, y$ & z components of displ. of the mid surface

$\left.\begin{array}{l} u_o = u_o(x,y) \\ v_o = v_o(x,y) \\ w_o = w_o(x,y) \end{array}\right\}$ ①

$x, u,$

y, v_o z, w_o

Ref. Plane

Rotations of x - axis

$\left.\begin{array}{l} \alpha_x = \dfrac{\partial w_o}{\partial x} \\ \text{Rotation of } y \text{ - axis} \\ \alpha_y = \dfrac{\partial w_o}{\partial y} \end{array}\right\}$ ②

From geometry, $\left.\begin{array}{l} u_B = u_o - z_B \dfrac{\partial w_o}{\partial x} = u_o - z_B \alpha_x \\ \text{Similarly,} \quad v_B = v_o - z_B \dfrac{\partial w_o}{\partial y} = v_o - z_B \alpha_y \end{array}\right\}$ ③

Any point at a distance of z- from the ref-plane

$\begin{array}{l} u = u_o - z \dfrac{\partial w_o}{\partial x} \\ v = v_o - z \dfrac{\partial w_o}{\partial y} \end{array} \Rightarrow \left.\begin{array}{l} u(x,y,z) = u_o - z \dfrac{\partial w_o}{\partial x} \\ v(x,y,z) = v_o - z \dfrac{\partial w_o}{\partial y} \\ w(x,y,z) = w_o(x,y) \end{array}\right\}$ ④

If we put $z=0$ in ④ we get the mid-surface displ.

If we put $z = \pm h/2 \rightarrow$ displ. of the bottom & top surface

Initial   Deformed

Considering an 'n' layer laminate the coordinate system is fixed at the mid surface of the laminate. Mid surface is nothing but the reference plane which is equidistant from the top and bottom surface of the laminate (as shown).

Suppose, u_o, v_o and w_o are the x, y and z components of displacement on the mid surface respectively. So, we can write

$$u_0 = u_0(x, y);$$
$$v_0 = v_0(x, y);$$
$$w_0 = w_0(x, y) \quad (1)$$

So, this is the mid surface displacement u_o, v_o, w_o along x, y and z are functions of x and y only. Considering a section in the x-z plane as shown, and considering a small element ABCD of the laminate ,where the straight line AD is in the x-z plane. A is the point on the bottom surface and the point D is the point on the top surface. C is the point where AD meets the reference plane and B is any point on this line which is at a distance of z_b from the mid surface. This is in the initial configuration (undeformed).

After deformation, point A moves to A', B moves to B', C moves to C' and D moves to D'. But then going by the assumption that the line AD still remains straight and perpendicular to the mid surface as shown this figure. This is because there is no transverse shear strain and $\gamma_{xz} = 0$.

So, in the x-z plane u_o is the x component of displacement of point of the reference plane (joining C and C'). Similarly A moves to A'; therefore joining A and A' in the x gives u_A, which is the x component of displacement of point A. , u_B is the x component of displacement of point B. Therefore we can write, rotation of x-axis (α_x) and rotation of y axis (α_y), α_x as

$$\alpha_x = \frac{\partial w_o}{\partial x}$$

$$\alpha_y = \frac{\partial w_o}{\partial y} \; ; \quad (2)$$

Analogous to taking a section in the x-z plane we can also take a section along y-z plane and then measure the y- component of displacement of A to A' as v_A and that of B to B' as v_B and distance between C and C' as v_o and the rotation will be α_y.

Now from the geometry,

$$\left. \begin{aligned} u_B &= u_0 - z_B \frac{\partial w_0}{\partial x} = u_0 - z_B \alpha_x \\ v_B &= v_0 - z_B \frac{\partial w_0}{\partial y} = v_0 - z_B \alpha_y \end{aligned} \right\} (3)$$

Therefore now we could correlate the displacement of any point B which is at a distance of z_B from the mid surface in terms of the mid surface displacement and the rotation.

