Theory of Composite Shells
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Week - 05
Lecture - 03
Development of Navier solution of finite shell

Dear learners welcome to week 5, lecture 3: Development of Navier solution for a finite shell.

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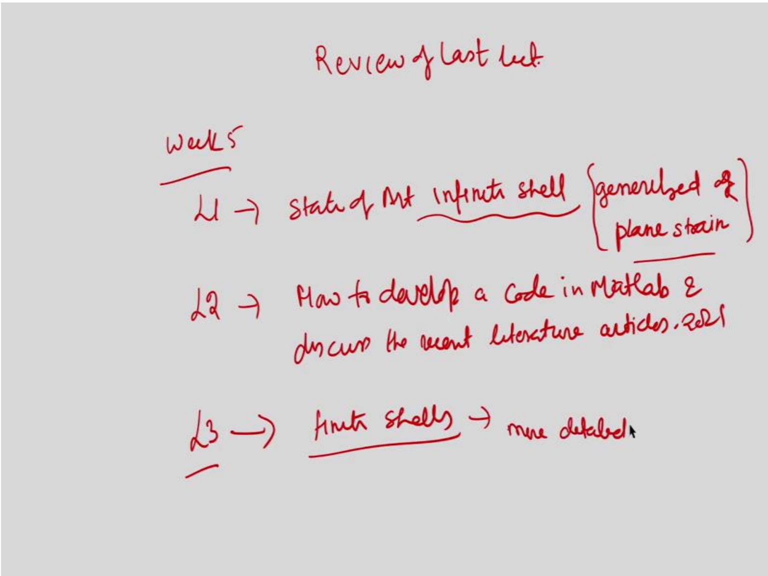

First, I will review the last lectures. In week 5, lecture- 1; I presented the state of art for infinite shell means, the concept of generalized plane strain. In lecture- 2: I described developing a code in MATLAB and discussed a recent literature article of 2021.

In this lecture, I will develop the state of art for a finite shell, the steps are similar to that I have presented in lecture 1, but is slightly more detailed.

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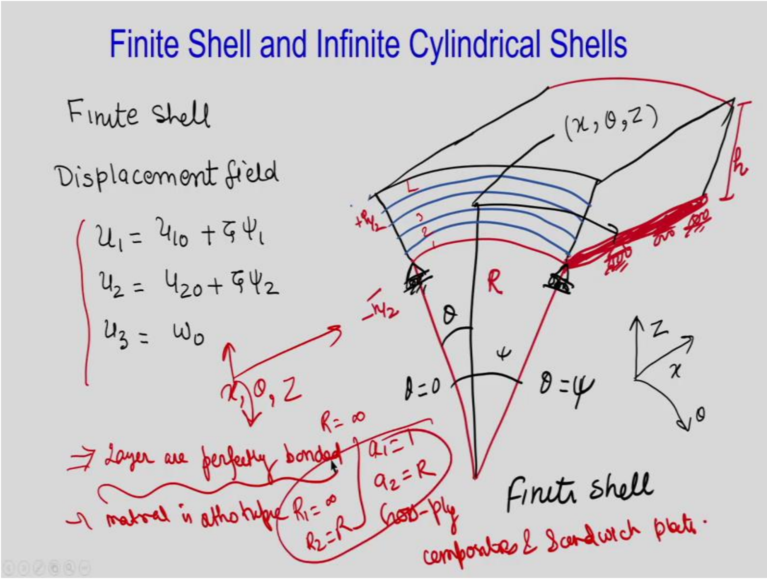

Let us consider, a finite shell made of composite materials, which is orthotropic in nature, and let us say, there are layers 1, 2, 3 and up to L, and they are perfectly bonded.

Sometimes, I may forget to discuss all the solutions or whatever I have discussed, even the formulation is valid when layers are perfectly bonded and the material is orthotropic. The solutions I am presenting for cross-ply composites and sandwich plates are is valid for that.

For this case, the same displacement field is considered, i.e.:

$$u_1 = u_{10} + \psi_1 \varsigma; \quad u_2 = u_{20} + \psi_2 \varsigma; \quad u_3 = w_0$$

Our coordinate system is x, θ, and z; where x is the longitudinal direction, θ is the circumferential direction, and z is the thickness direction. It is a singly curved surface, radius in one direction is ∞ and in the second direction it is R, i.e., $R_1 = \infty$ and $R_2 = R$. The lame's parameters $a_1 = 1$ and $a_2 = R$.

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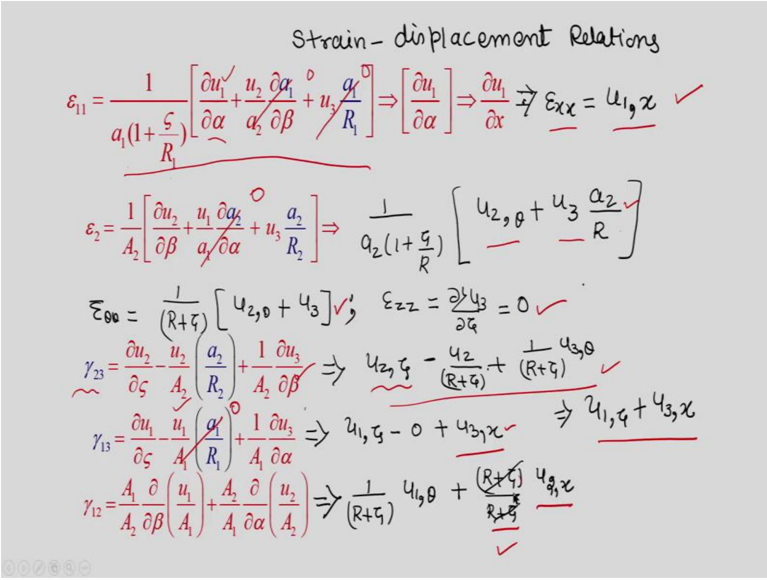

If we follow these things, we can find the strain components. There are two ways for a circular cylindrical shell; one way: you can find the strain components which are generally given in cylindrical coordinate system, most of the theory of elasticity book that strain displacement relations are given, one can use directly.

I already developed a strain displacement relation in a very general form. That can be found through a special case, a circular cylinder is a special case. Putting the value of lame's parameters and radius we can find the strains.

$$\varepsilon_{11} = \frac{1}{a_1\left(1+\frac{\varsigma}{R_1}\right)}\left[\frac{\partial u_1}{\partial \alpha} + \frac{u_2}{a_2}\frac{\partial a_1}{\partial \beta} + u_3\frac{a_1}{R_1}\right].$$

Here, $R_1 = \infty$ and a 1 = 1, $a_1\left(1+\frac{\varsigma}{R_1}\right)$ term will not contribute.

$\frac{u_2}{a_2}\frac{\partial a_1}{\partial \beta} = 0$, $u_3\frac{a_1}{R_1}$ = 0, the only contribution will be from this term $\frac{\partial u_1}{\partial \alpha}$.

Ultimately, ε_{xx} will be $u_{1,x}$.

Then, $\varepsilon_2 = \frac{1}{a_2\left(1+\frac{\varsigma}{R_2}\right)}\left(\frac{\partial u_2}{\partial \beta} + \frac{u_1}{a_1}\frac{\partial a_2}{\partial \alpha} + u_3\frac{a_2}{R_2}\right)$

Here, $\frac{u_1}{a_1}\frac{\partial a_2}{\partial \alpha}$ will not contribute, but $\frac{\partial u_2}{\partial \beta}$ and $u_3\frac{a_2}{R_2}$ will contribute, $a_2\left(1+\frac{\varsigma}{R_2}\right)$ will also contribute.

$\varepsilon_{\theta\theta}$ will be $\frac{1}{a_2\left(1+\frac{\varsigma}{R}\right)}\left[u_{2,\theta}+u_3\frac{a_2}{R}\right]$, where a_2 = R; R and R get cancelled.

Ultimately, $\varepsilon_{\theta\theta}$ will be $\frac{1}{(R+\varsigma)}\left(u_{2,\theta}+u_3\right)$.

Then ε_{zz} = $\frac{\partial u_3}{\partial \varsigma}$, here, u_3 is not a function of ς, therefore, it is going to be 0.

Now, we can find out the expression of γ_{23};

$$\gamma_{23}=\frac{\partial u_2}{\partial \varsigma}-\frac{u_2}{A_2}\left(\frac{a_2}{R_2}\right)+\frac{1}{A_2}\frac{\partial u_3}{\partial \beta}$$

In this expression, all terms will contribute, therefore it will be:

$$u_{2,\varsigma}-\frac{u_2}{(R+\varsigma)}+\frac{1}{(R+\varsigma)}u_{3,\theta}$$

In γ_{13} expression: $\gamma_{13}=\frac{\partial u_1}{\partial \varsigma}-\frac{u_1}{A_1}\left(\frac{a_1}{R_1}\right)+\frac{1}{A_1}\frac{\partial u_3}{\partial \alpha}$;

The term $\frac{u_1}{A_1}\left(\frac{a_1}{R_1}\right)$ = 0 because R_1 is ∞.

γ_{13} = $u_{1,\varsigma}+u_{3,x}$.

$$\gamma_{12}=\frac{A_1}{A_2}\frac{\partial}{\partial \alpha}\left(\frac{u_2}{A_2}\right)+\frac{A_2}{A_1}\frac{\partial}{\partial \beta}\left(\frac{u_1}{A_1}\right);$$

γ_{12} will be:

$$\frac{1}{(R+\varsigma)}u_{1,\theta}+\frac{(R+\varsigma)}{(R+\varsigma)}u_{2,x}$$

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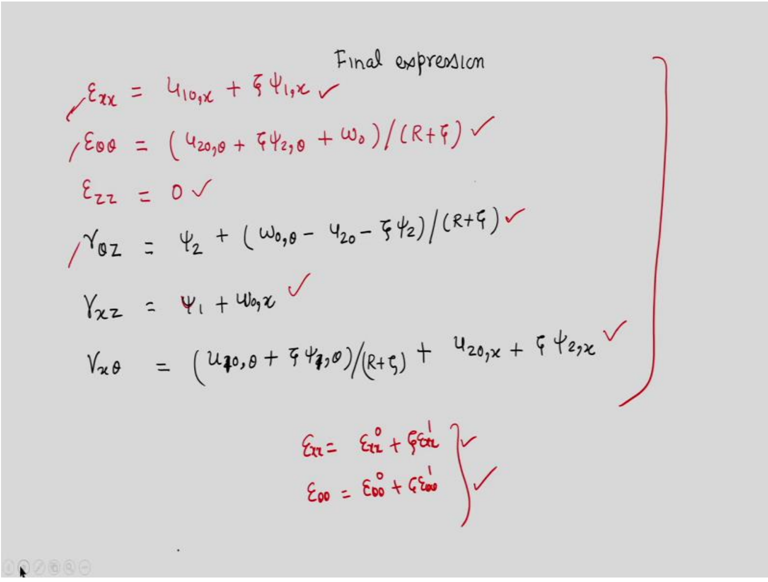

Finally, the non-zero strain components for the present case are written below:

$$\varepsilon_{xx} = u_{1,x} + \varsigma\psi_{1,x}$$
$$\varepsilon_{\theta\theta} = \left(u_{20,\theta} + \varsigma\psi_{2,\theta} + w_0\right)$$
$$\varepsilon_{zz} = 0$$
$$\gamma_{\theta z} = \psi_2 + \left(w_{0,\theta} - u_{20} - \varsigma\psi_2\right)/\left(R+\varsigma\right)$$
$$\gamma_{xz} = \psi_1 + w_{0,x}$$
$$\gamma_{x\theta} = \left(u_{10,\theta} + \varsigma\psi_{1,\theta}\right)/\left(R+\varsigma\right) + u_{20,x} + \varsigma\psi_{2,x}$$
$$\varepsilon_{xx} = \varepsilon_{xx}^0 + \varsigma\varepsilon_{xx}^1 \qquad \varepsilon_{\theta\theta} = \varepsilon_{\theta\theta}^0 + \varsigma\varepsilon_{\theta\theta}^1$$

If you remember, for the case of infinite shell: ε_{xx} = γ_{xz} , = $\gamma_{x\theta}$ = 0.

There are only three non-zero strains, ε_{xx} = γ_{xz} , = $\gamma_{x\theta}$ = 0, but for the case of a finite cylindrical shell only ε_{zz} = 0, other 5 strains are existing. This is the expression of strain displacement. If you want to arrange it in that form, let us say, $\varepsilon_{xx} = \varepsilon_{xx}^0 + \varsigma\varepsilon_{xx}^1$ and $\varepsilon_{\theta\theta} = \varepsilon_{\theta\theta}^0 + \varsigma\varepsilon_{\theta\theta}^1$.

But, in the present case, because it is very simple, I didn’t do it that way. One who is formulating only for a cylindrical shell can try in this way, programming will be easy.

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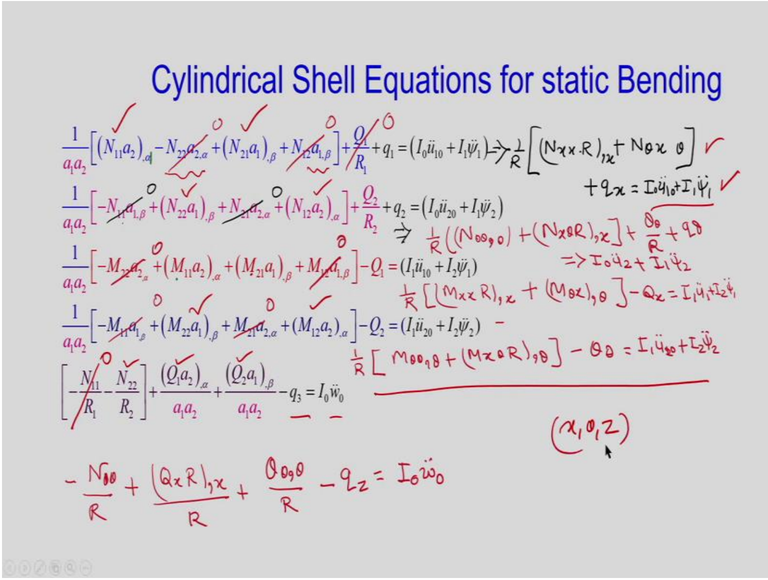

Can we convert the governing equations to the present case? If we talk about the first governing equation:

$$\frac{1}{a_1a_2}\left[\left(N_{11}a_2\right)_{,\alpha}-N_{22}a_{2,\alpha}+\left(N_{21}a_1\right)_{,\beta}+N_{12}a_{1,\beta}\right]+\frac{Q_1}{R_1}+q_1=\left(I_0\ddot{u}_{10}+I_1\ddot{\psi}_1\right)$$

Here, $\left(N_{11}a_2\right)_{,\alpha}$ and $\left(N_{21}a_1\right)_{,\beta}$ will contribute, derivatives of lame's parameters are going to be 0.

