Theory of Composite Shells

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Week - 02

Lecture - 04

Shell governing equation step – 1, 2, 3

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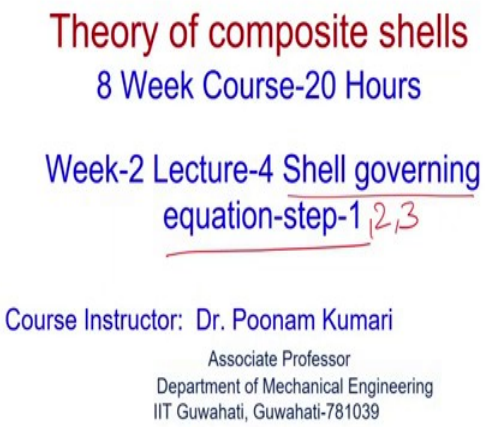

Dear learners welcome to Lecture 04 of Week 02, in this lecture, I will explain the Development of Shell Governing Equations.

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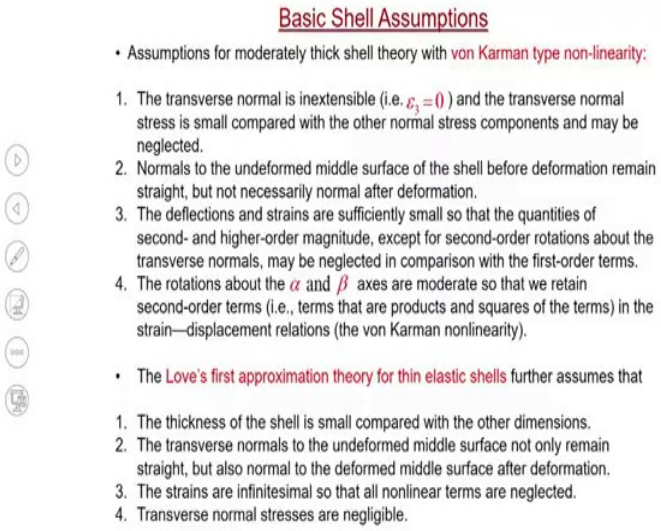

Already we have covered the basic assumptions in the last lecture for Love's first approximation theory or Love's Kirchhoff shell theory and the first-order shear deformation theory, where shear strains are not neglected.

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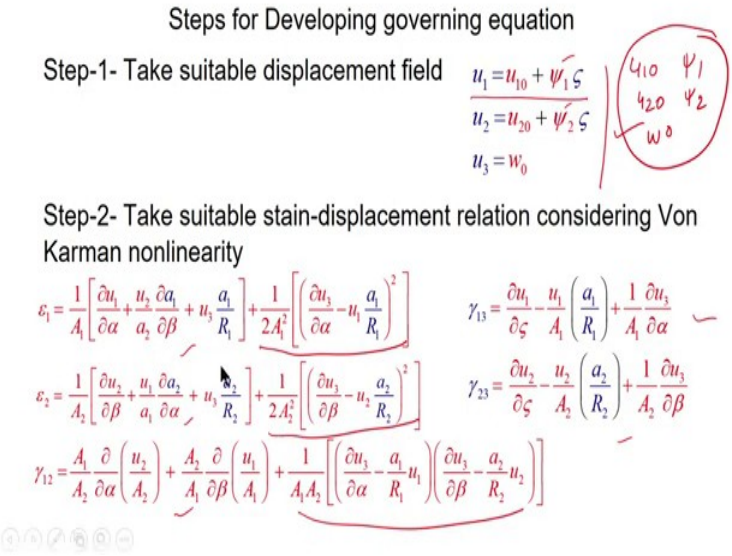

The initial step for developing governing equation or displacement-based governing equations is to choose a suitable displacement field. For the present case:

$u_1 = u_{10} + \psi_1 \varsigma$, $u_2 = u_{20} + \psi_2 \varsigma$, and $u_3 = w_0$

this displacement field is finalized.

In the last lecture, I have explained how to obtain the expression for the case of Love's Kirchhoff shell by putting γ_{13} and γ_{23} equal to 0 and integrating with respect to ς coordinates.

For this theory u_{10}, u_{20}, w_0, ψ_1 and ψ_2 are the primary variables, where ψ_1 and ψ_2 are known rotations and u_{10} and u_{20} cause membrane stretching effect and w_0 is the transfer displacement.

The next step is to choose a suitable strain displacement field. In this course, we are going to discuss the buckling of shell, for that purpose we need to consider geometrical nonlinearity so that at the end of the day, we can use the governing equations. If you start with the linear one, then we will not have a sufficient number of terms to go for buckling cases.

Already in lecture 02, in the complete strain displacement relations, we are having linear and non-linear terms and out of that, we said that when we talk about Von Karman nonlinearity that the nonlinearity effect is considered through the deflection u_3. The variation means u_3 differentiation with respect to α and β, the higher-order terms are neglected which are related to u_1, u_2 and the mixed one also.

We are going to have the non-linear terms in ε_{11}, ε_{22}, and ε_{33}, these 3 terms are extra when we consider the Von Karman nonlinearity other than that all these are linear terms.

$$\varepsilon_1 = \frac{1}{A_1}\left[\frac{\partial u_1}{\partial \alpha} + \frac{u_2}{a_2}\frac{\partial a_1}{\partial \beta} + u_3\frac{a_1}{R_1}\right] + \frac{1}{2A_1^{\,2}}\left(\frac{\partial u_3}{\partial \alpha} - u_1\frac{a_1}{R_1}\right)^2$$

$$\varepsilon_2 = \frac{1}{A_2}\left(\frac{\partial u_2}{\partial \beta} + \frac{u_1}{a_1}\frac{\partial a_2}{\partial \alpha} + u_3\frac{a_2}{R_2}\right) + \frac{1}{2A_1^{\,2}}\left(\frac{\partial u_3}{\partial \beta} - u_2\frac{a_2}{R_2}\right)^2$$

$$\gamma_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial \alpha}\left(\frac{u_2}{A_2}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \beta}\left(\frac{u_1}{A_1}\right) + \frac{1}{A_1A_2}\left[\left(\frac{\partial u_2}{\partial \alpha} - \frac{a_1}{R_1}u_1\right)\left(\frac{\partial u_3}{\partial \beta} - \frac{a_2}{R_2}u_2\right)\right]$$

$$\gamma_{13} = \frac{\partial u_1}{\partial \varsigma} - \frac{u_1}{A_1}\left(\frac{a_1}{R_1}\right) + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha}$$

$$\gamma_{23} = \frac{\partial u_2}{\partial \varsigma} - \frac{u_2}{A_2}\left(\frac{a_2}{R_2}\right) + \frac{1}{A_2}\frac{\partial u_3}{\partial \beta}$$

These are strain displacement relations, now we have to rewrite these strains considering the displacement fields.

Substituting $u_1 = u_{10} + \psi_1\varsigma$, $u_2 = u_{20} + \psi_2\varsigma$, and $u_3 = w_0$ like this. The strain displacement relations will be modified and will look like the final expressions for strains.