So, for any general point at a distance of z from the reference plane or mid surface, we can write

$$\left. \begin{aligned} u &= u_0 - z \frac{\partial w_0}{\partial x} \Rightarrow u(x, y, z) = u_0 - z \frac{\partial w_0}{\partial x} \\ v &= v_0 - z \frac{\partial w_0}{\partial y} \Rightarrow v(x, y, z) = v_0 - z \frac{\partial w_0}{\partial y} \\ \text{and} \quad w(x, y, z) &= w_o(x, y) \end{aligned} \right\} (4)$$

$w(x, y, z) = w_o(x, y)$ means the displacement along z at any point on the laminate it is same as that of the mid surface displacement. This is because it is thin and therefore the displacement of the whole laminate could actually be represented by the displacement of the mid surface. And the in-plane displacement u and v could be expressed in terms of the in-plane displacement of the mid surface and the distance of that point from the reference plane in the z direction.

So, the displacements of any point in the laminate could be expressed as a function of *x, y, z,* but the out of plane displacement *w* does not depend upon *z*. Putting $z = 0$ in (4) yields the mid surface displacement or reference plane displacement and putting z $=\pm$ *h/2*, yield the displacements of the bottom and top surface respectively (because the thickness of the laminate is *h*). These expressions for displacement are as per the assumptions. That means in-plane displacements actually vary linearly with *z* which clear here from this expression (4).

Classical Lamination Theory (CLT)

Strain-Displacements

From small displacement, classical strain displacement relationship

So, having obtained the displacement field now let us use the strain displacement relationship with the assumption that for small displacement the expressions for strain displacement as

$$\varepsilon_x = \frac{\partial u}{\partial x};$$

$$\varepsilon_y = \frac{\partial v}{\partial y};$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

and using the expressions for displacement from (4) we could write

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w_0}{\partial x\partial y} \quad (5)$$

These are the expressions for strains in terms of mid surface displacement and the derivative of the rotation. Suppose we would like to find out what is γ_{xz} or γ_{yz} (transverse shear strains). By the strain displacement relations, and using (4)

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$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial x} = 0$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial y} = 0.$$

It could be seen that the transverses shear strains are actually zer0 (again due the assumptions)

Now (5) could be written as

$$\left. \begin{aligned}
\epsilon_x &= \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} \Rightarrow \epsilon_x = \epsilon_x^0 + zK_x \\
\epsilon_y &= \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2} \Rightarrow \epsilon_y = \epsilon_y^0 + zK_y \\
\gamma_{xy} &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w_0}{\partial x\partial y} \Rightarrow \gamma_{xy} = \gamma_{xy}^0 + zK_{xy}
\end{aligned} \right\} (6)$$

where,

$$\left. \begin{aligned}
\epsilon_x^\circ &= \frac{\partial u_0}{\partial x} \\
\epsilon_y^\circ &= \frac{\partial v_0}{\partial y} \\
\partial_{xy}^\circ &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}
\end{aligned} \right\} (7) \leftarrow \text{Mid surface strains}$$

and

$$\left. \begin{aligned}
K_x &= -\frac{\partial^2 w_0}{\partial x^2} \\
K_y &= -\frac{\partial^2 w_0}{\partial y^2} \\
K_{xy} &= -2\frac{\partial^2 w_0}{\partial x\partial y}
\end{aligned} \right\} (8) \leftarrow \text{Mid surface curvatures}$$

But here this ϵ_x° ϵ_y° γ_{xy}° means it is the mid surface in-plane strain along x- and K_x, K_y, K_{xy} at are the mid surface curvature or simply curvature. Therefore equation (6) could be written as

$$
\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_x^{\circ} \\ \epsilon_y^{\circ} \\ \gamma_{xy}^{\circ} \end{Bmatrix} + z \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \quad (9)
$$

where

$\begin{Bmatrix} \epsilon_x^{\circ} \\ \epsilon_y^{\circ} \\ \gamma_{xy}^{\circ} \end{Bmatrix}$ are mid-surface strains and $\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$ are the curvatures

Therefore, the strain at any point could be established in terms of the mid surface strains and curvatures and is decided by what is the location of that point with respect to the reference plane. So, the displacement field, the in-plane strains at any point along z- is established. Let us now move to the constitutive relations or the stress strain relationship.

