

Theory of Rectangular Plates - Part 1
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Lecture - 04
Tutorial: Transformation of Tensors

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Example (1): - Consider a plate made of a graphite-epoxy material with the following material properties in the principal material coordinates as

$E_1 = 137.89 \text{ GPa}$, $E_2 = 8.96 \text{ GPa}$, $G_{12} = 7.10 \text{ GPa}$, $\nu_{12} = 0.3$, $\nu_{21} = 0.0194$

The elastic coefficients Q_{ij} (GPa) in the principal material coordinates can be calculated using following equations:

$Q_{11} = \frac{E_1}{(1 - \nu_{12}\nu_{21})} = \frac{137.89}{(1 - 0.3 \times 0.0194)} = 138.69 \text{ GPa}$

$Q_{12} = \frac{\nu_{12}E_2}{(1 - \nu_{12}\nu_{21})} = \frac{0.3 \times 8.96}{(1 - 0.3 \times 0.0194)} = 2.70 \text{ GPa}$

$Q_{22} = \frac{E_2}{(1 - \nu_{12}\nu_{21})} = \frac{8.96}{(1 - 0.3 \times 0.0194)} = 9.01 \text{ GPa}$

$Q_{66} = G_{12} = 7.10 \text{ GPa}$

Handwritten notes in red ink:

- $\nu_{21} = \frac{E_2 \nu_{12}}{E_1}$
- For isotropic $Q_{11} = \frac{E}{1 - \nu^2}$

Now I am going to solve some simple problems like if somebody gives you Young's modulus E_1 , E_2 , G_{12} , μ_{12} and μ_{21} . Even before that I may ask that E_2/E_1 and $\mu_{12} = \mu_{21}$. So using those relations, we can find out μ_{21} . If E_2 is known, E_1 is known, μ_{12} is known then you can find out μ_{21} . Why it is required? You see that elastic coefficient using these materials.

When we are going to develop a plate theory, we have to find out Q_{11} , Q_{12} numerically. So E_1 , $1 - \mu_{12}\mu_{21}$. For isotropic Q_{11} will be $E/(1 - \mu^2)$ okay. So in the assignments we may ask that how to evaluate a Q_{11} for a given material property, similarly how to evaluate Q_{12} , μ_{12} , E_2 . These are the very standard relations or I would like to say the basics of structural mechanics if we are not comfortable with this, we cannot proceed further. We must know that how to evaluate Q_{11} , Q_{12} , Q_{22} and Q_{66} is nothing but G_{12} .

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If the plate is of thickness $h = 0.0254$ mm and the plate coordinate axes coincide with the principal material coordinates, the (orthotropic) plate stiffness coefficients A_{ij} (KN/mm) and D_{ij} (KN-mm) can be calculated by using the following equations:

$$A_{11} = \frac{E_1 h}{(1 - \nu_{12}\nu_{21})} = \frac{137.89 \times 0.0254}{(1 - 0.3 \times 0.0194)} = 3.52 \text{ KN/mm}$$

$$A_{12} = \nu_{21} A_{11} = 0.0194 \times 3.52 = 0.068 \text{ KN/mm}$$

$$A_{22} = \frac{E_2 h}{E_1 \nu_{12}} = \frac{8.96}{137.89} \times 3.52 = 0.0184 \text{ KN/mm}$$

$$A_{66} = G_{12} h = 7.10 \times 0.0254 = 0.180 \text{ KN/mm}$$

$$D_{11} = \frac{E_1 h^3}{12(1 - \nu_{12}\nu_{21})} = \frac{137.89 \times 0.0254^3}{12(1 - 0.3 \times 0.0194)} = 1.89 \times 10^{-4} \text{ KN-mm}$$

$$D_{12} = \nu_{21} D_{11} = 0.0194 \times 1.89 \times 10^{-4} = 3.67 \times 10^{-6} \text{ KN-mm}$$

Handwritten notes on the slide:

- $N_{xx} = A_{11} \epsilon_{xx} + A_{12} \epsilon_{yy}$
- $A_{ij} \int_{-h/2}^{h/2} dz = Q_{ij} h$
- extensional stiffness
- Bending
- $M_{xx} = -D_{11} w_{,xx} - D_{12} w_{,yy}$

If you remember exactly, there I have said N_{xx} , what is that? A_{11} of ϵ_{xx} of 0 + A_{12} of ϵ_{yy} of 0 that is the plate constitutive relations. So for that we know how to evaluate A_{11} , A_{12} . So A_{11} the definition is $Q_{ij} dz$ $-h/2$ $h/2$ where Q_{ij} is a constant. So basically $Q_{ij} \cdot h$ times will be A_{ij} . Here ij goes from 1 to 2. So A_{11} will be nothing but $E_1 / (1 - \nu_{12}\nu_{21}) \cdot h$. So we can evaluate A_{11} .