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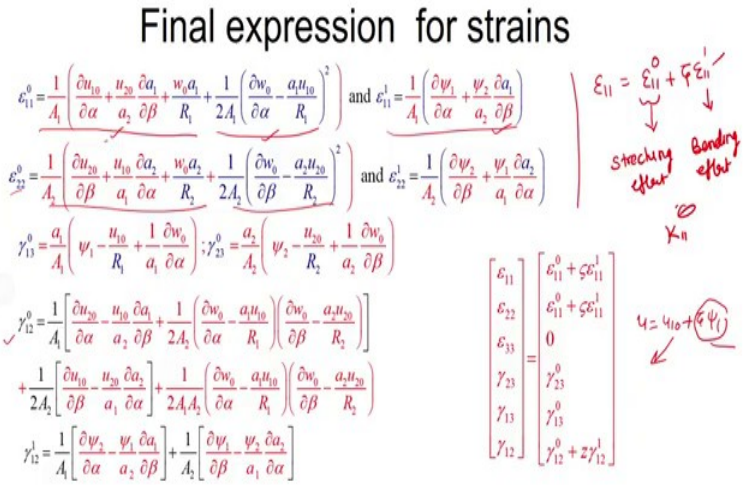

The whole strain components are arranged in this form of final expression for strains. Let us say if it is ε_{11} it will be arranged in this form:

$$\varepsilon_{11} = \varepsilon_{11}^{0} + \varsigma\varepsilon_{11}^{1}$$

Whereas ε_{11}^{0} cause stretching effect in the shell and ε_{11}^{1} will cause bending effect in the shell. In some books, it is also denoted by K_{11}^{0} related to curvature terms or the bending.

In ε_{11}^{0}: the explicit expression terms will be:

$$\varepsilon_{11}^{0} = \frac{1}{A_1}\left(\frac{\partial u_{10}}{\partial\alpha} + \frac{u_{20}}{a_2}\frac{\partial a_1}{\partial\beta} + \frac{w_0 a_1}{R_1} + \frac{1}{2A_1}\left(\frac{\partial w_0}{\partial\alpha} - \frac{a_1 u_{10}}{R_1}\right)^2\right)$$

$\frac{1}{A_1}\left(\frac{\partial u_{10}}{\partial\alpha} + \frac{u_{20}}{a_2}\frac{\partial a_1}{\partial\beta} + \frac{w_0 a_1}{R_1}\right)$ comes from linear term

$+\frac{1}{2A_1}\left(\frac{\partial w_0}{\partial\alpha} - \frac{a_1 u_{10}}{R_1}\right)^2$ is the contribution due to non-linear term

$$\varepsilon_{11}^{1} = \frac{1}{A_1}\left(\frac{\partial\psi_1}{\partial\alpha} + \frac{\psi_2}{a_2}\frac{\partial a_1}{\partial\beta}\right)$$

ε_{11}^{1} is the contribution due to the bending effect because ψ_1 and ψ_2 are the rotations if you take derivatives along that they will cause a bending effect.

Then, ε_{22} we can write like this:

$$\varepsilon_{22} = \varepsilon_{22}^{0} + \varsigma\varepsilon_{22}^{1}$$

$$\varepsilon_{22}^{0} = \frac{1}{A_2}\left(\frac{\partial u_{20}}{\partial\beta} + \frac{u_{10}}{a_1}\frac{\partial a_2}{\partial\alpha} + \frac{w_0 a_2}{R_2} + \frac{1}{2A_2}\left(\frac{\partial w_0}{\partial\beta} - \frac{a_2 u_{20}}{R_2}\right)^2\right)$$

$$\varepsilon_{22}^{1} = \frac{1}{A_2}\left(\frac{\partial\psi_2}{\partial\beta} + \frac{\psi_1}{a_1}\frac{\partial a_2}{\partial\alpha}\right)$$

It is a simple substitution and wherever required will do some derivatives or writing in a

more simplified manner means arranging the terms because ultimately when you have u = u_{10} + $\varsigma\,\psi_1$. Here, u_{10} will contribute to the curvature and $\varsigma\,\psi_1$ will contribute to the stretching, we must arrange it like this.

The main difference here is in the γ_{12}, In the previous expression γ_{12} is written in the general form. We have simplified it, then substitute the value of u_1 and u_2. This gives you a slightly big expression:

$$\gamma_{12}^{0} = \frac{1}{A_1}\left[\frac{\partial u_{20}}{\partial\alpha} - \frac{u_{10}}{a_2}\frac{\partial a_2}{\partial\beta} + \frac{1}{2A_2}\left(\frac{\partial w_0}{\partial\alpha} - \frac{a_1 u_{10}}{R_1}\right)\left(\frac{\partial w_0}{\partial\beta} - \frac{a_2 u_{20}}{R_2}\right)\right]$$
$$+\frac{1}{A_2}\left(\frac{\partial u_{10}}{\partial\beta} - \frac{u_{20}}{a_1}\frac{\partial a_2}{\partial\alpha}\right) + \frac{1}{2A_1A_2}\left(\frac{\partial w_0}{\partial\alpha} - \frac{a_1 u_{10}}{R_1}\right)\left(\frac{\partial w_0}{\partial\beta} - \frac{a_2 u_{20}}{R_2}\right)$$

where $\frac{1}{A_1}$ of linear terms plus $\frac{1}{2A_2}$ of non-linear contribution, here it will be $\frac{1}{A_2}$ of linear terms and $\frac{1}{2A_1A_2}$ of nonlinear terms.

There are some terms in which $\frac{1}{A_1}$ can be taken out and there are some terms in which $\frac{1}{A_2}$ can be taken out. These terms $\frac{1}{A_1}\left(\frac{\partial u_{20}}{\partial\alpha} - \frac{u_{10}}{a_2}\frac{\partial a_2}{\partial\beta}\right)$ and $\frac{1}{A_2}\left(\frac{\partial u_{10}}{\partial\beta} - \frac{u_{20}}{a_1}\frac{\partial a_2}{\partial\alpha}\right)$ are corresponding to the linear contribution and these terms

$\frac{1}{2A_2}\left(\frac{\partial w_0}{\partial\alpha} - \frac{a_1 u_{10}}{R_1}\right)\left(\frac{\partial w_0}{\partial\beta} - \frac{a_2 u_{20}}{R_2}\right)$ and $\frac{1}{2A_1A_2}\left(\frac{\partial w_0}{\partial\alpha} - \frac{a_1 u_{10}}{R_1}\right)\left(\frac{\partial w_0}{\partial\beta} - \frac{a_2 u_{20}}{R_2}\right)$ are corresponding to the non-linear contribution, then γ_{12}^{1} can be expressed as:

$$\gamma_{12}^{1} = \frac{1}{A_1}\left(\frac{\partial\psi_2}{\partial\alpha} - \frac{\psi_1}{a_2}\frac{\partial a_1}{\partial\beta}\right) + \frac{1}{A_2}\left(\frac{\partial\psi_1}{\partial\beta} - \frac{\psi_2}{a_1}\frac{\partial a_2}{\partial\alpha}\right)$$

In γ_{13} and γ_{23} there will be no curvature terms. Lean in-plane stretching terms are written like this:

$$\gamma_{13}^{0} = \frac{a_1}{A_1}\left(\psi_1 - \frac{u_{10}}{R_1} + \frac{1}{a_1}\frac{\partial w_0}{\partial \alpha}\right) \quad and \quad \gamma_{23}^{0} = \frac{a_2}{A_2}\left(\psi_2 - \frac{u_{20}}{R_2} + \frac{1}{a_2}\frac{\partial w_0}{\partial \beta}\right).$$

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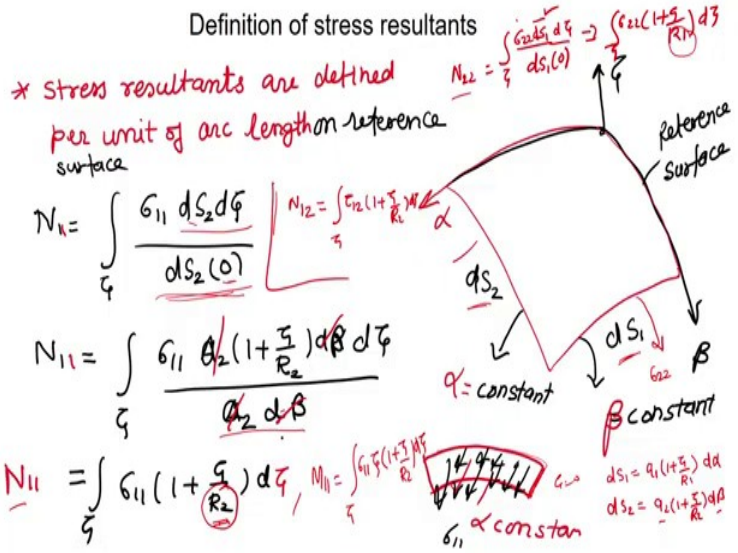

Now, we have to define the stress resultants. There are many ways to define stress resultant for the case of the shell. If you go through the book by Harry krauss; the stress resultants are defined per unit of arc length on the reference surface.