The first equation will be: $\left[\left(N_{xx}.R\right)_{,x}+N_{x\theta,\theta}\right]+q_1=\left(I_0\ddot{u}_{10}+I_1\ddot{\psi}_1\right)$

We have written the non-zero value of this and the dynamic is also taken. Because later on, in the 6th week, I am going to explain the free vibration of cylindrical shells or different shells. So, I have taken these terms, but for the present case, we are studying only the static part, we are going to put it 0.

In the second equation:

$$\frac{1}{a_1a_2}\left(-N_{11}a_{1,\beta}+\left(N_{22}a_1\right)_{,\beta}+N_{21}a_{2,\alpha}+\left(N_{12}a_2\right)_{,\alpha}\right)+\frac{Q_2}{R_2}+q_2=\left(I_0\ddot{u}_{20}+I_1\ddot{\psi}_2\right)$$

$-N_{11}a_{1,\beta}$ and $N_{21}a_{2,\alpha}=0$, $\left(N_{22}a_1\right)_{,\beta}$ and $\left(N_{12}a_2\right)_{,\alpha}$ will contribute and $\frac{Q_2}{R_2}$ will also contribute.

It will be: $\frac{1}{R}\left[N_{\theta\theta,\theta}+\left(N_{x\theta}R\right)_{,x}\right]+\frac{Q_\theta}{R}+q_2=\left(I_0\ddot{u}_{20}+I_1\ddot{\psi}_2\right)$

In the third equation:

$$\frac{1}{a_1 a_2}\left[-M_{22}a_{2,\alpha}+\left(M_{11}a_2\right)_{,\alpha}+\left(M_{21}a_1\right)_{,\beta}+M_{12}a_{1,\beta}\right]-Q_1=\left(I_1\ddot{u}_{10}+I_2\ddot{\psi}_1\right)$$

$-M_{22}a_{2,\alpha}$ and $M_{12}a_{1,\beta}$ = 0, $\left(M_{11}a_2\right)_{,\alpha}$ and $\left(M_{21}a_1\right)_{,\beta}$ will contribute. It will become:

$$\frac{1}{R}\left[\left(M_{xx}.R\right)_{,x}+\left(M_{\theta x}\right)_{,\theta}\right]-Q_x=\left(I_1\ddot{u}_{10}+I_2\ddot{\psi}_1\right)$$

In the fourth equation:

$$\frac{1}{a_1 a_2}\left[-M_{11}a_{1,\beta}+\left(M_{22}a_1\right)_{,\beta}+M_{21}a_{2,\alpha}+\left(M_{12}a_2\right)_{,\alpha}\right]-Q_2=\left(I_1\ddot{u}_{20}+I_2\ddot{\psi}_2\right)$$

$-M_{11}a_{1,\beta}$ and $M_{21}a_{2,\alpha}$ = 0, $\left(M_{22}a_1\right)_{,\beta}$ and $\left(M_{12}a_2\right)_{,\alpha}$ will contribute.

In the fifth equation: $\left(-\frac{N_{11}}{R_1}-\frac{N_{22}}{R_2}\right)+\frac{\left(Q_1a_2\right)_{,\alpha}}{a_1a_2}+\frac{\left(Q_2a_1\right)_{,\beta}}{a_1a_2}-q_3=I_0\ddot{w}_0$;

$-\frac{N_{11}}{R_1}$ = 0, $-\frac{N_{22}}{R_2}$ will contribute, $\frac{\left(Q_1a_2\right)_{,\alpha}}{a_1a_2}$ and $\frac{\left(Q_2a_1\right)_{,\beta}}{a_1a_2}$ will contribute.

It will become:

$$-\frac{N_{\theta\theta}}{R}+\frac{\left(Q_xR\right)_{,x}}{R}+\frac{Q_{\theta,\theta}}{R}-q_z=I_0\ddot{w}_0$$

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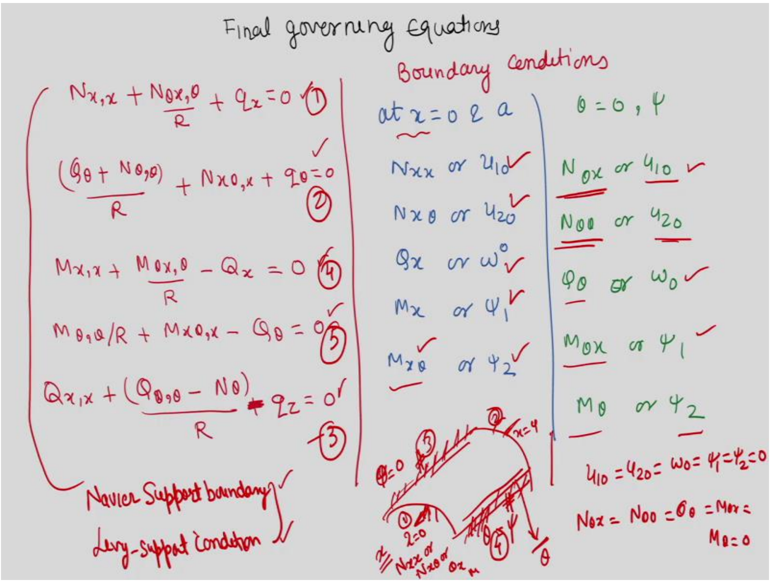

If we represent in the terms of x, θ, and z properly and write in a combined form, then, the following will be the final set of 5 partial differential equations for a finite cylindrical shell.

$$N_{x,x}+\frac{N_{x\theta,\theta}}{R}+q_x=0 \quad equation(1)$$

$$\frac{Q_\theta+N_{\theta,\theta}}{R}+N_{x\theta,x}+q_\theta=0 \quad equation(2)$$

$$M_{x,x}+\frac{M_{x\theta,\theta}}{R}-Q_x=0 \quad equation(3)$$

$$\frac{M_{\theta,\theta}}{R}+M_{x\theta,x}-Q_\theta=0 \quad equation(4)$$

$$Q_{x,x}+\frac{Q_{\theta,\theta}-N_\theta}{R}-q_z=0 \quad equation(5)$$

In most of the cases, the review articles or the general articles, the order is slightly changed. 5^{th} equation is treated as a 3^{rd} equation, 3^{rd} as 4^{th} equation, and 4^{th} as 5^{th} equation.

Now, the associated boundary conditions at edges: this is a finite cylindrical shell, therefore, for a boundary two faces are there, one is x = 0 and x = a.

If we talk about this, x = 0 and over this x = a, and other is θ = 0 and θ = ψ, over this edge. I am going to put hash here, the normal direction is x.

In the case of boundary condition: we can say that in-plane stress resultant in case of x is equal to 0 and x is equal to a will be:

N_{xx} *or* u_{10}, $N_{x\theta}$ *or* u_{20}, Q_x *or* w_0, M_{xx} *or* ψ_1, and $M_{x\theta}$ *or* ψ_2.

For the second case, θ is equal to 0 and θ is equal to ψ, in-plane stress resultants will be:

$N_{x\theta}$ *or* u_{10}, $N_{\theta\theta}$ *or* u_{20}, Q_θ *or* w_0, $M_{x\theta}$ *or* ψ_1, and $M_{\theta\theta}$ *or* ψ_2.

These are the variables, which are needed to be specified. Depending upon the boundary conditions, we can specify. For example, if you say that these edges are clamped, for that case our all displacements are going to be 0,

i.e., $u_{10}=u_{20}=w_0=\psi_1=\psi_2=0$, at θ equals to 0 and ψ.

If I say that this is clamped and it may be free, for that case, all stresses need to be 0.

Depending upon the boundary conditions we can specify our variables. I discussed several times and again I am going to discuss the analytical solution, the closed-form solutions are valid for simply supported boundary conditions.

If all edges are simply supported, let us say 1, 2, 3, and 4, if all edges are simply supported then the cylinder is said to be in Navier support boundary conditions.

If any two opposite edges like this are simply supported then we say that cylinder is subjected to levy support conditions. Analytical and closed-form solution is valid only for Navier support conditions and levy support conditions. If you are interested to solve a problem in which one edge is clamped and another is simply supported this edge is free and this is having some point support then we cannot get the analytical solution or closed-form solution.

I would like to say that even the approximate solution; which means the solution obtained through Ritz technique or Galerkin technique, where you can see that all edges are clamped or two edges are clamped and two edges are free. In that case, boundary conditions are placed in such a way that we can get some solution.

Other than the all-around simply supported and to oppose it as a simply supported we can get a solution through approximate techniques, and other techniques such as Ritz technique, Galerkin technique, and extended Kantorovich techniques.

Recently, you will find a lot of articles in which a cylindrical shell or a complete shell is studied using the extended Kantorovich method.

If you see, the loading and the boundary conditions are very arbitrary, in that case, the numerical solutions come into the picture, you call about finite element solution. Recently, DQM - Differential Quadrature Method and state-space finite element technique mean; combining of analytical as well as numerical technique are used. If we do so, we can get the solution for a variety of loading and boundary conditions.

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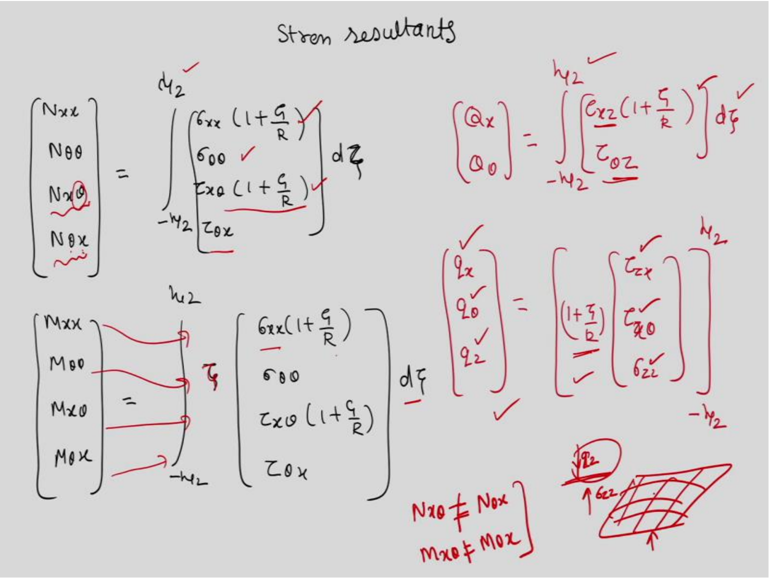

Now, let us define the stress resultants:

$$\begin{bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \\ N_{\theta x} \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_{xx}\left(1+\frac{\varsigma}{R}\right) \\ \sigma_{\theta\theta} \\ \tau_{x\theta}\left(1+\frac{\varsigma}{R}\right) \\ \tau_{\theta x} \end{bmatrix} d\varsigma$$

Here, we can see, for a cylindrical shell $N_{x\theta} \neq N_{\theta x}$.

Similarly, defining the moments:

$$\begin{bmatrix} M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \\ M_{\theta x} \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma \begin{bmatrix} \sigma_{xx}\left(1+\frac{\varsigma}{R}\right) \\ \sigma_{\theta\theta} \\ \tau_{x\theta}\left(1+\frac{\varsigma}{R}\right) \\ \tau_{\theta x} \end{bmatrix} d\varsigma$$

Here, $M_{x\theta} \neq M_{\theta x}$

Now, defining the shear stresses:

$$\begin{bmatrix} Q_{x} \\ Q_{\theta} \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \begin{bmatrix} \tau_{xz}\left(1+\frac{\varsigma}{R}\right) \\ \tau_{\theta z} \end{bmatrix} d\varsigma \ .$$

And the loading at the top and the bottom is equal to the traction, for example, let us say, in the case of a plate:

$$\begin{bmatrix} q_{x} \\ q_{\theta} \\ q_{z} \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \left(1+\frac{\varsigma}{R}\right) \begin{bmatrix} \tau_{zx} \\ \tau_{z\theta} \\ \sigma_{zz} \end{bmatrix}$$

Loading per unit area, it works on the complete area, if you do so, $\left(1+\frac{\varsigma}{R}\right)$ factor comes.

Ultimately, the applied traction is equal to the resisting traction or inside the stresses, at the boundary.

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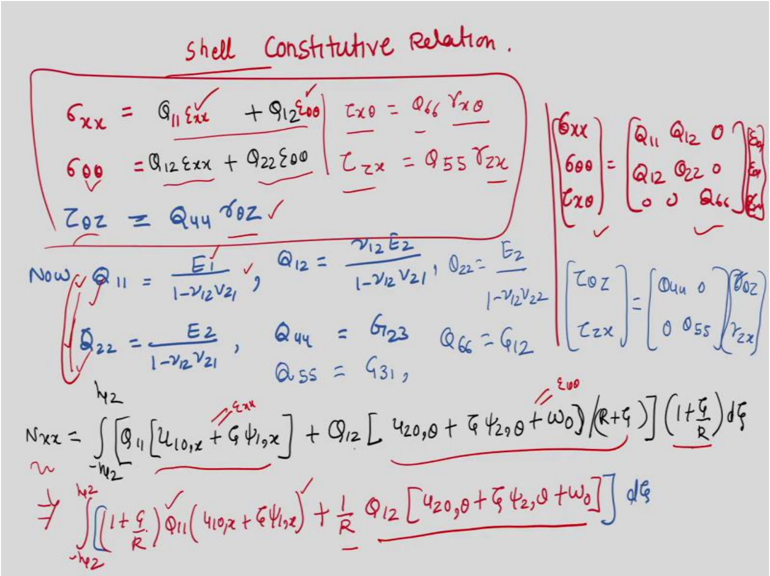

Now, we have to define the shell constitutive relation so, somebody has done for a plate constitutive relation. Now, I am saying that it is a shell constitutive relation.