If we know A_{11} , if we know A_{12} and through some experiments is there able to at any point the strain ϵ_{xx} 0, ϵ_{yy} 0 we can evaluate N_{xx} . A_{12} is $\nu_{21} \cdot A_{11}$, A_{22} is $E_2 / E_1 \cdot A_{11}$, then A_{66} up to here this is your extensional stiffness. These formulas are important, we must remember these things so that we can evaluate or we can calculate during the assignments or exams or some point of time.

Now we are talking about the bending stiffness. If you want to evaluate movement M_{xx} , what is that? It is $-D_{11}$ of $w_{,xx}$ - D_{12} of $w_{,yy}$. So this is your movement, so you can evaluate D_{11} is $E_1 h^3 / 12$ - or in another terms I can say that $Q_{11} \cdot h^3 / 12$, in other sense D_{11} is this thing. Somehow you know that Q_{11} you can calculate the $D_{11} h^3 / 12$ or you know the D_{11} you can calculate the Q_{11} .

Similarly, D_{12} one can calculate, so units should be small k, there is some mistake kilonewton.

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$$D_{11} = \frac{E_1}{E_1} D_{11} = \frac{8.96}{137.89} \cdot 1.89 \times 10^3 = 1.22 \times 10^3 \text{ KN-mm}$$

$$D_{66} = \frac{G_{12} h^3}{12} = \frac{7.10 \times 0.0254^3}{12} = 9.69 \times 10^{-6} \text{ KN-mm}$$

If the principal material axes x_1 and x_2 are oriented at 60° to the plane (x, y) axes and $x_3 = z$ then we have

$$\begin{aligned} Q_{11} &= 20.08, & Q_{22} &= 84.92, & Q_{12} &= 24.06, & Q_{66} &= 28.45 & (\text{GPa}) \\ A_{11} &= 5.0968, & A_{22} &= 21.556, & A_{12} &= 6.1073, & A_{66} &= 7.2233 & (\text{KN/mm}) \\ D_{11} &= 0.0269, & D_{22} &= 0.1179, & D_{12} &= 0.0322, & D_{66} &= 0.0381 & (\text{KN-mm}) \end{aligned}$$

If the plate is composed of three layers of the same thickness and material, but the top and bottom layers oriented at 0° and the middle layer at 90° [denoted $(0/90/0)$ laminate], then the plate stiffness coefficients are given by

$$\begin{aligned} A_{11} &= 24.234, & A_{22} &= 13.26, & A_{12} &= 0.6865, & A_{66} &= 1.8025 & (\text{KN/mm}) \\ D_{11} &= 0.1796, & D_{22} &= 0.0185, & D_{12} &= 3.629 \times 10^{-3}, & D_{66} &= 9.523 \times 10^{-3} & (\text{KN-mm}) \end{aligned}$$

Then D_{22} , D_{66} , now the question that material axes are oriented at 60 degree and $x_3=z$ then you have to evaluate bar transformed one which will be a \cos^4 and so on. Then based on this effective transform reduced stiffness A_{11} , A_{22} can be evaluated okay.

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Example (2):- Suppose that an orthographic plate is subjected to loads such that the only nonzero strain at a point (x,y) is $\epsilon_{xx}^0 = 10^3 \mu = 10^{-3} \text{ mm/mm}$. We wish to determine the state of stress $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ in the plate. Assuming that the plate is of thickness 0.254 mm, we wish to compute the plate forces and moments. The material properties and stresses (in GPa) are given by

$$E_1 = 137.89 \text{ GPa}, E_2 = 8.96 \text{ GPa}, G_{12} = 7.10 \text{ GPa}, \nu_{12} = 0.3, \nu_{21} = 0.0194$$

$$\sigma_{xx} = Q_{11} \epsilon_{xx}^0 = 1.3680 \times 10^{-7} \quad \sigma_{yy} = Q_{12} \epsilon_{xx}^0 = 2.665 \times 10^{-9} \quad \sigma_{xy} = Q_{16} \epsilon_{xx}^0 = 0 \quad (\text{in GPa})$$

where Q_{ij} are given as

$$Q_{11} = 138.69 \text{ GPa}, Q_{12} = 2.70 \text{ GPa}, Q_{21} = 9.01 \text{ GPa}, Q_{66} = 7.10 \text{ GPa}$$