At the reference surface, the curve length is slightly different, and let us say at ς height it is something different. If this is the reference surface and these are α and β coordinates, over this line, α will be constant over this line beta will be constant and ds_1 and ds_2 are the curve lengths.

What are ds_1 and ds_2? ds_1 is $a_1\left(1+\frac{\varsigma}{R_1}\right)\partial\alpha$ and ds_2 is equal to $a_2\left(1+\frac{\varsigma}{R_2}\right)\partial\beta$. The stress component σ_{11} is acting over this phase.

Now, we will find the resultant. Integrating over the thickness with its ς coordinate, the total force will be:

$$N_{11} = \int_{\varsigma} \frac{\sigma_{11} ds_2 d\varsigma}{ds_2(0)}$$

Arc length on the reference surface will be ds_2 where ς is 0, at the reference surface ς will be 0.

If you put zeta as 0, only a_2 and $\partial\beta$ will come up. We will write:

$$N_{11} = \int_{\varsigma} \frac{\sigma_{11} a_2 \left(1 + \frac{\varsigma}{R_2}\right) d\beta \, d\varsigma}{a_2 d\beta}$$

$a_2 d\beta$ will get canceled. N_{11} can be defined as :

$$N_{11} = \int_{\varsigma} \sigma_{11} \left(1 + \frac{\varsigma}{R_2}\right) d\varsigma$$

The same way we can define the M_{11} as:

$$M_{11} = \int_{\varsigma} \sigma_{11} \varsigma \left(1 + \frac{\varsigma}{R_2}\right) d\varsigma$$

N_{12} can be defined as:

$$N_{12} = \int_{\varsigma} \sigma_{12} \left(1 + \frac{\varsigma}{R_2}\right) d\varsigma,$$

We are talking about the definition here the normal component N_{22} is σ_{22}. If you want to define the stress resultant along this coordinate N_{22} will be:

$$N_{22} = \int_{\varsigma} \frac{\sigma_{22} ds_1 d\varsigma}{ds_1(0)} \Rightarrow N_{22} = \int_{\varsigma} \sigma_{22} \left(1 + \frac{\varsigma}{R_1}\right) d\varsigma$$

I would like to point out here, for N_{11}; $\left(1 + \frac{\varsigma}{R_2}\right)$, the radius of curvature in the second direction and $\left(1 + \frac{\varsigma}{R_1}\right)$ the radius in the first direction, these are interchanging. The point to be noted here is sometimes we ask that define the stress resultant for the case of shell or identify the coordinates properly.

Students may do mistake, in N_{11} they used to write R_1 or in N_{22} they used to write R_2. One has to be careful that when you are defining N_{11}, the radius will be the second direction radius, and in N_{22} the radius will be the first direction radius, if you define the coordinate system like this.

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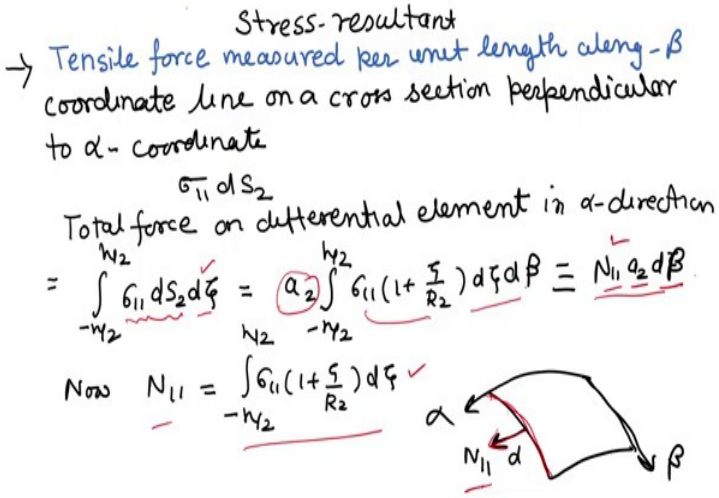

In the second book, from where I have taken some of the basic equations is J N Reddy's theory of plate and shell. In that or some older books also based on shell theories it is given that the tensile force measured per unit length along β coordinate line on a cross-section perpendicular to α coordinate can be represented like this. The force will be $\sigma_{11} ds_2$ where this line α is going to be constant.

The total force on a differential element can be written as $\frac{-h}{2}$ to $\frac{+h}{2}$. We have taken the shell element thickness $\frac{-h}{2}$ to $\frac{+h}{2}$ and total force will be into integrating over that. If you substitute the value of ds_2; we know that a_2 is not a function of ς it is only a function of α and β. We can write like this:

$$\sigma_{11} ds_2 = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11} ds_2 d\varsigma = a_2 \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11} \left(1 + \frac{\varsigma}{R_2}\right) d\varsigma d\beta = N_{11} a_2 d\beta .$$

If we compare this, a normal resultant force is acting in plane stress resultant N_{11}. Total over this reference surface will be $a_2 d\beta$ and total force. If we compare it with this, we see that $a_2 d\beta$ is here. N_{11} comes:

$$N_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11} \left(1 + \frac{\varsigma}{R_2}\right) d\varsigma$$

You can do it anyway; the results will be the same. In this, we are comparing N_{11} with the help of stress resultant, previously, we were directly or mathematically defining as taking stress resultant per unit length of reference surface arc length of the reference surface.

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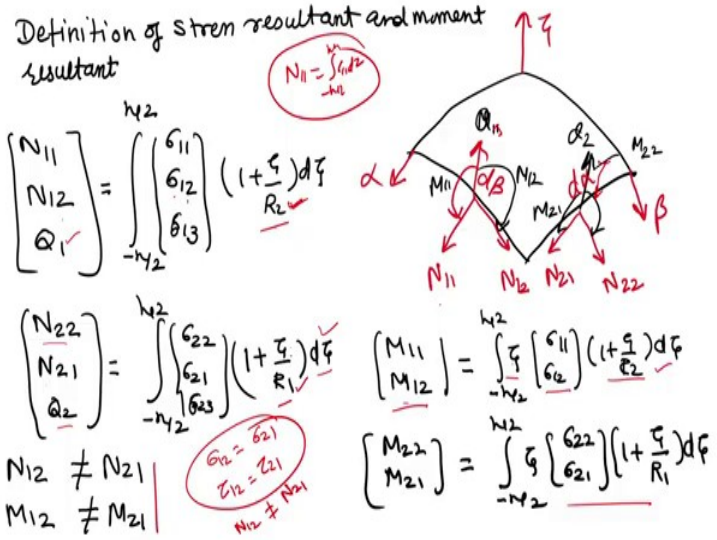

Already, I have discussed how to obtain N_{22}. Now, we are talking about a third component τ_{13} and τ_{23}.