Using the basic constitutive relations:

$$\sigma_{xx} = Q_{11}\varepsilon_{xx} + Q_{12}\varepsilon_{\theta\theta}$$

$$\sigma_{\theta\theta} = Q_{12}\varepsilon_{xx} + Q_{22}\varepsilon_{\theta\theta}$$

$$\tau_{\theta z} = Q_{44}\gamma_{\theta z}$$

$$\tau_{x\theta} = Q_{66}\gamma_{x\theta}$$

$$\tau_{zx} = Q_{55}\gamma_{zx}$$

Where, Q_{11}, Q_{12}, Q_{22}, Q_{44}, Q_{66} and Q_{55} can be represented in terms of engineering constants:

$Q_{11} = \dfrac{E_1}{1-\mu_{21}\mu_{12}}$; $Q_{12} = \dfrac{\mu_{12}E_2}{1-\mu_{12}\mu_{21}}$; $Q_{22} = \dfrac{E_2}{1-\mu_{12}\mu_{22}}$; $\bar{Q}_{22} = \dfrac{E_2}{1-\mu_{12}\mu_{21}}$; $Q_{44} = G_{23}$; $Q_{55} = G_{31}$; and $Q_{66} = G_{12}$.

For a composite material, you can find them all. Already, I discussed in the second lecture of week 5, to evaluate through a coding, that first, you give E_1, E_2, E_3, μ_{12}, through a program, then, you have to evaluate Q_1, Q_2, Q_3. And then, later on, I also gave the formula for a transformation of a $\bar{Q}_1$ and $\bar{Q}_2$.

I said that the present formulation is valid for a cross-ply cylindrical shell, one can develop a solution for an angle ply cylindrical shell, but here more generalized. Q_{16} and Q_{26} will also come into the picture. For symmetric and anti-symmetric cases, the analytical solutions are valid, but for an angle ply, finite shell analytical solution is not valid.

Then we have to think for an approximate or finite element solution.

Let us say, $N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\left(1+\frac{\varsigma}{R}\right)d\varsigma$

Here, σ_{xx} is replaced by $Q_{11}\varepsilon_{xx} + Q_{12}\varepsilon_{\theta\theta}$.

Now, explicitly writing, substituting the value of ε_{xx} and $\varepsilon_{\theta\theta}$. N_{xx} will be:

$$N_{xx} = \int_{-h/2}^{h/2}\left[Q_{11}\left[u_{10,x}+\varsigma\psi_{1,x}\right]+Q_{12}\left[\frac{\left(u_{20,\theta}+\varsigma\psi_{2,\theta}+w_0\right)}{(R+\varsigma)}\right]\right]\left(1+\frac{\varsigma}{R}\right)d\varsigma$$

If you put $\left(1+\frac{\varsigma}{R}\right)$ inside, then N_{xx} will be:

$$\int_{-h/2}^{h/2}\left[\left(1+\frac{\varsigma}{R}\right)Q_{11}\left[u_{10,x}+\varsigma\psi_{1,x}\right]+\frac{1}{R}Q_{12}\left[u_{20,\theta}+\varsigma\psi_{2,\theta}+w_0\right]\right]d\varsigma\ .$$

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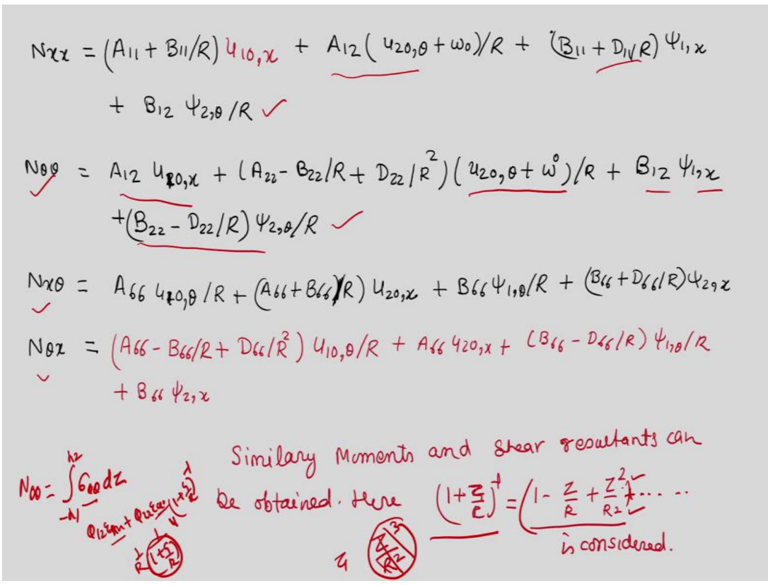

Here, one can see that we can write the coefficients. The first term gives you the definition of A_{11}, the second term gives you the definition of B_{11}. ς^2 is the definition of D_{11} and it is the constant, $Q_{12}\, u_{20,\theta}$ is the definition of A_{12} and if you multiply with $\varsigma\psi_{2,\theta}$ it will be B_{12} and the coefficient of w_0 will be also B_{12}.

Ultimately, N_{xx} can be represented as:

$$N_{xx} = \left(A_{11} + \frac{B_{11}}{R}\right)u_{10,x} + A_{12}\frac{\left(u_{20,\theta} + w_0\right)}{R} + \left(B_{11} + \frac{D_{11}}{R}\right)\psi_{1,x} + B_{12}\frac{\psi_{2,\theta}}{R}$$

This I have explained in lecture 1 of week 5 also. The only thing is that now, we have a slightly bigger form, the basic idea remains the same, but now it is a slightly bigger form because we have more terms.

In the previous case, we do not have ε_{xx}, we have only $\varepsilon_{\theta\theta}$, now the terms corresponding to ε_{xx} are slightly increased when we are going to solve for $N_{\theta\theta}$.

$N_{\theta\theta} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{\theta\theta}\, d\varsigma$. Here, $\sigma_{\theta\theta} = Q_{12}\varepsilon_{xx} + Q_{22}\varepsilon_{\theta\theta}$.

$Q_{12}\varepsilon_{xx}$ is fine, but when you talk about $Q_{22}\varepsilon_{\theta\theta}$, it contains a term $\frac{1}{R}\frac{1}{\left(1+\frac{\varsigma}{R}\right)}$.

If you take upside over there, then it will be $\left(1+\frac{\varsigma}{R}\right)^{-1}$.

In the previous lecture, $\left(1+\frac{\varsigma}{R}\right)^{-1}$ opened up in infinite series, but we are going to consider up to quadratic terms $\left(1+\frac{\varsigma}{R}\right)^{-1} = 1 - \frac{\varsigma}{R} + \frac{\varsigma^2}{R^2}$.

In most of the literature, the Flugge theory has considered the term up to quadratic.

This gives a more accurate solution if we take terms up to quadratic. If we take more terms in cubic and then there is a less contribution for that. $1 - \frac{\varsigma}{R} + \frac{\varsigma^2}{R^2}$ term is considered. If during the integration, ς^3 comes, then we are not going to integrate that.

By following those approaches, $N_{\theta\theta}$ is written like this:

$$N_{\theta\theta} = A_{12}u_{10,x} + \left(A_{22} - \frac{B_{22}}{R} + \frac{D_{22}}{R^2}\right)\frac{\left(u_{20,\theta} + w_0\right)}{R} + B_{12}\psi_{1,x} + \left(B_{22} - \frac{D_{22}}{R}\right)\frac{\psi_{2,\theta}}{R} \ .$$

I have explained for N_{xx} and $N_{\theta\theta}$. By following a similar procedure, one can get the expression for $N_{x\theta}$ and $N_{\theta x}$.

$$N_{x\theta} = A_{66}\frac{u_{10,\theta}}{R} + \left(A_{66} + \frac{B_{66}}{R}\right)u_{20,x} + \frac{B_{66}\psi_{1,\theta}}{R} + \left(B_{66} + \frac{D_{66}}{R}\right)\psi_{2,x}$$

$$N_{\theta x} = \left(A_{66} - \frac{B_{66}}{R} + \frac{D_{66}}{R^2}\right)\frac{u_{10,\theta}}{R} + A_{66}u_{20,x} + \left(B_{66} - \frac{D_{66}}{R}\right)\frac{\psi_{1,\theta}}{R} + B_{66}\psi_{2,x}$$

When we get an expression like this then this expression is known as shell constitutive relations.

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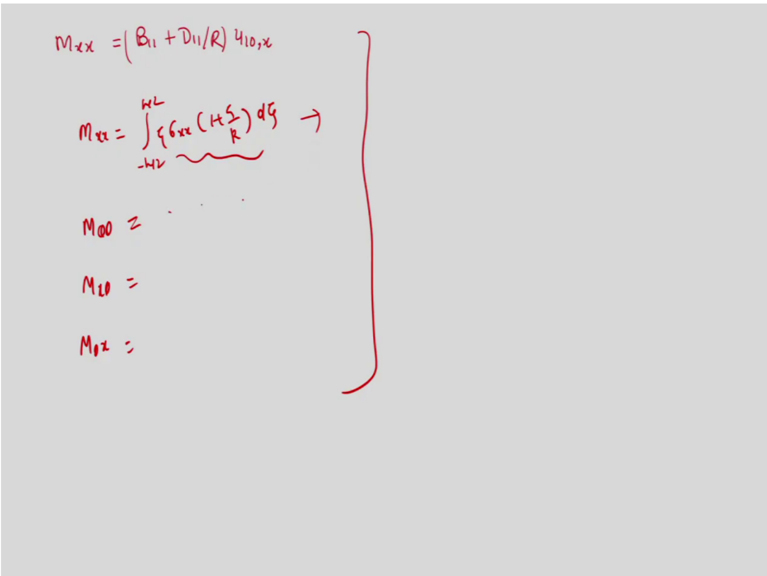

Similarly, one can get the relations for M_{xx}:

$$M_{xx} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma\sigma_{xx}\left(1 + \frac{\varsigma}{R}\right)d\varsigma \ .$$

If you substitute all these things, $M_{xx} = \left(B_{11} + \frac{D_{11}}{R}\right)u_{10,x}$.

And similarly, one can derive for $M_{\theta\theta}$, $M_{x\theta}$, and $M_{\theta x}$, it's very easy, one has to substitute the terms and finally, writing the coefficients.

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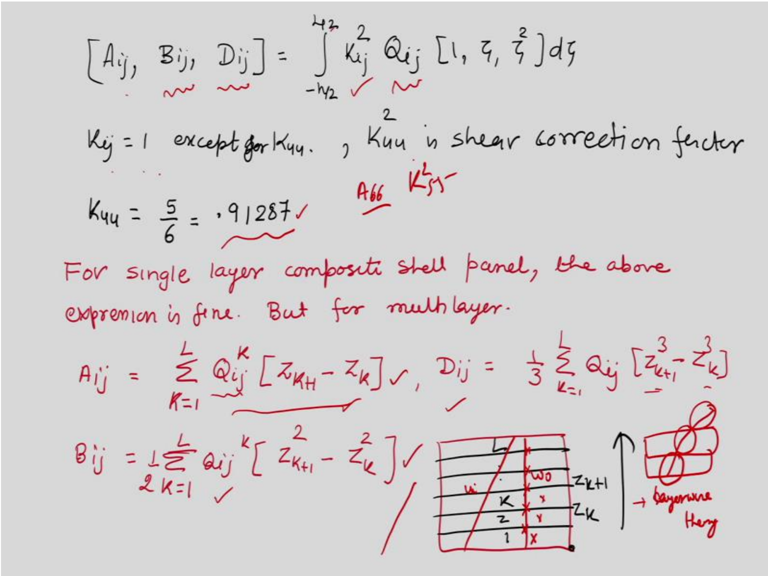

Ultimately, $\left[A_{ij}, B_{ij}, D_{ij}\right] = \int_{\frac{-h}{2}}^{\frac{h}{2}} k_{ij}^{2} Q_{ij} \left(1, \varsigma, \varsigma^{2}\right) d\varsigma$,

When it is multiplied with 1, it will be known as A_{ij}, when it is multiplied with ς, it will be known as B_{ij} , and when it is multiplied with ς^{2}, it will be known as D_{ij}.

Here, the term k_{ij}^{2} is used, this is saying that this k_{ij} = 1, except for k_{44}. When we say that A_{66}, A_{44}, A_{55}, k_{55}^{2}, the shear correction factor is going to be 1, otherwise it is taken 0.91287.

For a single layer, one can integrate and find the value. But for a composite panel; as I have discussed in programming also, we have to take the coordinate system that each layer thickness and the cube difference of the cube and divided by $\frac{1}{3}$. Ultimately, these are the discrete layers, we are adding summation of all the layers material property.

$$A_{ij} = \sum_{K=1}^{L} Q_{ij}^{K} \left(Z_{K+1} - Z_{K}\right);$$

$$B_{ij} = \frac{1}{2} \sum_{K=1}^{L} Q_{ij}^{K} \left(Z_{K+1}^{2} - Z_{K}^{2}\right);$$

$$D_{ij} = \frac{1}{3}\sum_{K=1}^{L} Q_{ij}^{K}\left(Z_{K+1}^{3} - Z_{K}^{3}\right)$$

Kth means the Kth layer, it goes from 1 to L, we can add all this, though it is equivalent single layer theory which means we are solving a single layer, using the concept of summation we can get a solution for a composite layer. If somebody is interested to have a very accurate solution like at each layer interface what is the shear stress variation and all these things.

For those cases, layer-wise theories are used, in which each layer has a number of variables. In the present case, the variation of u is assumed linear across the thickness and w is constant along the thickness, w_0 and u_i ($u_i = u_1$ and u_2). But if you talk about a layer-wise theory, in each layer it is linear, it may have some kinkiness, it can represent the local behavior.

The theory I have discussed here is valid for a thick shell theory, but it may not give a very accurate result for inter-laminar shear stresses.