The tensile stress σ_{yy} is the reaction of the laminate trying to contract in the y -direction due to the Poisson effect. Since the strain $\epsilon_{yy} = 0$ by assumption, a tensile stress σ_{yy} is required to maintain the zero strain condition.

The forces and moments in the plate are

$$\begin{pmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{pmatrix} = \begin{pmatrix} A_{11} \\ A_{12} \\ 0 \end{pmatrix} \epsilon_{xx}^0 = \begin{pmatrix} 0.0352 \\ 6.864 \times 10^{-4} \\ 0 \end{pmatrix} \text{ KN/mm} \quad \begin{pmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ KN-mm}$$

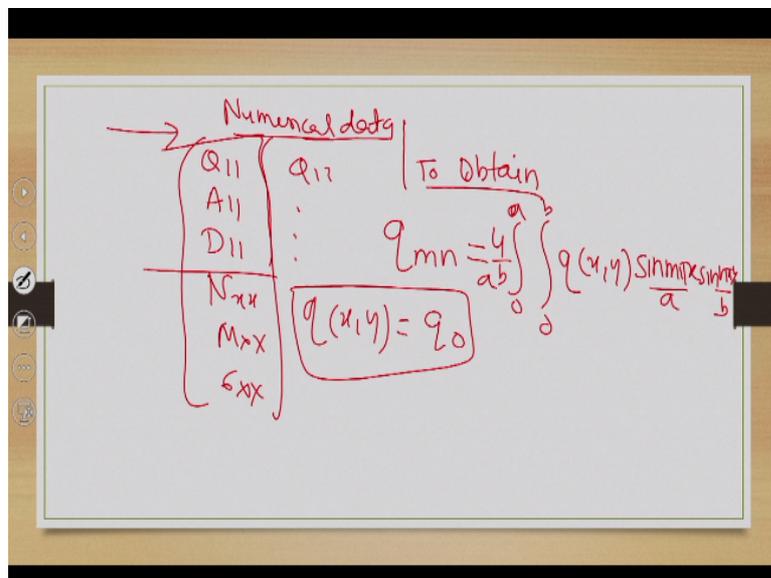
I have already told you that if your E_1 , E_2 , E_3 all things are given and then at any some point of time ϵ_{xx}^0 is given, so it is we are going to read the statement that an orthotropic plate is subjected to only nonzero strain is this and we have to be interested to find out that state of stress at that point and thickness of the plate is given, material properties are given so you can evaluate σ_{xx} axis Q_{11} of ϵ_{xx}^0 .

So for this purpose, we know how to evaluate Q_{11} , Q_{11} is what? Q_{11} is $E_1 / (1 - \nu_{12} \nu_{21})$ and what is that Q_{12} ? And similarly Q_{16} later on, so they have evaluation and if you

substitute it there you can evaluate the sigma xx, sigma yy, sigma xy. Then stress resultant N_{xx} , N_{yy} , N_{xy} , only epsilon xx 0 is nonzero, so only contribution will be A_{11} . So if you multiply with that you can get what will be the movement.

For the case of movement, it should be epsilon xx of 1 so there is no epsilon xx of 1 so movements will be 0. So these types of questions or the numericals when you are going to make a program for a plate, so you must know these formulas that how to evaluate this, you are going to use this, at the end of the day you need some number that deflection should be in some numbers, movements should be in some numbers. So you have to evaluate these things.