Here, $Q_1 = \sigma_{13}\left(1+\frac{\zeta}{R_2}\right)d\zeta$ and $Q_2 = \sigma_{23}\left(1+\frac{\zeta}{R_1}\right)d\zeta$.

Here point to be noted that for the definition

$N_{12} = \sigma_{12}\left(1+\frac{\zeta}{R_2}\right)d\zeta$ and $N_{21} = \sigma_{21}\left(1+\frac{\zeta}{R_1}\right)$.

You may say that τ_{12} or σ_{12} is equal to τ_{21} or σ_{21}, but $N_{12} = N_{21}$ because here R_2 and R_1 are different.

For the case of spherical shell and plate, they will be the same because R_1 and R_2 both are ∞. But for other cases like a cylinder, they will not be the same because in one direction radius is ∞ and in another direction it is R. You can take some conical shell or any other type of shell where R_1 and R_2 are not equal, there $N_1 \neq N_2$.

In the same way, we can define the moment in one direction as:

$$M_{11} = \int_{\varsigma} \sigma_{11} \varsigma \left(1 + \frac{\varsigma}{R_2}\right) d\varsigma$$

M_{12}, M_{22} and M_{21} can be defined as:

$$M_{12} = \int_{\varsigma} \sigma_{12} \varsigma \left(1 + \frac{\varsigma}{R_2}\right) d\varsigma$$

$$\begin{bmatrix} M_{22} \\ M_{21} \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma \begin{bmatrix} \sigma_{22} \\ \sigma_{21} \end{bmatrix} \left(1 + \frac{\varsigma}{R_1}\right) d\varsigma$$

Here, $M_{12} \neq M_{21}$

This type of expression you come to know when you are going to derive the equations for a shell. If you have done till plate only then you will not think that ok this may be different or if you see that these stress resultants can be a special case like if you put R_2

for a case of a plate. For a case of a plate, it will be $\int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11} d\varsigma$.

This is the definition of stress resultant for the case of a plate over the surface element reference member; N_{11}, N_{12}, N_{22}, N_{21}, M_{11}, M_{12}, M_{22}, M_{21}, Q_1, and Q_2 are shown here. This is a very important part; we are going to use these definitions and point to

remember is that for the case of N_{11} it will be $\left(1 + \frac{\varsigma}{R_2}\right)$ for case of the second direction it

will be $\left(1 + \frac{\varsigma}{R_1}\right)$

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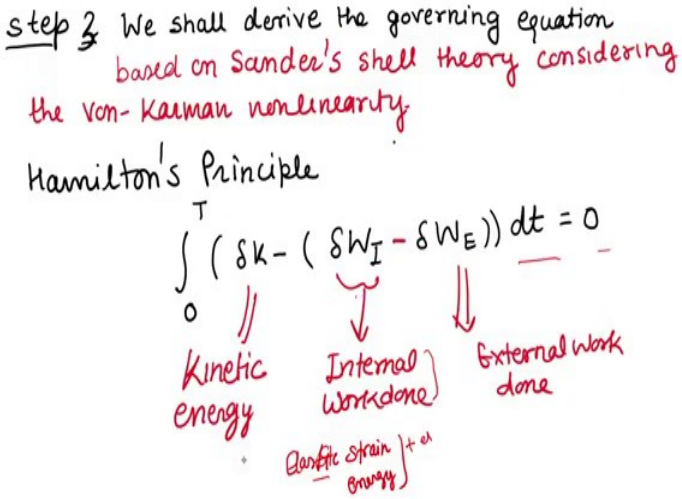

In the third step, we must derive the governing equations based on Sander's shell theory and considering the Von Karman nonlinearity. The Hamilton principle states:

$$\int_0^T \left(\partial K - \left(\partial W_I - \partial W_E \right) \right) dt = 0$$

What is the total energy? The first variation in kinetic energy minus internal work done or elastic strain energy, it is very convenient or generalized to say internal work done because later if somebody wants to study a piezoelectric shell or some functionally graded shell made of some other different material. Instead of elastic energy, they may have some electrical energy and some other form of energy. They will also be clubbed together here under the term of internal work done.

Then the external work done over the surfaces is the work done by the traction forces that may be the loading or the traction forces. We can find the external work done from there so that is going to be 0. First, we will calculate what is the kinetic energy for the case of a shell element.

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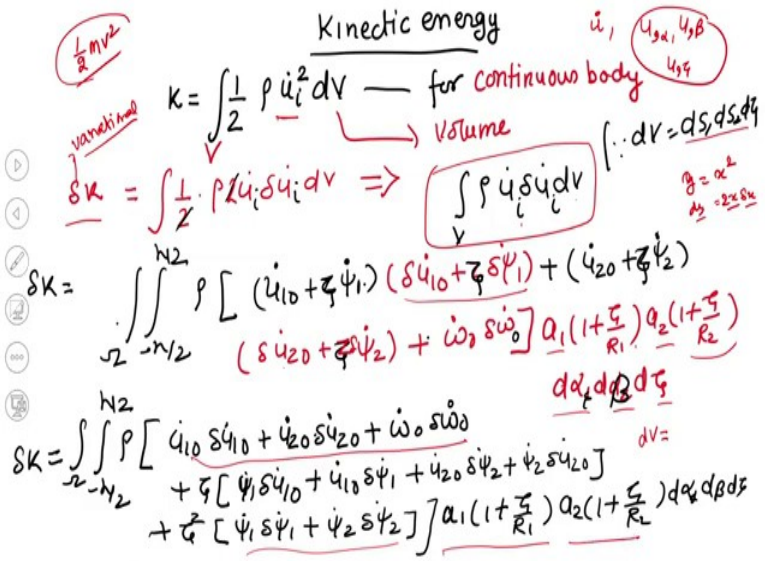

For a continuous body, the kinetic energy expression can be written as:

$$K = \int_v \frac{1}{2}\rho \dot{u}_i^2 dv,$$

where u is displacement $\dot{u}$ means $\frac{du}{dt}$. It is a standard expression that whenever you have a time derivative you take the derivative with respect to (.) and whenever you have a space derivative then you put a (,), like $(,\alpha)$, $(,\beta)$, $(,\varsigma)$, and so on.

We have a time derivative; you can say that $\frac{1}{2}mv^2$ for a discrete system like a car of a mass (m) moving with the velocity (v), what will be the kinetic energy? Then we can say that it will be $\frac{1}{2}mv^2$, but in the same way, if a beam or a plate or a shell is vibrating, or it is having motion then $K = \int_v \frac{1}{2}\rho \dot{u}_i^2 dv$

We have to take the first variation, I assume that you are aware of taking variations in a quantity, ∂ is a variational operator and it behaves as our differential operator. If we

want to take the first variation in this, we will say $\dot{u}_i^2$ will be $2\dot{u}_i\ \partial\dot{u}_i$ variation in that quantity.

For example, you want to say $y = x^2$ then, what is $\frac{dy}{dx}$. It will be $\frac{dy}{dx} = 2x\partial x$.

Similarly, we have written that. 2 will get canceled and ultimately the expression will be:

$$\partial K = \int \frac{1}{2}\rho 2\dot{u}_i \partial\dot{u}_i dv \Rightarrow \int \rho \dot{u}_i \partial\dot{u}_i dv$$

This is the expression of kinetic energy, the first variation of kinetic energy in the index form.