For that case, either one should try for a 3-dimensional solution or the layer-wise theories. But in general, if you are interested in the deflection behavior or in-plane bending, bending stresses are pretty good, but the transverse shear stresses are slightly away from the three-dimensional solutions

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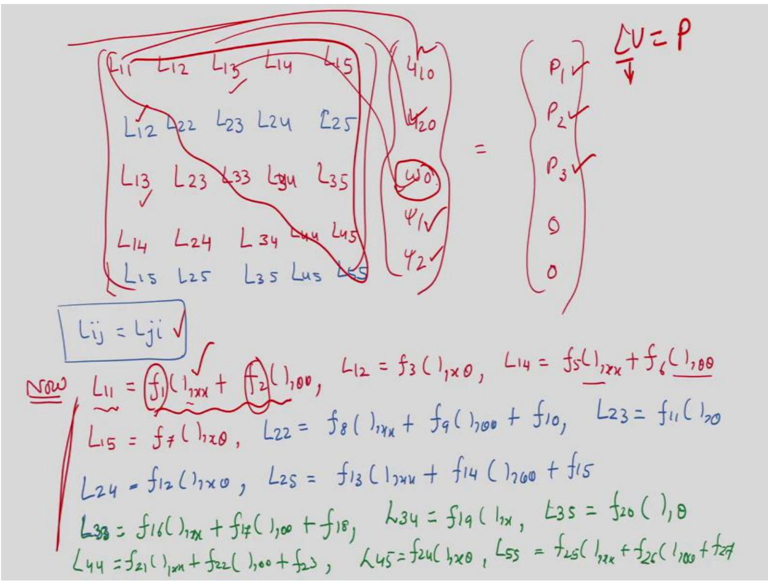

Ultimately, we get five differential equations. If we substitute the value of $M_{x\theta}$, $N_{x\theta}$, q_x, q_θ, if you substitute all in the main 5 partial differential equations that leads to this set of equations that LU = P, where L is a differential operator, it is a 5 by 5 matrix:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{12} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{13} & L_{23} & L_{33} & L_{34} & L_{35} \\ L_{14} & L_{24} & L_{34} & L_{44} & L_{45} \\ L_{15} & L_{25} & L_{35} & L_{45} & L_{55} \end{bmatrix} \begin{bmatrix} u_{10} \\ u_{20} \\ w_0 \\ \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 0 \\ 0 \end{bmatrix}.$$

The important part is that here, $L_{ij} = L_{ji}$, i.e., $L_{12} = L_{21}$, $L_{13} = L_{31}$

We need to get the non-zero value of the upper half and then we can say that the lower half is the same. $L_{11} = f_1(\;),_{xx} + f_2(\;),_{\theta\theta}$

f_1 and f_2 I have explained in the previous lecture.

$f_1 = A_{22} + \frac{B_{22}}{R} + \frac{D_{22}}{R^2}$ and so on.

f_1 and f_2 are nothing but the coefficients of any term, which is having $(\;),_{xx}$, if you are talking about particularly

$L_{11} = f_1(u_{10}),_{xx} + f_2(u_{10}),_{\theta\theta}$. $L_{12} = f_3(u_{20}),_{x\theta}$.

In that way you can find them all:

$L_{11} = f_1(\;),_{xx} + f_2(\;),_{\theta\theta}$; $L_{12} = f_3(\;),_{x\theta}$; $L_{14} = f_5(\;),_{xx} + f_6(\;),_{\theta\theta}$; $L_{13} = f_4(\;),_{\theta\theta} + f_5$
$L_{15} = f_7(\;),_{x\theta}$; $L_{22} = f_8(\;),_{xx} + f_9(\;),_{\theta\theta} + f_{10}$; $L_{23} = f_{11}(\;),_{\theta}$; $L_{24} = f_{12}(\;),_{x\theta}$;
$L_{25} = f_{13}(\;),_{xx} + f_{14}(\;),_{\theta\theta} + f_{15}$; $L_{33} = f_{16}(\;),_{xx} + f_{17}(\;),_{\theta\theta} + f_{18}$; $L_{34} = f_{19}(\;),_{x}$; $L_{35} = f_{20}(\;),_{\theta}$
$L_{44} = f_{21}(\;),_{xx} + f_{22}(\;),_{\theta\theta} + f_{23}$; $L_{45} = f_{24}(\;),_{x\theta}$; $L_{55} = f_{25}(\;),_{xx} + f_{26}(\;),_{\theta\theta} + f_{27}$
$L_{14} = f_5(w_0),_{xx} + f_6(w_0),_{\theta\theta}$, only thing is that the fifth equation is placed here at the third position.

We have obtained ψ_1 and ψ_2, and these are loading p_1, p_2, and p_3. I have not presented all the individual components like f_1 and f_2 because I have already given the details in lecture 1. One can derive or find the actual component of f_1 and f_2. In this lecture, I am just presenting the overall picture to proceed with a solution of a finite cylindrical shell.

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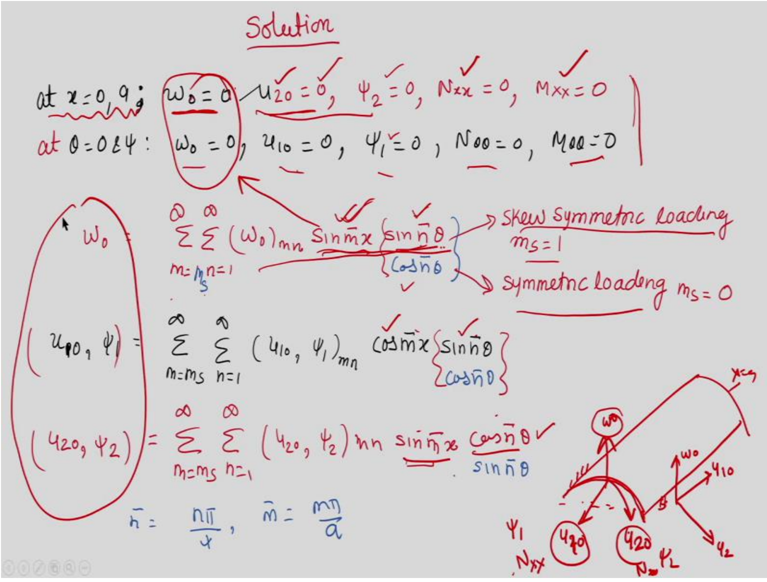

For the solution for a simply supported shell; for the boundaries, x = 0 and x = a, the following variables need to be specified;

w_0, transverse deflection = 0

The longitudinal deflection u_{20} = 0, if you talk about a shell like this: over this edge it will be u_{10}, this is u_{20}, and w_0.

If you are talking about x = 0; this is the edge where we are going to specify the boundary condition and this is the edge where, x = a, w_0 = 0, u_{20} = 0.

N_{xx} = 0 and M_{xx} = 0.

ψ_1 and ψ_2 are rotations, here instead of $N_{x\theta}$, u_{20} this longitudinal = 0, ψ_2 = 0, and w_0 = 0; then only we can assume the solution in a Fourier series, otherwise, we cannot get the solution.

This is the hard simply supported condition. At θ = 0, and θ = ψ; there you have simply supported case. Here, w_0 = 0, u_{10} = 0, ψ_1 = 0, $N_{\theta\theta}$ = 0, and $M_{\theta\theta}$ = 0.

If you see that at x = 0 and x = θ, w_0 and ψ_2 = 0.

Straightforward we can assume: $w_0 = \sum_{m=ms}^{\infty}\sum_{n=1}^{\infty}(w_0)_{mn}\sin\bar{m}x\begin{Bmatrix}\sin\bar{n}\theta\\ \cos\bar{n}\theta\end{Bmatrix}$; w_0 in x and θ

direction is sin series, if you put x = 0, then sin = 0;

If you put x = a, then sin = 0;

If you put θ = 0, then sin = 0;

If you put $\theta = \psi$, then again sin = 0.

In this way, the expression satisfies the boundary condition exactly that is why sometimes it is called is the exact solution. And then the variable u_{20} = 0, where x = 0 and a.

$$(u_{20},\psi_2) = \sum_{m=ms}^{\infty}\sum_{n=1}^{\infty}(u_{20},\psi_2)_{mn} \sin \bar{m}x \begin{Bmatrix} \cos \bar{n}\theta \\ \sin \bar{n}\theta \end{Bmatrix} \quad \bar{n} = \frac{n\pi}{\psi} \quad \bar{m} = \frac{m\pi}{a}$$

Along the x-axis, u_{20} is assumed as sin series, but along θ direction, it can be assumed cos series. And the same way u_{10} = 0 along θ direction. u_{10} is assumed sin series along θ direction and cosine along x direction, and ψ_1 and ψ_2, follow the similar procedure.

$$(u_{10},\psi_1) = \sum_{m=ms}^{\infty}\sum_{n=1}^{\infty}(u_{10},\psi_1)_{mn} \cos \bar{m}x \begin{Bmatrix} \sin \bar{n}\theta \\ \cos \bar{n}\theta \end{Bmatrix}$$

When a cylinder is subjected to skew-symmetric loading, then the solution is expressed like this and ms = 1 here, if the cylinder is having symmetric loading which means that around the θ, the loading is symmetric, then the solution will be assumed in sin and cos. This is the most important part.

Generally, in most of the skew-symmetric cases, the problem is solved in the literature. For the symmetric case, it becomes further special, it independent of θ and then an axisymmetric case can be done, and the problem will be simpler. The most general problem is the skew-symmetric loading where loading is expressed like this.

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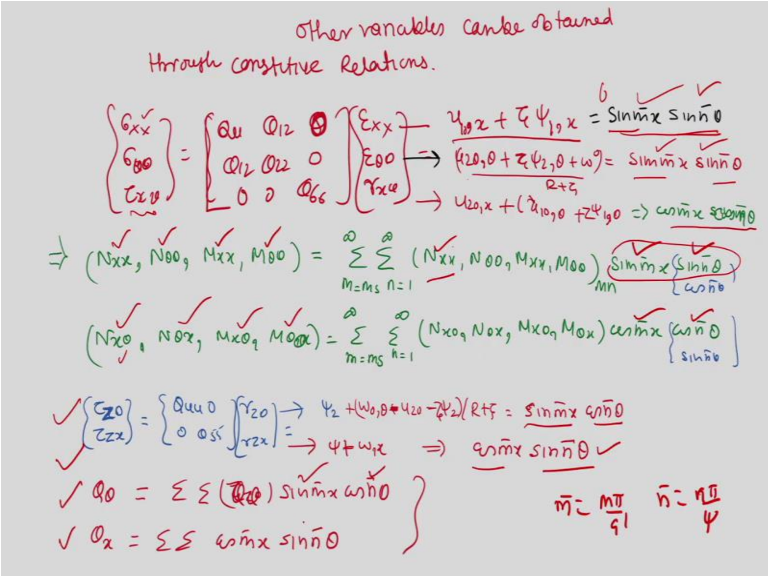

Now, we are interested to represent $\left(N_{xx}, N_{\theta\theta}, M_{xx}, M_{\theta\theta}\right)$ over $\left(N_{x\theta}, N_{\theta x}, M_{x\theta}, M_{\theta x}\right)$; they will be sin cosine function or cos and cos function. For getting this so, first, based on the boundary conditions, we can only express these 5 variables displacement fields; u_{10}, u_{20}, w_0, ψ_1, and ψ_2. After that, if you are sure that displacement field would be such that, then using the constitutive relations, one can find σ_{xx}, $\sigma_{\theta\theta}$, and $\tau_{x\theta}$.

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{\theta\theta} \\ \tau_{x\theta} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{\theta\theta} \\ \gamma_{x\theta} \end{bmatrix} \qquad \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{\theta\theta} \\ \gamma_{x\theta} \end{bmatrix} = \begin{bmatrix} u_{10,x} + \varsigma \psi_{1.x} \\ \left(u_{20,x} + \varsigma \psi_{2.\theta} + w_0\right) / \left(R + \varsigma\right) \\ u_{20,x} + u_{10,\theta} + \varsigma \psi_{1.\theta} \end{bmatrix} = \begin{bmatrix} \sin \bar{m}x \sin \bar{n}\theta \\ \sin \bar{m}x \sin \bar{n}\theta \\ \cos \bar{m}x \cos \bar{n}\theta \end{bmatrix}$$

We can see that N_{xx} is just an integration in the thickness direction, it is not changing along in-plane direction.

N_{xx}, $N_{\theta\theta}$, M_{xx}, and $M_{\theta\theta}$ can be expressed in terms of sin series as w_0 is expressed:

$$\left(N_{xx}, N_{\theta\theta}, M_{xx}, M_{\theta\theta}\right) = \sum_{m=ms}^{\infty} \sum_{n=1}^{\infty} \left(N_{xx}, N_{\theta\theta}, M_{xx}, M_{\theta\theta}\right)_{mn} \sin \bar{m}x \begin{Bmatrix} \sin \bar{n}\theta \\ \cos \bar{n}\theta \end{Bmatrix}$$

Now, we talk about $N_{x\theta}$; $\tau_{x\theta}$ gave you the expression of $\cos \bar{m}x \cos \bar{n}\theta$ So, therefore, $N_{x\theta}, N_{\theta x}, M_{x\theta}, and\ M_{\theta x}$ will be represented as cos cos series:

$$\left(N_{x\theta}, N_{\theta x}, M_{x\theta}, M_{\theta x}\right) = \sum_{m=ms}^{\infty} \sum_{n=1}^{\infty} \left(N_{x\theta}, N_{\theta x}, M_{x\theta}, M_{\theta x}\right)_{mn} \cos \bar{m}x \begin{Bmatrix} \cos \bar{n}\theta \\ \cos \bar{n}\theta \end{Bmatrix}$$

Some of you may ask why are you worrying about stresses when ultimately, we are going to solve only displacement. Yes, we will solve the displacement, but at the end of the day, we are interested to find the variation of stress resultants or the stresses also. This will give you the actual variation, there will be some magnitude, but they will follow this path, mode of type.

Then,

$$\begin{bmatrix} \tau_{z\theta} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{bmatrix} \gamma_{z\theta} \\ \gamma_{zx} \end{bmatrix} \quad \begin{bmatrix} \gamma_{z\theta} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \psi_2 + \left(w_{0,\theta} - u_{20} - \varsigma\psi_2\right)/\left(R+\varsigma\right) \\ \psi + w_{0,x} \end{bmatrix} = \begin{bmatrix} \sin\bar{m}x\cos\bar{n}\theta \\ \cos\bar{m}x\sin\bar{n}\theta \end{bmatrix} \mathrm{W}$$

here, $\bar{m}$ is equal to $\dfrac{m\pi}{a}$ and $\bar{n}$ is equal to $\dfrac{n\pi}{\psi}$.