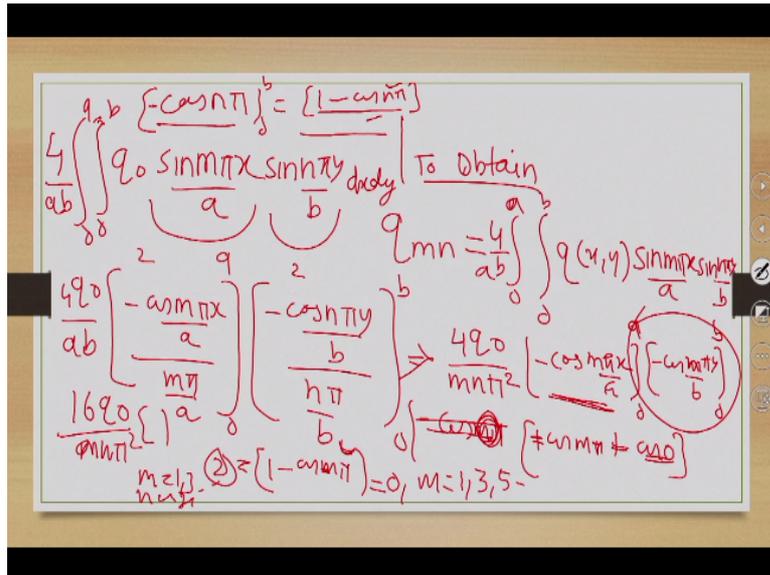
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Now I am talking about the now you have known that how to evaluate Q_{11} , how to evaluate A_{11} , how to evaluate D_{11} and corresponding some Q_{12} and so on and how to evaluate N_{xx} and M_{xx} for a given point or a sigma xx and so on for a given numerical data. The next is to evaluate or obtain if you remember with q_{mn} that q_{mn} definition was that is also required for the solution.

If you want to solve a plate problem, you have to at least up to here all material constants and q_{mn} , so q_{mn} definition wise was $4/ab \int_0^a \int_0^b q(x,y) \sin(m\pi x/a) \sin(n\pi y/b)$. Now I am saying let us say my $Q(x,y)$ is nothing but just a UDL kind of thing q_0 , not a function of anything then what will be my q_{mn} ? So you have to solve this integration.

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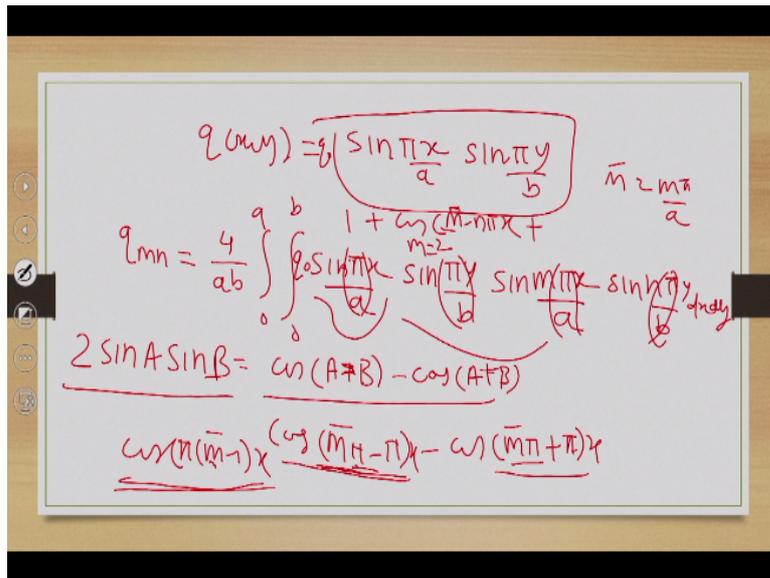
What will be that just put it there. This is very, very much important. Most of the time, we understood the theory that how to represent or how to develop a governing equation for a plate but for making a program for getting the solutions, we must know these basic things that how to evaluate. Program will not give you the final solution. You have to write your own code or your own program that this will be equal to something and that we are going to use that.

Let us say $Q_0 \sin m \pi x/a \sin n \pi y/b dx dy$ 0 to a 0 to b, so this is only a function of x, this is only a function of y, so we can integrate. So basically $q_0 4/ab$ you put it here and then sin will be $-\cos m \pi x/a / m \pi/a$ units 0 to a. Then $-\cos n \pi y/b / n \pi/b$ 0 to b. So this you further simplify a and b will get out cancel, so basically $4 q_0/mn$ again π square will come up here and here $-\cos m \pi x/a$ 0 to a and $-\cos n \pi y/b$ 0 to b.