We will write in the explicit form ∂K will be $\dot{u}_i$ means u_1 plus u_2 plus u_3. We know that $\dot{u}_1 = \dot{u}_{10} + \varsigma\dot{\psi}_1$; $\dot{u}_2 = \dot{u}_{20} + \varsigma\dot{\psi}_2$ and $\dot{u}_3 = \dot{w}_0 \partial\dot{w}_0$. Now, putting them in the equation we get:

$$\partial K = \int_{\Omega}\int_{\frac{-h}{2}}^{\frac{h}{2}} \rho\left[\left(\dot{u}_{10} + \varsigma\dot{\psi}_1\right)\left(\partial\dot{u}_{10} + \varsigma\partial\dot{\psi}_1\right) + \left(\dot{u}_{20} + \varsigma\dot{\psi}_2\right)\left(\partial\dot{u}_{20} + \varsigma\partial\dot{\psi}_2\right) + \dot{w}_0\partial\dot{w}_0\right] X$$

$$a_1\left(1 + \frac{\varsigma}{R_1}\right) a_2\left(1 + \frac{\varsigma}{R_2}\right)\left(d\alpha\right)\left(d\beta\right)\left(d\varsigma\right)$$

Volume dv is ds_1, ds_2, $d\varsigma$

and ds_1 is $a_1\left(1 + \frac{\varsigma}{R_1}\right) d\alpha$ and ds_2 is $a_2\left(1 + \frac{\varsigma}{R_2}\right) d\beta$. If we multiply these terms and mathematically simplify.

There will be some terms with ς and some without ς. The term without ς will be kept together, and the terms where ς comes, there we will take ς common and put those terms together.

$$\partial K = \int_{\Omega}\int_{\frac{-h}{2}}^{\frac{h}{2}} \rho\left(\dot{u}_{10}\partial\dot{u}_{10} + \dot{u}_{20}\partial\dot{u}_{20} + \dot{w}_0\partial\dot{w}_0\right) + \varsigma\left(\dot{\psi}_1\partial\dot{u}_{10} + \dot{u}_{10}\partial\dot{\psi}_1 + \dot{u}_{20}\partial\dot{\psi}_2 + \dot{\psi}_2\partial\dot{u}_{20}\right)$$

$$+\varsigma\left(\dot{\psi}_1\partial\dot{\psi}_1 + \dot{\psi}_2\partial\dot{\psi}_2\right) a_1\left(1+\frac{\varsigma}{R_1}\right) a_2\left(1+\frac{\varsigma}{R_2}\right)\left(d\alpha\right)\left(d\beta\right)\left(d\varsigma\right)$$

What is the moment of inertia?

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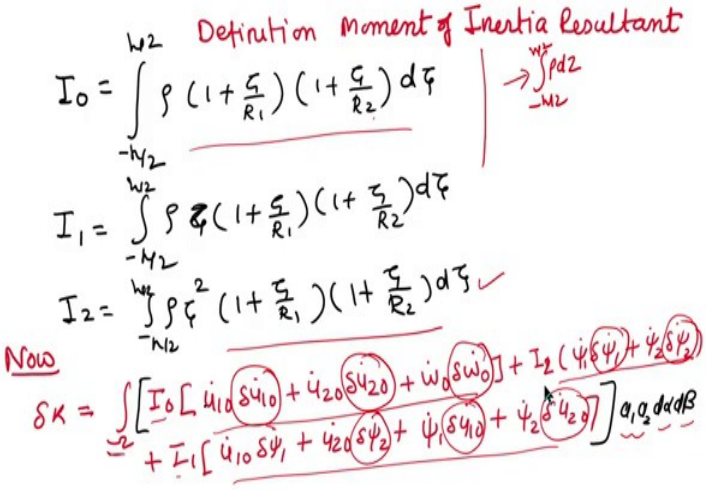

I_0 will be defined as:

$$I_0 = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho\left(1+\frac{\varsigma}{R_1}\right)\left(1+\frac{\varsigma}{R_2}\right) d\varsigma$$

This is the definition of the moment of inertia resultant for the case of the doubly curved shell.

You can see that this expression is entirely different if you talk about the case of a plate,

If you put $R_2 = \infty$ and $R_1 = \infty$, this expression becomes this $I_0 = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho d\varsigma$

$$I_1 = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho\varsigma \left(1 + \frac{\varsigma}{R_1}\right)\left(1 + \frac{\varsigma}{R_2}\right) d\varsigma$$

$$I_2 = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho\varsigma^2 \left(1 + \frac{\varsigma}{R_1}\right)\left(1 + \frac{\varsigma}{R_2}\right) d\varsigma$$

By defining the moment of inertia resultant, we are converting a volume integral to an area integral. I would like to say that this is an essential step in any 2-dimensional theory whether you are interested in a shell or a plate or a beam; you need to follow this step. Without defining the inertia resultants or stress resultants or moment resultants you cannot convert a volume integral into the area integral.

By following this procedure, ∂K becomes:

$$\partial K = \int_{\Omega} \left[\begin{array}{l} I_0\left(\dot{u}_{10}\partial\dot{u}_{10} + \dot{u}_{20}\partial\dot{u}_{20} + \dot{w}_0\partial\dot{w}_0\right) + I_2\left(\dot{\psi}_1\partial\dot{\psi}_1 + \dot{\psi}_2\partial\dot{\psi}_2\right) \\ + I_1\left(\dot{u}_{10}\partial\dot{\psi}_1 + \dot{u}_{20}\partial\dot{\psi}_2 + \dot{\psi}_1\partial\dot{u}_{10} + \dot{\psi}_2\partial\dot{u}_{20}\right) \end{array} \right] a_1 a_2 \left(d\alpha\right)\left(d\beta\right)$$

Here the term like $\partial\dot{u}_{10}$, $\partial\dot{u}_{20}$, $\partial\dot{w}_0$, $\partial\dot{\psi}_1$, and $\partial\dot{\psi}_2$, we have to get rid of them because we do not want any derivative in the variation.

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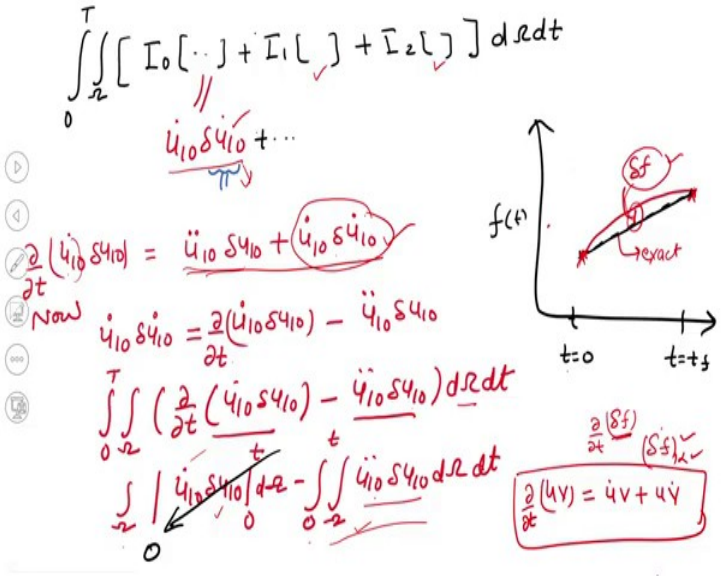

If I explain the first variation in the case of time, let us say time t = 0 and time t = f and a function ft is given. So, let us take a point where time t = 0 and time t = f.

The exact solution can be expressed by this black line and we are taking the first variation. We are taking some variations because we do not know the exact solution. We are taking an arbitrary variation in that function and we are going to minimize this. It is ∂ f; the first variation and at the boundary this term is going to be 0.

We have assumed a virtual variation in a quantity f, but we do not have any information about the $\frac{\partial}{\partial t}\partial f$. We do not have any information of its time derivative, we want to convert it to its basic form.