Similarly,

$$Q_\theta = \sum_{m=ms}^{\infty}\sum_{n=1}^{\infty}\left(Q_\theta\right)_{mn}\sin\bar{m}x\cos\bar{n}\theta$$

$$Q_x = \sum_{m=ms}^{\infty}\sum_{n=1}^{\infty}\left(Q_z\right)_{mn}\cos\bar{m}x\sin\bar{n}\theta$$

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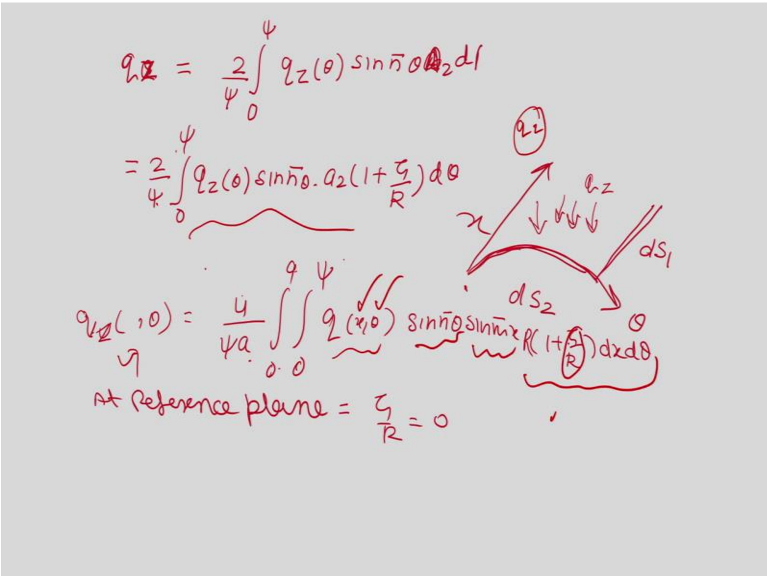

The loading that q_z or q_θ : if you find the Fourier series for the case of infinite, it will be:

$$q_z = \frac{2}{\psi}\int_0^{\psi} q_z\left(\theta\right)\sin\bar{n}\theta A_2 d\varsigma \Rightarrow \frac{2}{\psi}\int_0^{\psi} q_z\left(\theta\right)\sin\bar{n}\theta . a_2\left(1+\frac{\varsigma}{R}\right)d\theta .$$

Then, $q_z\left(,_\theta\right) = \dfrac{4}{\psi_a}\displaystyle\int_0^{q}\int_0^{\psi} q\left(x,\theta\right)\sin\bar{n}\theta\sin\bar{m}xR\left(1+\frac{\varsigma}{R}\right)dxd\theta$, most of the time for the 2-

dimensional theory q_z is presented at the reference plane where, $\frac{\varsigma}{R}$ = 0.

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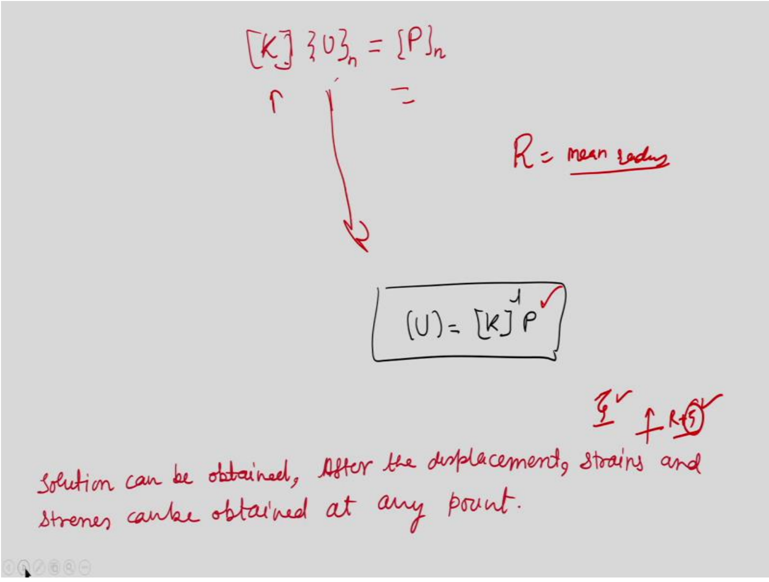

If you substitute all the sin expressions into l, that gives you a matrix $[K]\{U\}_n$ = P. From here, $U = [K]^{-1} P$.

The solution for a simply supported finite shell can be obtained, where only terms is R, R is the mean radius at any location. $(R+\varsigma)$, if you put there the value of ς, then you can obtain the strain and stresses at any point.

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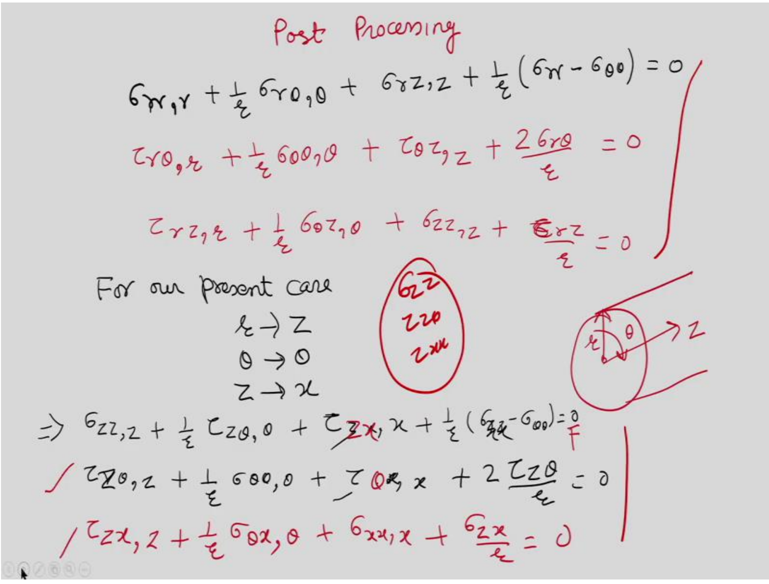

The post-processing techniques; the stresses, if you talk about that $\sigma_{zz,z}$ or $\tau_{zz,\theta}$ or $\tau_{zx,z}$, all are transverse stresses, we are interested to find σ_{zz}, $\tau_{z\theta}$, and τ_{zx}. Already we are using the shear correction factor that is fine, but still, there is some inaccuracy.

If we do the post-processing technique following are the general 3-dimensional equation for equilibrium:

$$\sigma_{rr,r}+\frac{1}{r}\sigma_{r\theta,\theta}+\sigma_{rz,z}+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta})=0$$

$$\tau_{r\theta,r}+\frac{1}{r}\sigma_{\theta\theta,\theta}+\tau_{\theta z,z}+\frac{2\sigma_{r\theta}}{r}=0$$

$$\tau_{rz,r}+\frac{1}{r}\sigma_{\theta z,\theta}+\sigma_{zz,z}+\frac{\tau_{rz}}{r}=0$$

If we convert to our system for the present case, then those equations will be:

$$\sigma_{zz,z}+\frac{1}{r}\tau_{z\theta,\theta}+\tau_{zx,x}+\frac{1}{r}(\sigma_{xx}-\sigma_{\theta\theta})=0$$

$$\tau_{z\theta,z}+\frac{1}{r}\sigma_{\theta\theta,\theta}+\tau_{\theta x,x}+\frac{2\tau_{z\theta}}{r}=0$$

$$\tau_{zx,z}+\frac{1}{r}\sigma_{\theta x,\theta}+\sigma_{xx,x}+\frac{\sigma_{zx}}{r}=0$$

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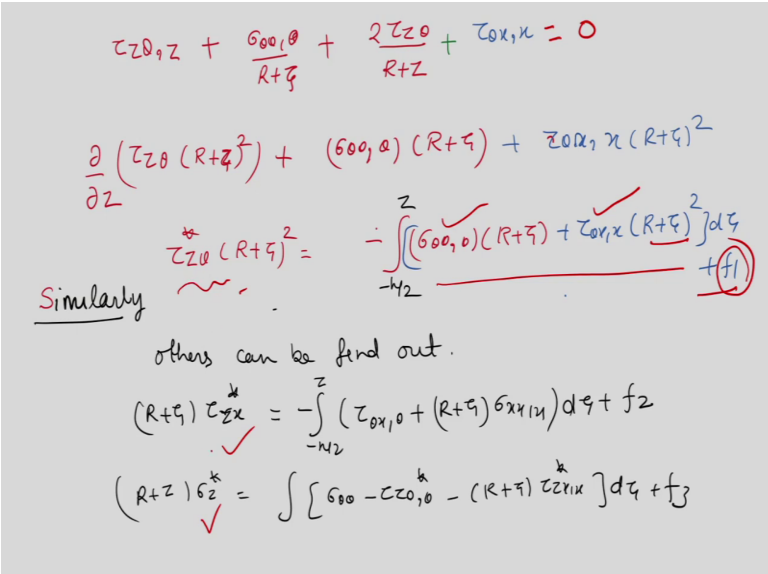

Ultimately, the second equation for a static case is going to be 0:

$$\tau_{z\theta,z}+\frac{\sigma_{\theta\theta,\theta}}{R+\varsigma}+\frac{2\tau_{z\theta}}{R+\varsigma}+\tau_{\theta x,x}=0\,.$$

$$\tau^*_{z\theta}\left(R+\varsigma\right)^2 = -\int_{-h/2}^{z}\left[\left(\sigma_{\theta\theta,\theta}\right)\left(R+\varsigma\right)+\tau_{\theta x,x}\left(R+\varsigma\right)^2\right]d\varsigma + f_1$$

And the same way τ^*_{zx} and σ^*_z can be obtained:

$$\left(R+\varsigma\right)\tau^*_{zx} = -\int_{-h/2}^{z}\left[\tau_{\theta x,\theta}+\left(R+\varsigma\right)\sigma_{xx,x}\right]d\varsigma + f_2$$

$$\left(R+\varsigma\right)\sigma^*_{z} = -\int_{-h/2}^{z}\left[\sigma_{\theta\theta}-\sigma^*_{z\theta,\theta}-\left(R+\varsigma\right)\sigma^*_{zx,x}\right]d\varsigma + f_3$$

We put star because they are different as we have obtained these through constitutive relations. They are slightly different, so we put star here. We have obtained these through post-processing techniques.

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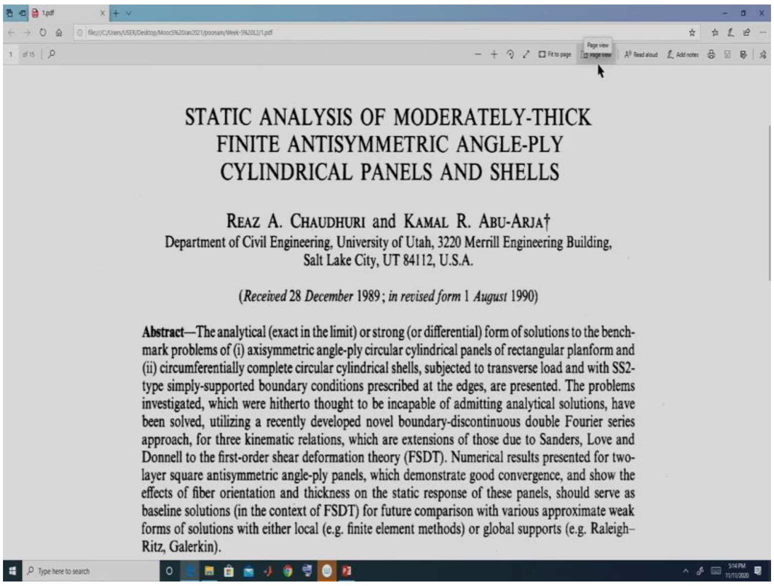

STATIC ANALYSIS OF MODERATELY-THICK FINITE ANTISYMMETRIC ANGLE-PLY CYLINDRICAL PANELS AND SHELLS

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Abstract—The analytical (exact in the limit) or strong (or differential) form of solutions to the benchmark problems of (i) axisymmetric angle-ply circular cylindrical panels of rectangular planform and (ii) circumferentially complete circular cylindrical shells, subjected to transverse load and with SS2-type simply-supported boundary conditions prescribed at the edges, are presented. The problems investigated, which were hitherto thought to be incapable of admitting analytical solutions, have been solved, utilizing a recently developed novel boundary-discontinuous double Fourier series approach, for three kinematic relations, which are extensions of those due to Sanders, Love and Donnell to the first-order shear deformation theory (FSDT). Numerical results presented for two-layer square antisymmetric angle-ply panels, which demonstrate good convergence, and show the effects of fiber orientation and thickness on the static response of these panels, should serve as baseline solutions (in the context of FSDT) for future comparison with various approximate weak forms of solutions with either local (e.g. finite element methods) or global supports (e.g. Raleigh–Ritz, Galerkin).

The very first paper in this field “static bending of cylindrical shells or the composite shells” was published in 1964. There is another paper “static analysis of moderately-thick finite antisymmetric angle-ply cylindrical panels and shells, had published in 1989, 1991.

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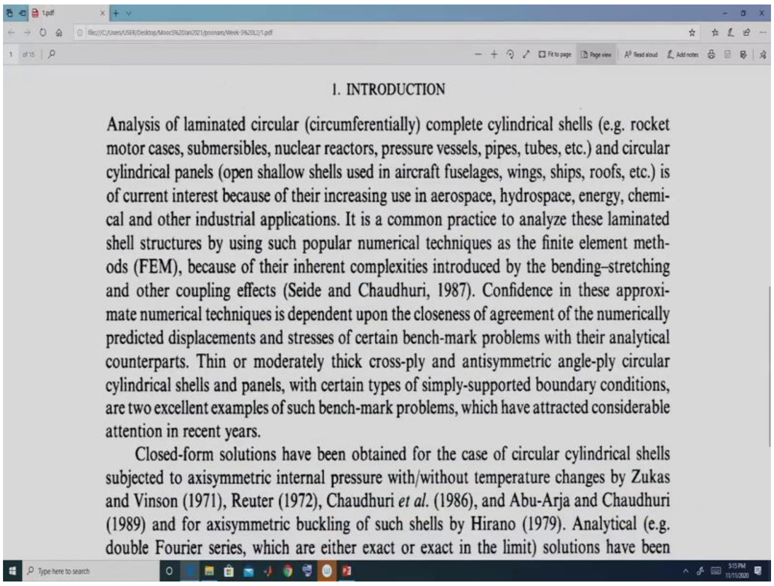

1. INTRODUCTION

Analysis of laminated circular (circumferentially) complete cylindrical shells (e.g. rocket motor cases, submersibles, nuclear reactors, pressure vessels, pipes, tubes, etc.) and circular cylindrical panels (open shallow shells used in aircraft fuselages, wings, ships, roofs, etc.) is of current interest because of their increasing use in aerospace, hydrospace, energy, chemical and other industrial applications. It is a common practice to analyze these laminated shell structures by using such popular numerical techniques as the finite element methods (FEM), because of their inherent complexities introduced by the bending–stretching and other coupling effects (Seide and Chaudhuri, 1987). Confidence in these approximate numerical techniques is dependent upon the closeness of agreement of the numerically predicted displacements and stresses of certain bench-mark problems with their analytical counterparts. Thin or moderately thick cross-ply and antisymmetric angle-ply circular cylindrical shells and panels, with certain types of simply-supported boundary conditions, are two excellent examples of such bench-mark problems, which have attracted considerable attention in recent years.