Then, what is that? Put a so it gets cancel out $-\cos m \pi$ okay and then lower limit- upper limit, you first put it 0, $\cos m \pi$ only and then this $\cos 0$. So when you put $1 m \pi$ so basically I am going to put here 1 then $-$, $+$ so $\cos 0$ is $1 -$ of $\cos m \pi$. Then if you put $\pi m=1$, $\cos \pi$ will be -1 then it will become 2. When you put m is 2, then it will be $\cos 2 \pi$ which is $+1$ then it becomes 0.

So this series is valid only $m=1, 3,$ and 5 and so on. Similarly, here $\cos n \pi$ 0 to b $-\cos n \pi$ you can put like that. So if you put 1, it becomes 2. So ultimately 2 from this, 2 from this, so $16 q_0/mn \pi$ square and that is valid for $m=1, 3$ and $n=1, 3$ and so on. So we can evaluate this function like this.

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Now come to the second most important function, if your $Q(x, y)$ is represented $\sin \pi x/a$ and $\sin \pi y/b$ sin loading then your q_{mn} will be $4/ab$ 0 to a 0 to b, put it this thing and you say q_{00} something value $\sin \pi x/a \sin \pi y/b$ and then $\sin m \pi x/a \sin n \pi y/b dx dy$. So here you see that $\sin x$ and $\sin c$ is basically if you remember $\sin A$ and $\sin B$ formula which is basically $\cos A - \cos B$ like this some formula.

So if you put it there, so I would like to say let us say $m \pi$ is greater so I would like to write like that into this $-\cos m \pi + \pi$ of x and so on. So you can say that \cos of $\pi m - 1 * x$. So this first term when you put $m=1$ it becomes 0, when you put $m=2$ okay then series goes like that, so we can say this term can be written as $1 + \cos m \pi x$. So basically you will say that m bar where m bar is nothing but $m \pi/a$ okay.

Here also π/a is there, π/b these things will come out. So here this goes from $m=$ similarly for this case. So if you do that integration of a \cos function, it will be \sin , you take any value $\sin \pi$, $\sin 2 \pi$, $\sin 3 \pi$, $\sin 4 \pi$ contribution will be 0.

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$$q(x,y) = q_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \quad \bar{m} = \frac{m\pi}{a}$$

$$q_{mn} = \frac{4}{ab} \int_0^a \int_0^b q_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx dy$$

$$= \frac{4}{ab} \left[\int_0^a \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx \right] \left[\int_0^b \sin\left(\frac{\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right) dy \right]$$

$$= \frac{4}{ab} \left[\frac{a}{2} \delta_{m1} \right] \left[\frac{b}{2} \delta_{n1} \right] = q_0 \delta_{m1} \delta_{n1}$$

$0: \quad \frac{1}{2} \quad \dots$

So I would like to say the gist of it is this that for such a case it is just becoming $q_0 + \cos$ of something + \cos of something and so on, so there will be integration so 0 contribution, only q_0 come up in there, nonzero contribution 1. So q_{mn} a, b they will cancel out and 2 in this and you can take 2 again, so q_{mn} is just q_0 . So that is the easiest one. When I say that a plate is subjected to its sinusoidal loading then you understood that q_{mn} must be just q_0 .

For most of the time for simplification or just to expose you that what will be the stresses or moments, we use to assume that our loading is by sinusoidal along x and y axis. So q_{mn} will be q_0 and we can solve that deflection, moment and stresses so on.

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$$q(x,y) = q_0 \frac{x}{a}$$

$$q_{mn} = q_0 \frac{xy}{ab}$$

We may give q x, y is like that $q_0 x/a$, a linear function or sometimes $q_0 xy/ab$ any kind, so then you are supposed to evaluate that q_{mn} . Even evaluating this q_{mn} is important or I would

like to say that sometimes that making a function is a challenging task that if you see in a book that have to represent if this is the only a triangular loading increasing that to say that along x axis sometimes we call it is a hydrostatic loading of $q_0 x/a$.

But if you have a loading like this in pyramid shape then how to make this function, so it will be two series (()) (18:51) one is this, then combining together so one function will be valid up to here and other function will be valid up to here but if you have a sin series 1 or if you have a function like this then you can assume in terms of sin series or some polynomial functions for this case.

So our aim is to find out first how the loading is there and based on that $q_m n$. So evaluating of $q_m n$ is a very important task if you are interested to get the deflection and moment of a plate.