Wherever you have variation terms either it is a time derivative or space derivative we have to get rid of this. I am explaining with the help of the first term the same way the other terms can be converted.

We can write this term as:

$$\frac{\partial}{\partial t}\dot{u}_{10}\partial u_{10} = \ddot{u}_{10}\partial u_{10} + \dot{u}_{10}\partial\dot{u}_{10}$$

This is the simple rule of differentiation when you have a 2-function u and v and you want to do differentiation with respect to time, what do you do?

You say that differentiation of the first term and second term as it is plus the first term as it is and differentiation of the second term. By following the same we have obtained this term $\dot{u}_{10}\partial\dot{u}_{10}$. You can express $\dot{u}_{10}\partial\dot{u}_{10}$:

$$\dot{u}_{10}\partial\dot{u}_{10} = \frac{\partial}{\partial t}\left(\dot{u}_{10}\partial u_{10}\right) - \ddot{u}_{10}\partial u_{10}$$

$\ddot{u}_{10}$ becomes the acceleration term second-time derivative.

If you substitute this equation and take the time integration, you will see that if there is a time derivative and integration over time. We can integrate with respect to time and from using this term $\ddot{u}_{10}\partial u_{10}$ we do not get any advantage. Hence, we will keep it same on the integration time and area integration.

$$\int_0^T\int_\Omega\left[\frac{\partial}{\partial t}\left(\dot{u}_{10}\partial u_{10}\right) - \ddot{u}_{10}\partial u_{10}\right]d\Omega dt = \int_\Omega\left|0\dot{u}_{10}\partial u_{10}\right|_0^t d\Omega - \int_0^t\int_{-2}\ddot{u}_{10}\partial u_{10}d\Omega dt$$

But if you do integrate with respect to time and take the limit 0 to T at the initial time and final time the variation is 0. this term $\dot{u}_{10}\partial u_{10}$ will not contribute. This term $\dot{u}_{10}\partial\dot{u}_{10}$ will give only one contribution that - $\ddot{u}_{10}\partial u_{10}$. We will get one contribution the other time derivative is going to be 0.

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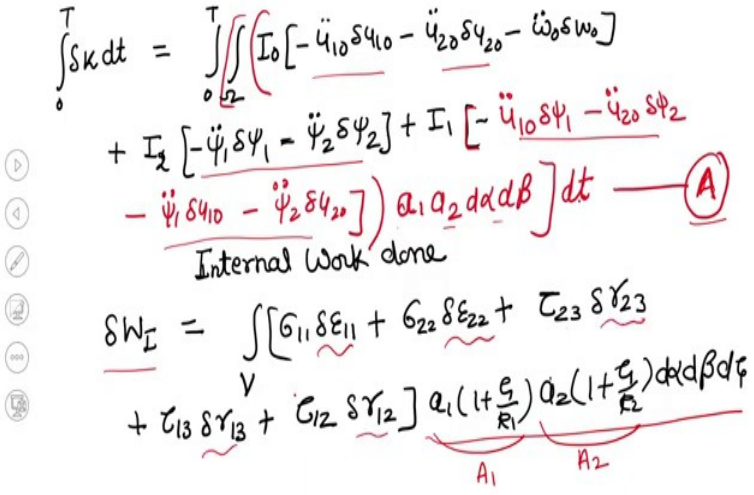

By following the same procedure, the first variation in kinetic energy all the terms are written like this:

$$\int_0^T \partial K dt = \int_0^T \left\{ \int_\Omega \left[\begin{array}{l} I_0\left(\dot{u}_{10}\partial\dot{u}_{10} + \dot{u}_{20}\partial\dot{u}_{20} + \dot{w}_0\partial\dot{w}_0\right) + I_2\left(\dot{\psi}_1\partial\dot{\psi}_1 + \dot{\psi}_2\partial\dot{\psi}_2\right) \\ + I_1\left(\dot{u}_{10}\partial\dot{\psi}_1 + \dot{u}_{20}\partial\dot{\psi}_2 + \dot{\psi}_1\partial\dot{u}_{10} + \dot{\psi}_2\partial\dot{u}_{20}\right) \end{array} \right] a_1 a_2 \left(d\alpha\right)\left(d\beta\right) \right\} dt$$

It is the perfect term that can be directly going to the governing equations. We defined the equation number (A). We have to calculate the internal work done, external work done, and later we will club them together.

What is the internal work done? For the case of an elastic body, ∂W_I will be:

$$\partial W_I = \int_v \left(\sigma_{11}\partial\varepsilon_{11} + \sigma_{22}\partial\varepsilon_{22} + \tau_{23}\partial\gamma_{23} + \tau_{13}\partial\gamma_{13} + \tau_{12}\partial\gamma_{12} \right) a_1\left(1+\frac{\varsigma}{R_1}\right) a_2\left(1+\frac{\varsigma}{R_2}\right)\left(d\alpha\right)\left(d\beta\right)\left(d\varsigma\right)$$

We can denote $a_1\left(1+\frac{\varsigma}{R_1}\right)$ as A_1 and $a_2\left(1+\frac{\varsigma}{R_2}\right)$ as A_2. We have to write this variation form of strains.

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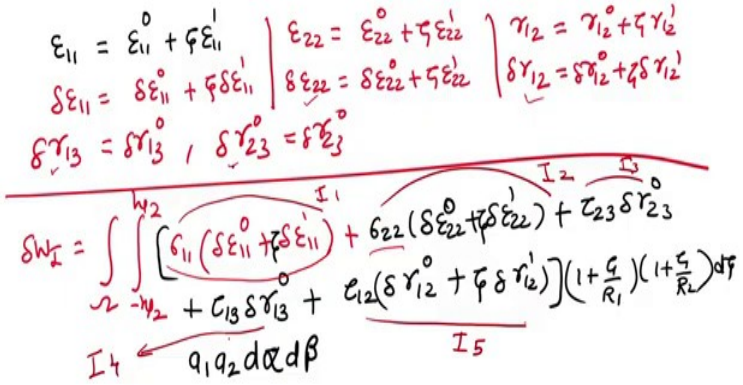

We know $\varepsilon_{11} = \varepsilon_{11}^{0} + \varsigma\varepsilon_{11}^{1}$, if we take the first variation it will be $\partial\varepsilon_{11}^{0} + \varsigma\partial\varepsilon_{11}^{1}$.

Similarly, substituting $\partial\varepsilon_{22}$, $\partial\gamma_{23}$, $\partial\gamma_{13}$, and $\partial\gamma_{12}$ in this equation.

Working with this equation altogether will be very lengthy; we will work one by one. I_1 $= \sigma_{11}\left(\partial\varepsilon_{11}^{0} + \varsigma\partial\varepsilon_{11}^{1}\right)$

$I_2 = \sigma_{22}\left(\partial\varepsilon_{22}^{0} + \varsigma\partial\varepsilon_{22}^{1}\right)$

$I_3 = \tau_{23}\partial\gamma_{23}^{0}$

$I_4 = \tau_{13}\partial\gamma_{13}^{0}$

$I_5 = \tau_{12}\left(\partial\gamma_{12}^{0} + \varsigma\partial\gamma_{12}^{1}\right)$

We are dividing them into small integration, later, we will club them together.