Closed-form solutions have been obtained for the case of circular cylindrical shells subjected to axisymmetric internal pressure with/without temperature changes by Zukas and Vinson (1971), Reuter (1972), Chaudhuri *et al.* (1986), and Abu-Arja and Chaudhuri (1989) and for axisymmetric buckling of such shells by Hirano (1979). Analytical (e.g. double Fourier series, which are either exact or exact in the limit) solutions have been

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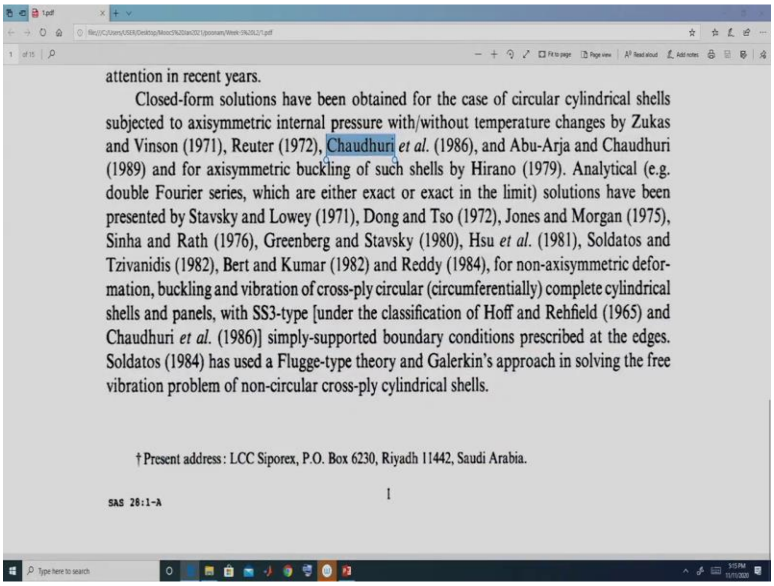

attention in recent years.

Closed-form solutions have been obtained for the case of circular cylindrical shells subjected to axisymmetric internal pressure with/without temperature changes by Zukas and Vinson (1971), Reuter (1972), Chaudhuri *et al.* (1986), and Abu-Arja and Chaudhuri (1989) and for axisymmetric buckling of such shells by Hirano (1979). Analytical (e.g. double Fourier series, which are either exact or exact in the limit) solutions have been presented by Stavsky and Lowey (1971), Dong and Tso (1972), Jones and Morgan (1975), Sinha and Rath (1976), Greenberg and Stavsky (1980), Hsu *et al.* (1981), Soldatos and Tzivanidis (1982), Bert and Kumar (1982) and Reddy (1984), for non-axisymmetric deformation, buckling and vibration of cross-ply circular (circumferentially) complete cylindrical shells and panels, with SS3-type [under the classification of Hoff and Rehfield (1965) and Chaudhuri *et al.* (1986)] simply-supported boundary conditions prescribed at the edges. Soldatos (1984) has used a Flugge-type theory and Galerkin's approach in solving the free vibration problem of non-circular cross-ply cylindrical shells.

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In this paper a cylindrical shell is studied, simply supported boundary conditions prescribed at the edges. it is clearly written here that they have used Flugge-type shell theory and Galerkin’s approach.

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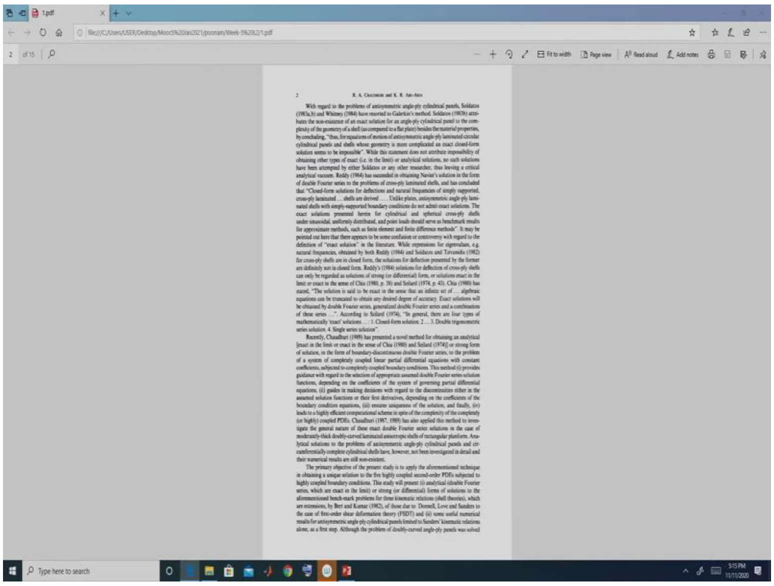

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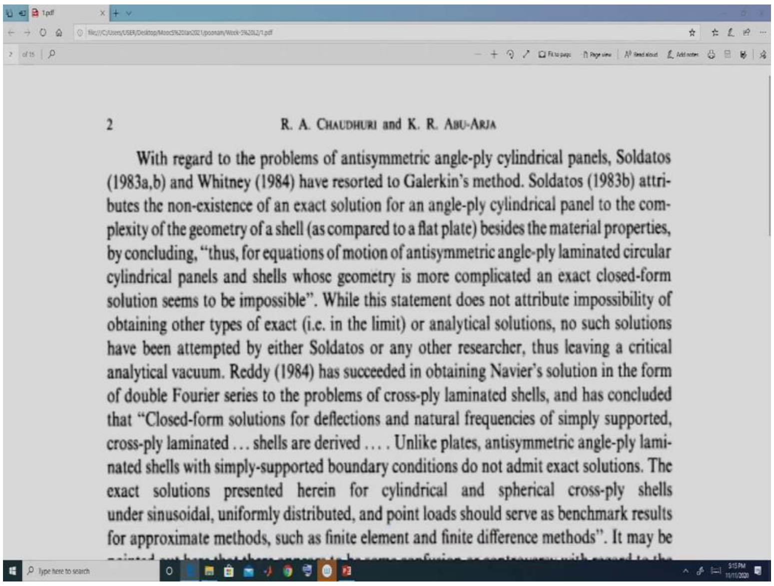

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With regard to the problems of antisymmetric angle-ply cylindrical panels, Soldatos (1983a,b) and Whitney (1984) have resorted to Galerkin’s method. Soldatos (1983b) attributes the non-existence of an exact solution for an angle-ply cylindrical panel to the complexity of the geometry of a shell (as compared to a flat plate) besides the material properties, by concluding, “thus, for equations of motion of antisymmetric angle-ply laminated circular cylindrical panels and shells whose geometry is more complicated an exact closed-form solution seems to be impossible”. While this statement does not attribute impossibility of obtaining other types of exact (i.e. in the limit) or analytical solutions, no such solutions have been attempted by either Soldatos or any other researcher, thus leaving a critical analytical vacuum. Reddy (1984) has succeeded in obtaining Navier’s solution in the form of double Fourier series to the problems of cross-ply laminated shells, and has concluded that “Closed-form solutions for deflections and natural frequencies of simply supported, cross-ply laminated . . . shells are derived . . . . Unlike plates, antisymmetric angle-ply laminated shells with simply-supported boundary conditions do not admit exact solutions. The exact solutions presented herein for cylindrical and spherical cross-ply shells under sinusoidal, uniformly distributed, and point loads should serve as benchmark results for approximate methods, such as finite element and finite difference methods”. It may be

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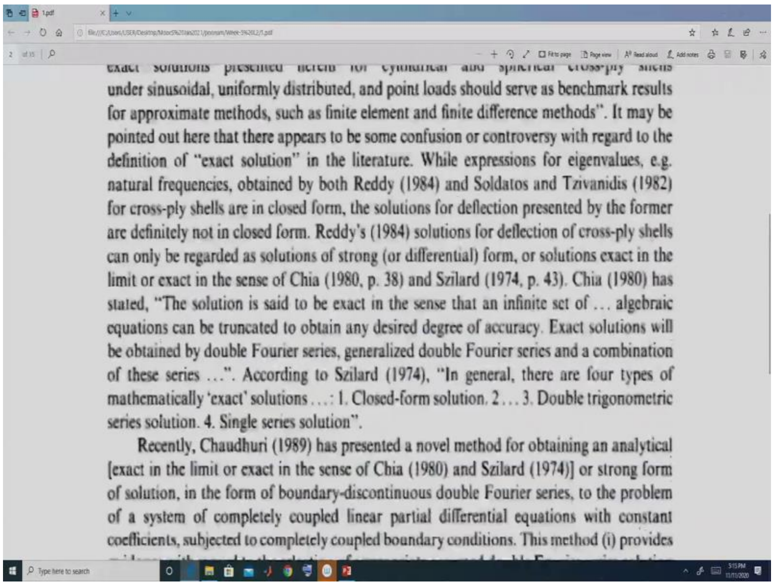
exact solutions presented herein for cylindrical and spherical cross-ply shells under sinusoidal, uniformly distributed, and point loads should serve as benchmark results for approximate methods, such as finite element and finite difference methods". It may be pointed out here that there appears to be some confusion or controversy with regard to the definition of "exact solution" in the literature. While expressions for eigenvalues, e.g. natural frequencies, obtained by both Reddy (1984) and Soldatos and Tzivanidis (1982) for cross-ply shells are in closed form, the solutions for deflection presented by the former are definitely not in closed form. Reddy's (1984) solutions for deflection of cross-ply shells can only be regarded as solutions of strong (or differential) form, or solutions exact in the limit or exact in the sense of Chia (1980, p. 38) and Szilard (1974, p. 43). Chia (1980) has stated, "The solution is said to be exact in the sense that an infinite set of ... algebraic equations can be truncated to obtain any desired degree of accuracy. Exact solutions will be obtained by double Fourier series, generalized double Fourier series and a combination of these series ...". According to Szilard (1974), "In general, there are four types of mathematically 'exact' solutions ...: 1. Closed-form solution. 2 ... 3. Double trigonometric series solution. 4. Single series solution".

Recently, Chaudhuri (1989) has presented a novel method for obtaining an analytical [exact in the limit or exact in the sense of Chia (1980) and Szilard (1974)] or strong form of solution, in the form of boundary-discontinuous double Fourier series, to the problem of a system of completely coupled linear partial differential equations with constant coefficients, subjected to completely coupled boundary conditions. This method (i) provides

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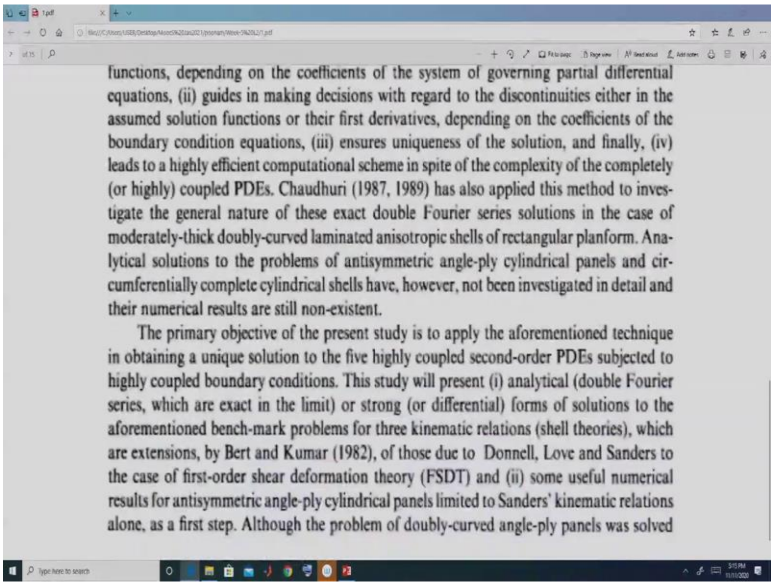
functions, depending on the coefficients of the system of governing partial differential equations, (ii) guides in making decisions with regard to the discontinuities either in the assumed solution functions or their first derivatives, depending on the coefficients of the boundary condition equations, (iii) ensures uniqueness of the solution, and finally, (iv) leads to a highly efficient computational scheme in spite of the complexity of the completely (or highly) coupled PDEs. Chaudhuri (1987, 1989) has also applied this method to investigate the general nature of these exact double Fourier series solutions in the case of moderately-thick doubly-curved laminated anisotropic shells of rectangular planform. Analytical solutions to the problems of antisymmetric angle-ply cylindrical panels and circumferentially complete cylindrical shells have, however, not been investigated in detail and their numerical results are still non-existent.

The primary objective of the present study is to apply the aforementioned technique in obtaining a unique solution to the five highly coupled second-order PDEs subjected to highly coupled boundary conditions. This study will present (i) analytical (double Fourier series, which are exact in the limit) or strong (or differential) forms of solutions to the aforementioned bench-mark problems for three kinematic relations (shell theories), which are extensions, by Bert and Kumar (1982), of those due to Donnell, Love and Sanders to the case of first-order shear deformation theory (FSDT) and (ii) some useful numerical results for antisymmetric angle-ply cylindrical panels limited to Sanders' kinematic relations alone, as a first step. Although the problem of doubly-curved angle-ply panels was solved

And literature survey is presented.