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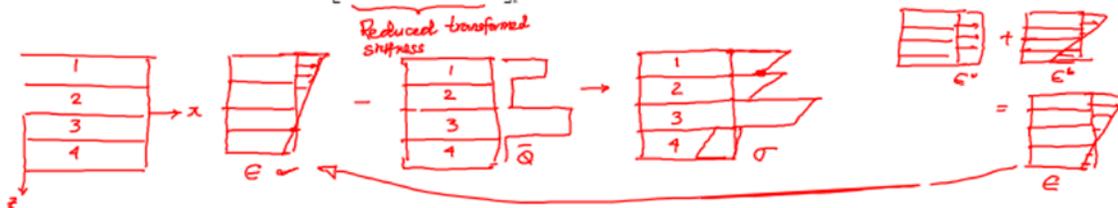

Classical Lamination Theory (CLT)
Constitutive Relations– Stress-Strain

Considering an individual layer 'k' in an 'n' layer laminate.
z : mid-surface of the kth layer from the laminate reference plane

The stress-strain relation for the layer :

Now in the 'n' layer laminate, considering any k^{th} (k=1,2,…,n) layer, if the distance of this k^{th} layer is z, then the stress strain relationship for that layer is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ & \bar{Q}_{22} & \bar{Q}_{26} \\ & & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k \quad (10)$$

The in-plane stresses in the x-y plane could be related to the corresponding in-plane strains in the x-y plane for the kth lamina. $\left[\underline{Q}\right]$ is the reduced transformed stiffness matrix for an orthotropic lamina; it is in the global x-y plane.

Therefore using equation number (9) in (10) we can write this as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ & \bar{Q}_{22} & \bar{Q}_{26} \\ & & \bar{Q}_{66} \end{bmatrix}_k \left[\begin{Bmatrix} \varepsilon_x^\circ \\ \varepsilon_y^\circ \\ \gamma_{xy}^\circ \end{Bmatrix} + z \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \right] \quad (11)$$

$[\bar{Q}]_k$ is the reduced transform stiffness matrix is for k^{th} layer.

The stress-strain relationship is for a particular layer but the laminate actually consists of a number of layers. Say in this case with 'n' layer laminate.
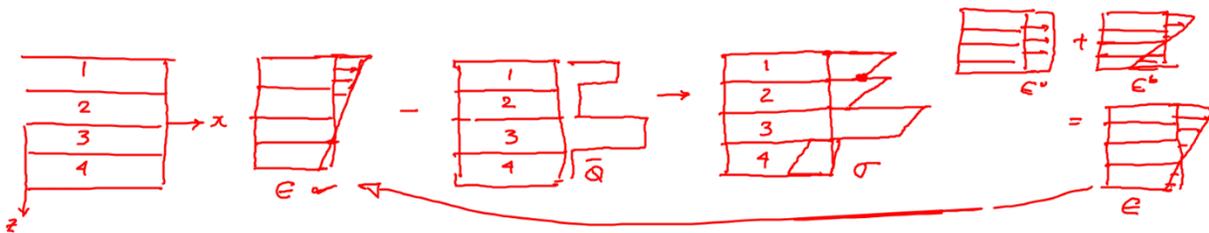
As shown, the strain variation across the thickness is linear i.e., it varies linearly with z. Only four layers have been shown and **in** each of these four layers, the stiffness $\left[\underline{Q}\right]$ will be different as shown in the stiffness variation across the layers. This leads to the stress $\{\sigma\} = \left[\underline{Q}\right]\{\varepsilon\}$ variation across the thickness as, shown. So, while the strain variation is linear along the thickness, the variation in stresses are not so. Because the stiffness are different in different layers and therefore the stress variation is not linear also it is not continuous, at the interface the stresses are discontinuous. Note that in this case the mid surface may not be the neutral axis, why? Because the total strain is superposition of the bending strain and the axial strain unlike pure bending of beams where it is only bending strain.