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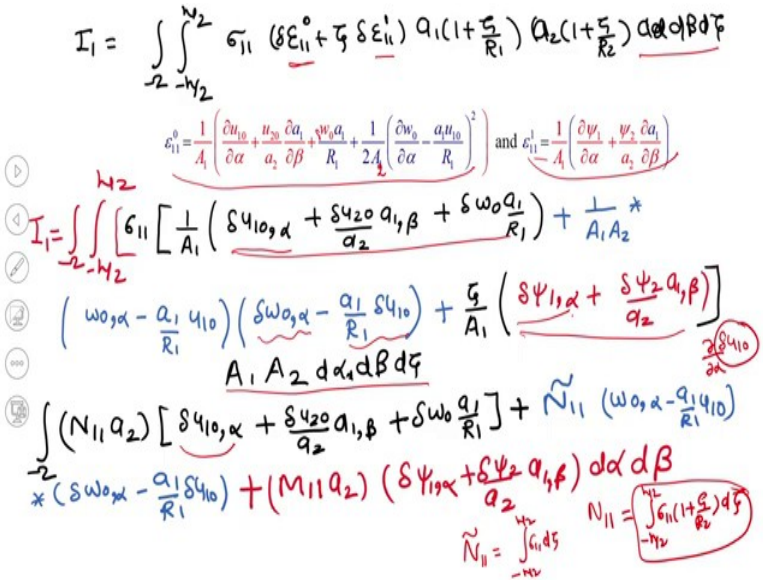

The first integration I_1:

$$I_1 = \int_{\Omega}\int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11}\left(\partial\varepsilon_{11}^{0} + \varsigma\partial\varepsilon_{11}^{1}\right) a_1\left(1+\frac{\varsigma}{R_1}\right) a_2\left(1+\frac{\varsigma}{R_2}\right)(d\alpha)(d\beta)(d\varsigma)$$

Now, we need the explicit expression of ε_{11}^{0} and ε_{11}^{1}. I have written the explicit expression of ε_{11}^{0} and if we take the variation, ∂ will come here and again ∂ will come here and so on.

$$I_1 = \int_{\Omega}\int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11}\left[\begin{array}{l}\dfrac{1}{A_1}\left(\partial u_{10,\alpha} + \dfrac{\partial u_{20}}{a_2}a_{1,\beta} + \partial w_0\dfrac{a_1}{R_1}\right) + \dfrac{1}{A_1A_2}\left(w_{0,\alpha} - \dfrac{a_1}{R_1}u_{10}\right) \\ \left(\partial w_{0,\alpha} - \dfrac{a_1}{R_1}\partial u_{10}\right) + \dfrac{\varsigma}{A_1}\left(\partial\psi_{1,\alpha} + \dfrac{\partial\psi_2}{a_2}a_{1,\beta}\right)\end{array}\right] A_1A_2(d\alpha)(d\beta)(d\varsigma)$$

We have to do it carefully, If we write $\dfrac{\partial}{\partial\alpha}$ it takes huge space and working with that is slightly difficult. I have written in a small form.

$\dfrac{1}{A_1}\left(\partial u_{10,\alpha} + \dfrac{\partial u_{20}}{a_2}a_{1,\beta} + \partial w_0\dfrac{a_1}{R_1}\right)$ is the linear contribution.

In a non-linear one, it will be:

$$\frac{1}{A_1 A_2}\left(w_{0,\alpha} - \frac{a_1}{R_1} u_{10} \right)\left(\partial w_{0,\alpha} - \frac{a_1}{R_1} \partial u_{10} \right) + \frac{\varsigma}{A_1}\left(\partial \psi_{1,\alpha} + \frac{\partial \psi_2}{a_2} a_{1,\beta} \right)$$

Here $\frac{1}{A_1}$ is outside from the linear term and $\frac{1}{A_1 A_2}$ is outside from the non-linear term.

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11} \frac{1}{A_1} = N_{11} a_2 .$$

Because, $N_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11} \left(1 + \frac{\varsigma}{R_2} \right) d\varsigma$.

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11} \frac{1}{A_1 A_2} = \tilde{N}_{11} \text{constant} ; \; \tilde{N}_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11} d\varsigma$$

They will get canceled later. We can multiply and do something.

Instead of $\frac{\varsigma}{A_1}$ we write $M_{11} a_2$.

You see here $\partial u_{10,\alpha}$; we only have information about ∂u_{10}, we have taken variation in primary displacement ∂u_{10}. We do not have any information about its space derivative with respect to α. We have to reduce it to ∂u_{10}.

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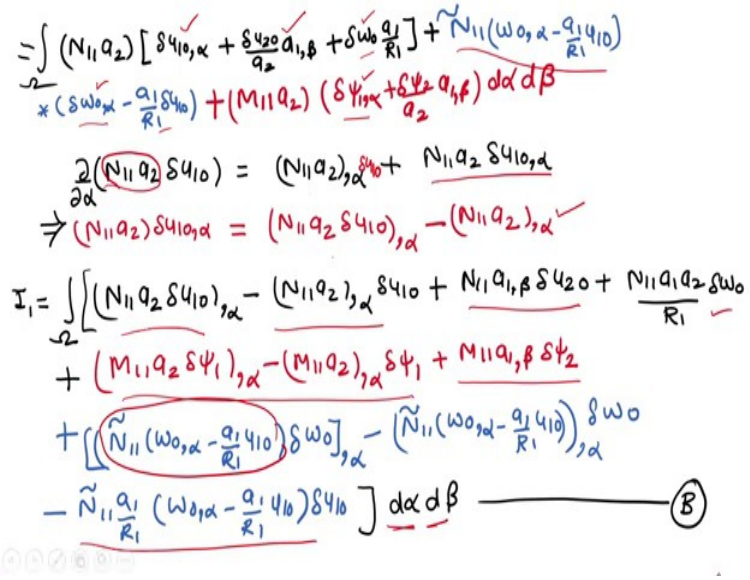

By following the same technique which we have followed for the case of time derivative we can write $N_{11}a_2$, here we consider $N_{11}a_2$ is one variable.

$$\frac{\partial}{\partial\alpha}\left(N_{11}a_2\partial u_{10}\right)=\left(N_{11}a_2\right)_{,\alpha}\partial u_{10}+\left(N_{11}a_2\right)\partial u_{10,\alpha}$$

$$\left(N_{11}a_2\right)\partial u_{10,\alpha}=\left(N_{11}a_2\partial u_{10}\right)_{,\alpha}-\left(N_{11}a_2\right)_{,\alpha}$$

This term will give you two terms, the term $\left(N_{11}a_2\partial u_{10}\right)_{,\alpha}$ will go to the boundary and $\left(N_{11}a_2\right)_{,\alpha}$ term will remain on the area.

Same way, here $\left(\frac{\partial u_{20}}{a_2}a_{1,\beta}+\partial w_0\frac{a_1}{R_1}\right)$ in this term, we do not need to do anything and here also we do not do anything because there is no derivative. But here is the derivative in the non-linear term, derivative in the first term and derivative in the moment term.

The first two terms and then the second and third will be as it is, then the moment gives these terms $\left(M_{11}a_2\partial\psi_1\right)_{,\alpha}$, $\left(\left(M_{11}a_2\right)_{,\alpha}\partial\psi_1\right)$ and $\left(M_{11}a_{1,\beta}\partial\psi_2\right)$. In the same way, the

non-linear terms will be expressed. This whole $\tilde{N}_{11}\left(w_{0,\alpha}-\frac{a_1}{R_1}u_{10}\right)$ will be treated as one term.