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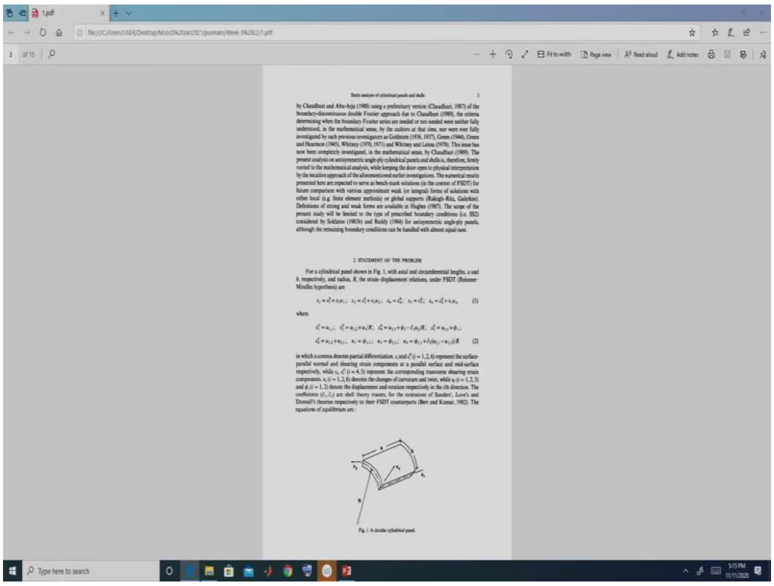

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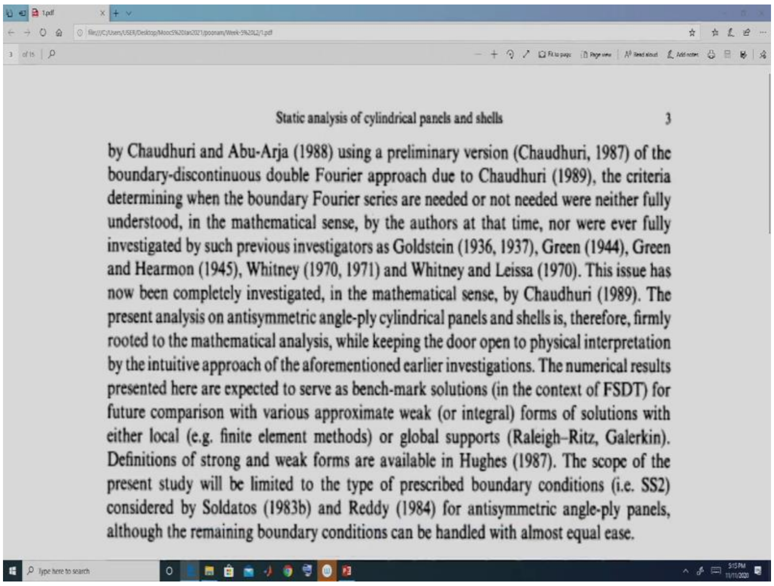

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by Chaudhuri and Abu-Arja (1988) using a preliminary version (Chaudhuri, 1987) of the boundary-discontinuous double Fourier approach due to Chaudhuri (1989), the criteria determining when the boundary Fourier series are needed or not needed were neither fully understood, in the mathematical sense, by the authors at that time, nor were ever fully investigated by such previous investigators as Goldstein (1936, 1937), Green (1944), Green and Hearmon (1945), Whitney (1970, 1971) and Whitney and Leissa (1970). This issue has now been completely investigated, in the mathematical sense, by Chaudhuri (1989). The present analysis on antisymmetric angle-ply cylindrical panels and shells is, therefore, firmly rooted to the mathematical analysis, while keeping the door open to physical interpretation by the intuitive approach of the aforementioned earlier investigations. The numerical results presented here are expected to serve as bench-mark solutions (in the context of FSDT) for future comparison with various approximate weak (or integral) forms of solutions with either local (e.g. finite element methods) or global supports (Raleigh–Ritz, Galerkin). Definitions of strong and weak forms are available in Hughes (1987). The scope of the present study will be limited to the type of prescribed boundary conditions (i.e. SS2) considered by Soldatos (1983b) and Reddy (1984) for antisymmetric angle-ply panels, although the remaining boundary conditions can be handled with almost equal ease.

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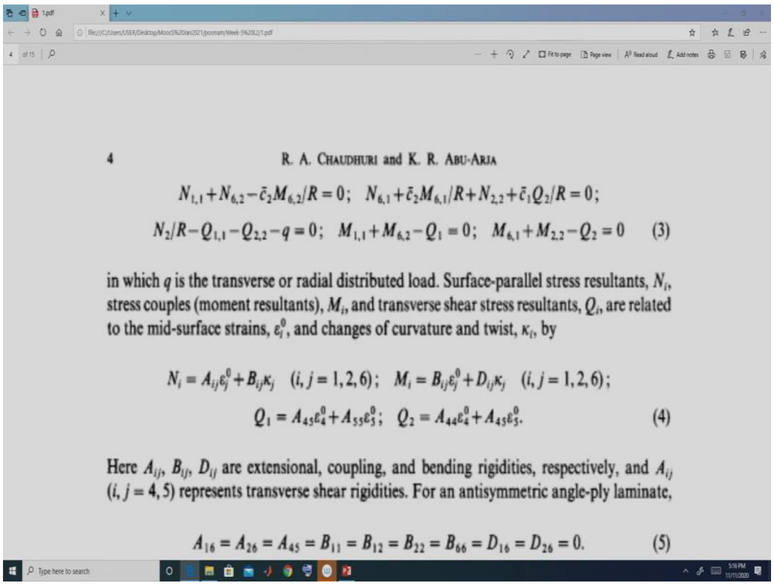

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$$N_{1,1}+N_{6,2}-\bar{c}_2M_{6,2}/R=0;\quad N_{6,1}+\bar{c}_2M_{6,1}/R+N_{2,2}+\bar{c}_1Q_2/R=0;$$
$$N_2/R-Q_{1,1}-Q_{2,2}-q=0;\quad M_{1,1}+M_{6,2}-Q_1=0;\quad M_{6,1}+M_{2,2}-Q_2=0 \qquad (3)$$

in which q is the transverse or radial distributed load. Surface-parallel stress resultants, N_i, stress couples (moment resultants), M_i, and transverse shear stress resultants, Q_i, are related to the mid-surface strains, ε_i^0, and changes of curvature and twist, κ_i, by

$$N_i=A_{ij}\varepsilon_j^0+B_{ij}\kappa_j\quad (i,j=1,2,6);\quad M_i=B_{ij}\varepsilon_j^0+D_{ij}\kappa_j\quad (i,j=1,2,6);$$
$$Q_1=A_{45}\varepsilon_4^0+A_{55}\varepsilon_5^0;\quad Q_2=A_{44}\varepsilon_4^0+A_{45}\varepsilon_5^0. \qquad (4)$$

Here A_{ij}, B_{ij}, D_{ij} are extensional, coupling, and bending rigidities, respectively, and A_{ij} $(i,j=4,5)$ represents transverse shear rigidities. For an antisymmetric angle-ply laminate,

$$A_{16}=A_{26}=A_{45}=B_{11}=B_{12}=B_{22}=B_{66}=D_{16}=D_{26}=0. \qquad (5)$$

One can see that the strains are expressed: $\varepsilon_{1,}\varepsilon_{2,}\varepsilon_{3,}\varepsilon_{4,}\varepsilon_{5,}\varepsilon_6$ *and* $\varepsilon=0$; and in terms of the displacement field, they are represented like this. k_1, k_2, ε_{11} of 1, ε_{22} of 1 and so on., It is FSDT type theory, these are the governing equations obtained.

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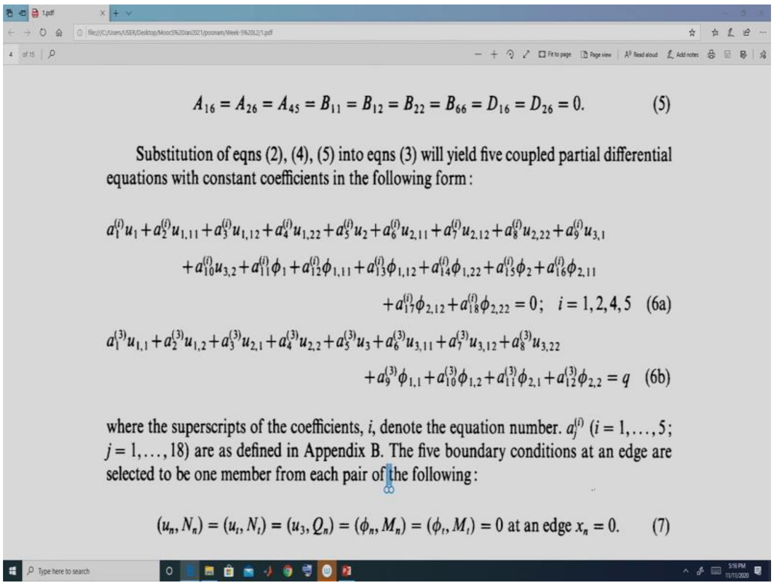

$$A_{16}=A_{26}=A_{45}=B_{11}=B_{12}=B_{22}=B_{66}=D_{16}=D_{26}=0. \qquad (5)$$

Substitution of eqns (2), (4), (5) into eqns (3) will yield five coupled partial differential equations with constant coefficients in the following form:

$$a_1^{(i)}u_1+a_2^{(i)}u_{1,11}+a_3^{(i)}u_{1,12}+a_4^{(i)}u_{1,22}+a_5^{(i)}u_2+a_6^{(i)}u_{2,11}+a_7^{(i)}u_{2,12}+a_8^{(i)}u_{2,22}+a_9^{(i)}u_{3,1}$$
$$+a_{10}^{(i)}u_{3,2}+a_{11}^{(i)}\phi_1+a_{12}^{(i)}\phi_{1,11}+a_{13}^{(i)}\phi_{1,12}+a_{14}^{(i)}\phi_{1,22}+a_{15}^{(i)}\phi_2+a_{16}^{(i)}\phi_{2,11}$$
$$+a_{17}^{(i)}\phi_{2,12}+a_{18}^{(i)}\phi_{2,22}=0;\quad i=1,2,4,5 \qquad (6a)$$
$$a_1^{(3)}u_{1,1}+a_2^{(3)}u_{1,2}+a_3^{(3)}u_{2,1}+a_4^{(3)}u_{2,2}+a_5^{(3)}u_3+a_6^{(3)}u_{3,11}+a_7^{(3)}u_{3,12}+a_8^{(3)}u_{3,22}$$
$$+a_9^{(3)}\phi_{1,1}+a_{10}^{(3)}\phi_{1,2}+a_{11}^{(3)}\phi_{2,1}+a_{12}^{(3)}\phi_{2,2}=q \qquad (6b)$$

where the superscripts of the coefficients, i, denote the equation number. $a_j^{(i)}$ $(i=1,\ldots,5;\ j=1,\ldots,18)$ are as defined in Appendix B. The five boundary conditions at an edge are selected to be one member from each pair of the following:

$$(u_n,N_n)=(u_t,N_t)=(u_3,Q_n)=(\phi_n,M_n)=(\phi_t,M_t)=0 \text{ at an edge } x_n=0. \qquad (7)$$

Here you can see that for an antisymmetric angle-ply; I said that one can develop for an angle ply case, but we have to be very particular, sometimes anti-symmetric or symmetric is such that some coefficients can be going to be 0. If you are interested in a general angle ply, then it is very difficult to develop an analytical solution. These are the governing equations expressed.

If you remember that N_i, M_i, the coefficients are expressed as A_{ij}, B_{ij}. These are the standard way of representing the coefficients.

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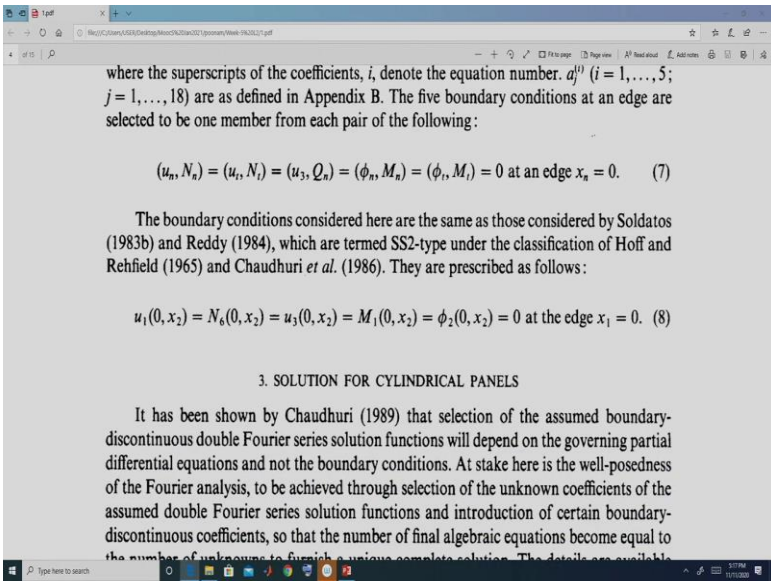

where the superscripts of the coefficients, i, denote the equation number. $a_j^{(i)}$ ($i = 1,\ldots,5$; $j = 1,\ldots,18$) are as defined in Appendix B. The five boundary conditions at an edge are selected to be one member from each pair of the following:

$$(u_n, N_n) = (u_t, N_t) = (u_3, Q_n) = (\phi_n, M_n) = (\phi_t, M_t) = 0 \text{ at an edge } x_n = 0. \quad (7)$$

The boundary conditions considered here are the same as those considered by Soldatos (1983b) and Reddy (1984), which are termed SS2-type under the classification of Hoff and Rehfield (1965) and Chaudhuri *et al.* (1986). They are prescribed as follows:

$$u_1(0, x_2) = N_6(0, x_2) = u_3(0, x_2) = M_1(0, x_2) = \phi_2(0, x_2) = 0 \text{ at the edge } x_1 = 0. \quad (8)$$

3. SOLUTION FOR CYLINDRICAL PANELS

It has been shown by Chaudhuri (1989) that selection of the assumed boundary-discontinuous double Fourier series solution functions will depend on the governing partial differential equations and not the boundary conditions. At stake here is the well-posedness of the Fourier analysis, to be achieved through selection of the unknown coefficients of the assumed double Fourier series solution functions and introduction of certain boundary-discontinuous coefficients, so that the number of final algebraic equations become equal to

How to represent these coefficients A_{44}, A_{55}, so on and if we substitute all these things into the partial equation and the boundary conditions that either this is u_n or N_n, either u_t or N_t at x = 0 and then so on.