Whole derivative comma α, we will consider it as a single term then only we can get rid of their derivatives and the second term again as it is and $(d\alpha)(d\beta)$.

$$I_1=\int_\Omega\left(N_{11}a_2\partial u_{10}\right)_{,\alpha}-\left(\left(N_{11}a_2\right)_{,\alpha}\partial u_{10}\right)+\left(N_{11}a_{1,\beta}\,\partial u_{20}\right)+\left(\frac{N_{11}a_1a_2}{R_1}\partial w_0\right)$$
$$+\left(M_{11}a_2\partial\psi_1\right)_{,\alpha}-\left(\left(M_{11}a_2\right)_{,\alpha}\partial\psi_1\right)+\left(M_{11}a_{1,\beta}\,\partial\psi_2\right)$$
$$+\left[\tilde{N}_{11}\left(w_{0,\alpha}-\frac{a_1}{R_1}u_{10}\right)\partial w_0\right]_{,\alpha}-\left[\tilde{N}_{11}\left(w_{0,\alpha}-\frac{a_1}{R_1}u_{10}\right)_{,\alpha}\partial w_0\right]-\left[\tilde{N}_{11}\frac{a_1}{R_1}\left(w_{0,\alpha}-\frac{a_1}{R_1}u_{10}\right)\partial u_{10}\right](d\alpha)(d\beta)$$

This is the expression required for the differential equation for I_1 denoted as equation (B).

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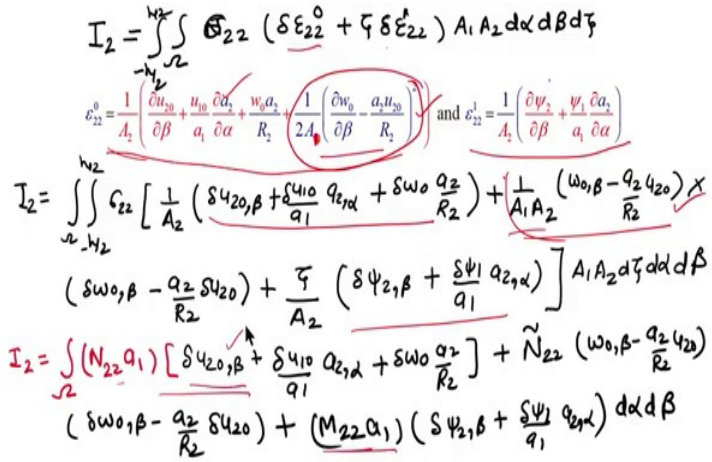

Now, the expression for I_2:

$$I_2=\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_\Omega\sigma_{22}\left(\partial\varepsilon_{22}^0+\varsigma\partial\varepsilon_{22}^1\right)A_1A_2\left(d\alpha\right)\left(d\beta\right)\left(d\varsigma\right)$$

We will substitute the expression of $\partial\varepsilon_{22}^{0}$ and $\varsigma\partial\varepsilon_{22}^{1}$.

$\partial\varepsilon_{22}$ expression I put here. Here, the linear term is right, and the curvature is right only the non-linear terms have some issue. I will explain in the next lecture, the exact contribution of the non-linear terms.

We can write:

$$I_2 = \int_{\Omega}\int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{22} \begin{bmatrix} \frac{1}{A_2}\left(\partial u_{20},_{\beta} + \frac{\partial u_{10}}{a_1} a_2,_{\alpha} + \partial w_0 \frac{a_2}{R_2}\right) + \frac{1}{A_1 A_2}\left(w_0,_{\beta} - \frac{a_2}{R_2} u_{20}\right) \\ \left(\partial w_0,_{\beta} - \frac{a_2}{R_2}\partial u_{20}\right) + \frac{\varsigma}{A_2}\left(\partial \psi_2,_{\beta} + \frac{\partial \psi_1}{a_1} a_2,_{\alpha}\right) \end{bmatrix} A_1 A_2 (d\alpha)(d\beta)(d\varsigma)$$

Using the concept of plane stress resultant

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{22} \frac{1}{A_1} = N_{22} a_1$$

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{22} \frac{1}{A_1 A_2} = \tilde{N}_{22} \text{ and}$$

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{22} \frac{\varsigma}{A_2} = M_{22} a_1 .$$

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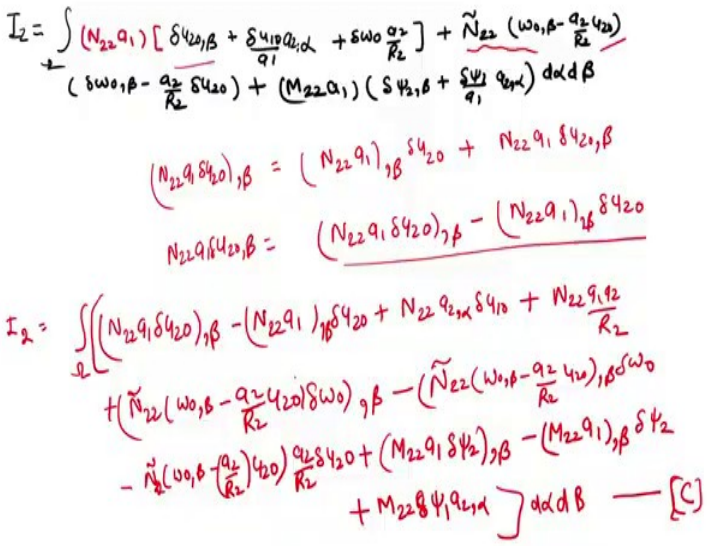

$$I_2 = \int_{\Omega} \left(N_{22} a_1 \right) \left(\partial u_{20},_{\beta} + \frac{\partial u_{10}}{a_1} a_2,_{\alpha} + \partial w_0 \frac{a_2}{R_2} \right) + \tilde{N}_{22} \left(w_0,_{\beta} - \frac{a_2}{R_2} u_{20} \right) \left(\partial w_0,_{\beta} - \frac{a_2}{R_2} \partial u_{20} \right)$$
$$+ \left(M_{22} a_1 \right) \left(\partial \psi_2,_{\beta} + \frac{\partial \psi_1}{a_1} a_2,_{\alpha} \right) \left(d\alpha \right) \left(d\beta \right)$$

Similarly, here is derivative with respect to β, we must get rid of that. With the help of using the basic concept

$$\left(N_{22} a_1 \partial u_{20} \right),_{\beta} = \left(N_{22} a_1 \right),_{\beta} \partial u_{20} + \left(N_{22} a_1 \right) \partial u_{20},_{\beta}$$
$$\left(N_{22} a_1 \right) \partial u_{20},_{\beta} = \left(N_{22} a_1 \partial u_{20} \right),_{\beta} - \left(N_{22} a_1 \right),_{\beta} \partial u_{20}$$

I_2 expression will be:

$$I_2 = \int_{\Omega} \left(N_{22} a_1 \partial u_{20} \right),_{\beta} - \left(\left(N_{22} a_1 \right),_{\beta} \partial u_{20} \right) + \left(N_{22} a_2,_{\alpha} \partial u_{10} \right) + \left(\frac{N_{22} a_1 a_2}{R_1} \partial w_0 \right)$$
$$+ \left(M_{22} a_1 \partial \psi_2 \right),_{\beta} - \left(\left(M_{22} a_1 \right),_{\beta} \partial \psi_2 \right) + \left(M_{22} a_2,_{\alpha} \partial \psi_1 \right)$$
$$+ \left[\tilde{N}_{22} \left(w_0,_{\beta} - \frac{a_2}{R_2} u_{20} \right) \partial w_0 \right]_{,\beta} - \left[\tilde{N}_{22} \left(w_0,_{\beta} - \frac{a_2}{R_2} u_{20} \right)_{,\beta} \partial w_0 \right] - \left[\tilde{N}_{22} \frac{a_2}{R_2} \left(w_0,_{\beta} - \frac{a_2}{R_2} u_{20} \right) \partial u_{20} \right] \left(d\alpha \right) \left(d\beta \right)$$

In this way, we get the expression (C).

In next week lecture, I will explain I_3, I_4, I_5, external work done and clarify the issue in

the definition of non-linear terms.

Thank you very much.