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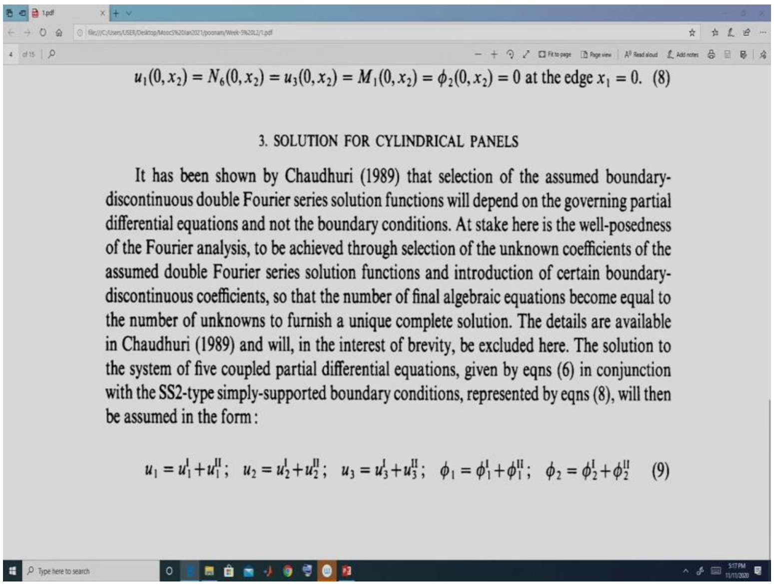

$$u_1(0, x_2) = N_6(0, x_2) = u_3(0, x_2) = M_1(0, x_2) = \phi_2(0, x_2) = 0 \text{ at the edge } x_1 = 0. \quad (8)$$

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$$u_1 = u_1^{I} + u_1^{II}; \quad u_2 = u_2^{I} + u_2^{II}; \quad u_3 = u_3^{I} + u_3^{II}; \quad \phi_1 = \phi_1^{I} + \phi_1^{II}; \quad \phi_2 = \phi_2^{I} + \phi_2^{II} \quad (9)$$

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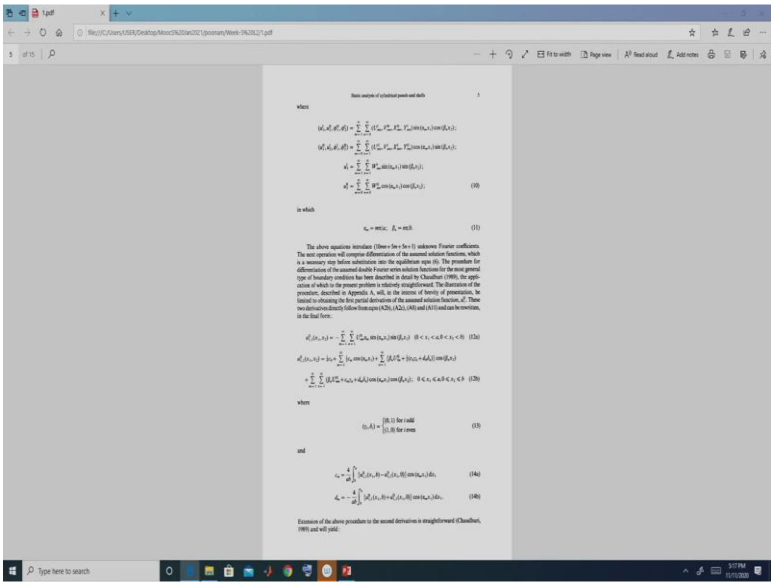

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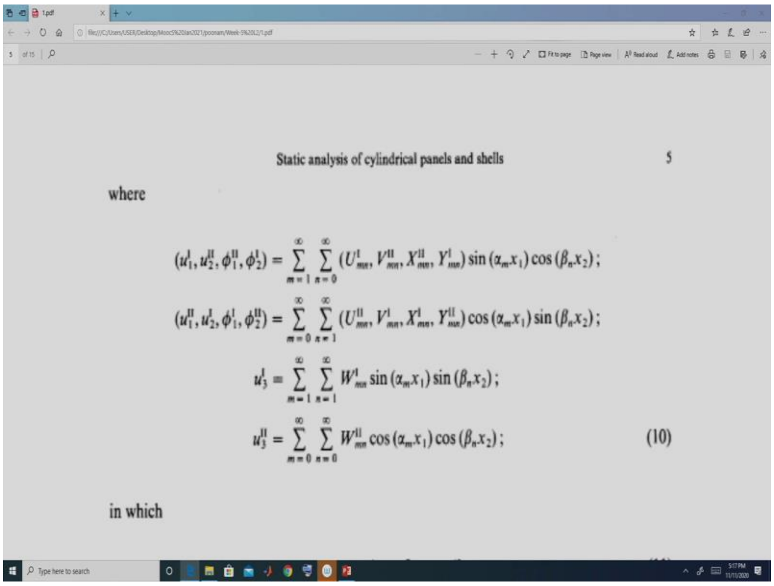

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where

$$(u_1^{I}, u_2^{II}, \phi_1^{II}, \phi_2^{I}) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} (U_{mn}^{I}, V_{mn}^{II}, X_{mn}^{II}, Y_{mn}^{I}) \sin(\alpha_m x_1)\cos(\beta_n x_2);$$

$$(u_1^{II}, u_2^{I}, \phi_1^{I}, \phi_2^{II}) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} (U_{mn}^{II}, V_{mn}^{I}, X_{mn}^{I}, Y_{mn}^{II}) \cos(\alpha_m x_1)\sin(\beta_n x_2);$$

$$u_3^{I} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}^{I} \sin(\alpha_m x_1)\sin(\beta_n x_2);$$

$$u_3^{II} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} W_{mn}^{II} \cos(\alpha_m x_1)\cos(\beta_n x_2); \quad (10)$$

in which

They have solved this term. My aim is to tell you that we have done the same way they did.

u_3, deflection is assumed in sin and sin, u_2 is assumed in cos and sin, and u_1 is assumed as sin and cos. In this way, the solutions are presented. The students or the learners can go through these papers and can understand. I would like to discuss some results here.

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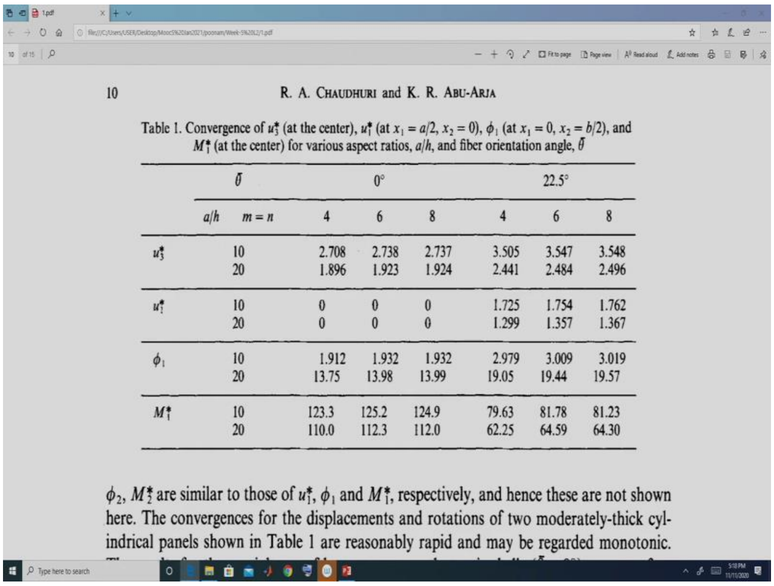

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Table 1. Convergence of u_3^* (at the center), u_1^* (at $x_1 = a/2, x_2 = 0$), ϕ_1 (at $x_1 = 0, x_2 = b/2$), and M_1^* (at the center) for various aspect ratios, a/h, and fiber orientation angle, $\bar{\theta}$

	$\bar{\theta}$		0°			22.5°	
	a/h	$m = n$	4	6	8	4	6	8
u_3^*	10		2.708	2.738	2.737	3.505	3.547	3.548
	20		1.896	1.923	1.924	2.441	2.484	2.496
u_1^*	10		0	0	0	1.725	1.754	1.762
	20		0	0	0	1.299	1.357	1.367
ϕ_1	10		1.912	1.932	1.932	2.979	3.009	3.019
	20		13.75	13.98	13.99	19.05	19.44	19.57
M_1^*	10		123.3	125.2	124.9	79.63	81.78	81.23
	20		110.0	112.3	112.0	62.25	64.59	64.30

ϕ_2, M_2^* are similar to those of u_1^*, ϕ_1 and M_1^*, respectively, and hence these are not shown here. The convergences for the displacements and rotations of two moderately-thick cylindrical panels shown in Table 1 are reasonably rapid and may be regarded monotonic.

You can see that the results are presented: for a by h, m is equal to n, then taking the terms, m_{10} and m_{20}, then we get the convergence. The number of terms is required and this is $\theta°$, 0° and 22.5°. And u_1, u_3, ϕ_1, and m_1 are compared.

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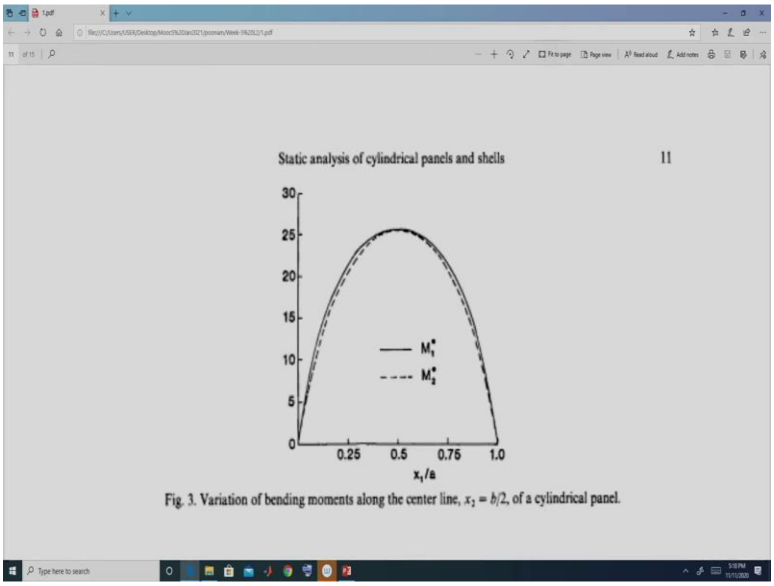

Here one can see that moments are compared, plotted along $\frac{x}{a}$ and they are calculated at $\frac{b}{2}$.

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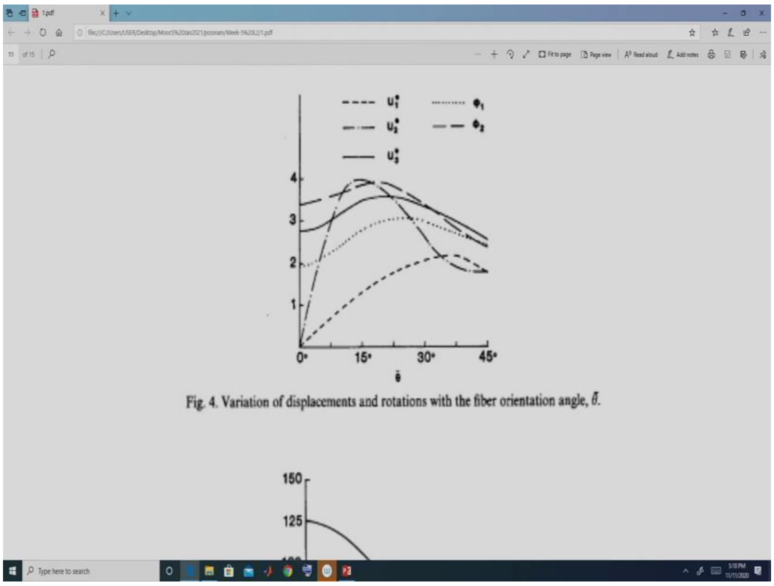

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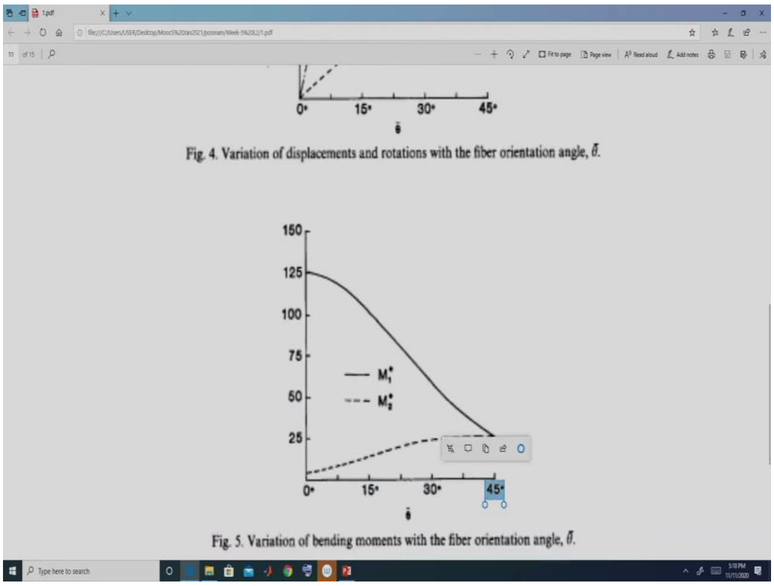

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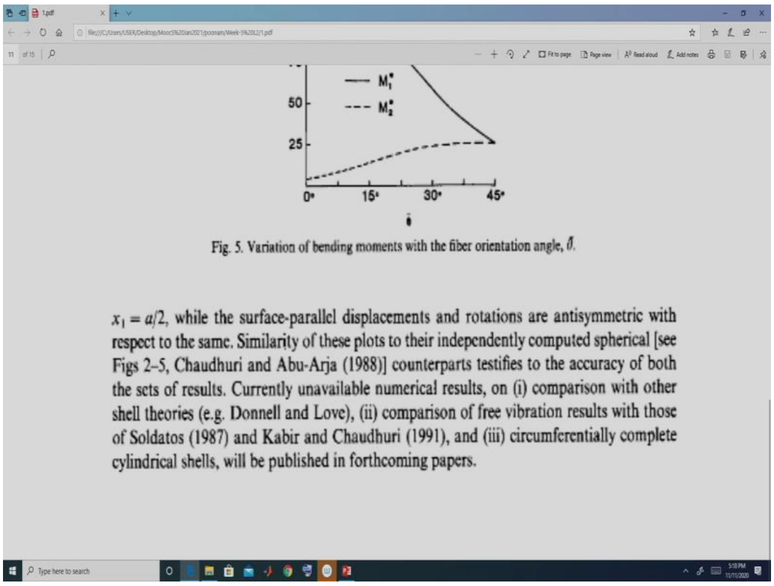

The same way, along the θ, different moments are calculated. Those days in the 90s, 80s, or 70s when the computer was just started. The very preliminary results were presented like, the bending moments or the deflection. But after 95; when the use of computers has increased, we can find more detailed results and various parametric solutions.

Even these days, you can directly use C packages like; anysys or nastran or any structural analysis software, to directly get the deflection stress resultants. The aim is to tell you the basics behind these different software's.

Behind that, there are simple theories, for research, these days if some people are working on a shell made of carbon nanotube or on a shell made of graphene sheets, what will be the deflection? For this, some different material model is required, but the basic will remain same. The functionally graded shell is currently studied with the basic models we can go for a complex one.

With this, I would like to end this lecture.

Thank you very much.