Theory and Practice of Rotor Dynamics
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Module - 5
Torsional Vibrations
Lecture - 21
Finite Element Method

Till now in the torsional vibration, we have studied simple by Newton's second law how to obtain the equation of motion. Also we have dealt with transfer matrix method for multi degree of freedom system. In these cases, all the rotors, which we consider generally, the shaft of such rotors had only the stiffness property. Thus we negated the mass of the shaft, but in practical notice we find that not only the stiffness also the mass of the mass, or the inertia property of the shaft is distributed throughout the length.

So, these shafts are very heavy and because of that they have appreciable amount of polar moment of inertia, and that is distributed throughout the length. In such rotors generally, when we want to model using discrete mass analysis, the analysis is not that much accurate and for such cases, we have approached that is called continuum, continuous system approach. Generally, we deal with such system with continuous system approach.

And today, we will see very briefly this particular approach, how to obtain the covered equation for the torsional case, when the both mass and elastic property of the shaft is distributed. And our main focus would be in the subsequent lectures to have finite element formulation of such system. So, that it can be applied to real system. So, let us see what are the things we will be covering today?

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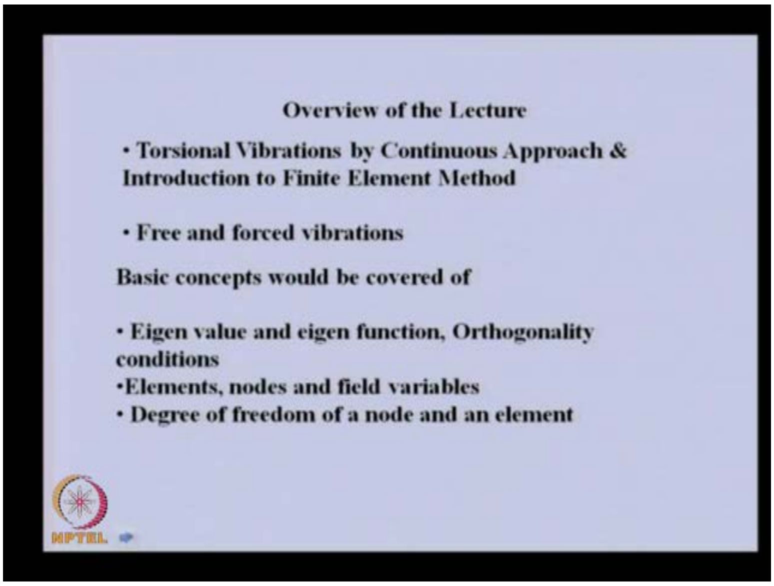

So, basically we will be developing the torsional vibration analysis using continuous approach. And we will introduce some basic terminology of the finite element method, in this free and forced vibration is the main analysis which we do, and some of the concept like Eigen value which is nothing but natural frequency. Eigen function in discrete system it was nothing but the mode shape some other property, like orthogonality property we will see which is there for the Eigen function especially, for the continuous system.

Then some basic definitions, which we use in the finite element method like elements, nodes, felid variable, degree of freedom of node and element, these are the basic definitions will be introducing in this particular lecture and in subsequent lecture, we will be dealing with finite element method in more detail. So, let us see how this particular continuous system, continuous system approach works and for this let us see what are the type of shafts we have considered.

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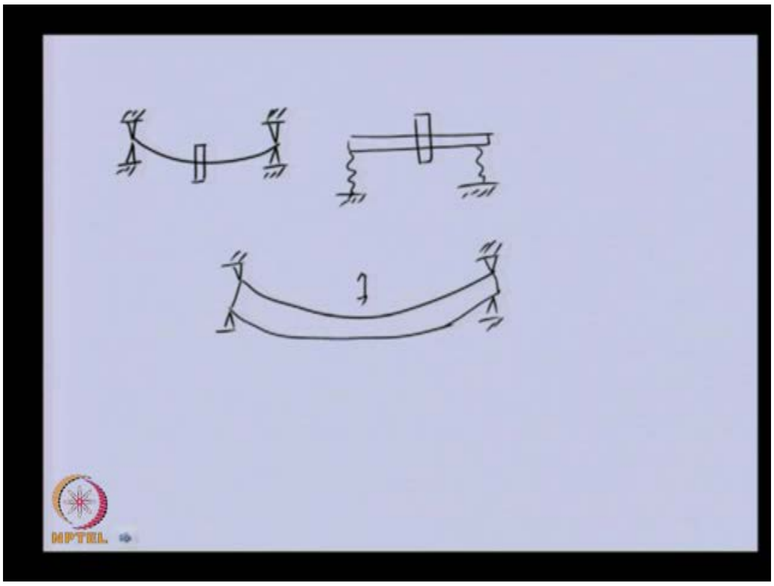

Earlier, we had one flexible shaft especially, for transverse vibration it is more clear. So, I am trying to explain this continuous system approach by a transverse vibration, and subsequent then we will be analyzing. So, this is mass less flexible shaft and a heavy disk we placed, and such rotors we analyzed by considering only the flexibility of the shaft and the mass of the shaft. Another case we took rigid shaft and flexible bearing in this particular case, the shaft we considered as rigid only the flexibility was there in the bearing, but in some cases the rotors are relatively heavy and flexible.

So, when we found them on bearings they not only have appreciable amount of inertia also they have elasticity. For such rotors we cannot able to distinguish, where a particular mass we should able to concentrate at one location or several locations. So, generally in this particular case we will see that elastic property on the mass property of the shaft, we need to consider continuously. This was the example given for the transverse vibration in which we are interested in this kind of vibration, but the present study we are interested in the torsional vibration of such rotor system. So, let us see how this can be modeled.

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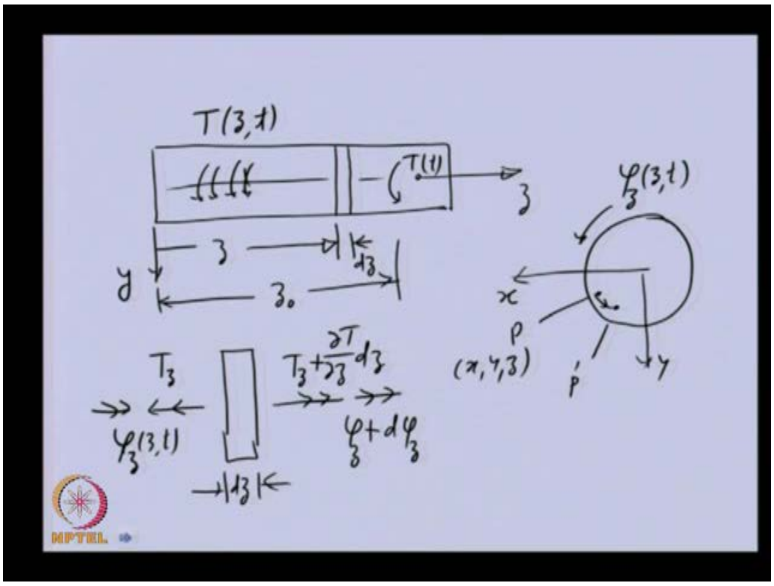

So, for this particular case if we have a shaft which is let us say this is the axis of the shaft z, and this is the one of the plain of the shaft shown here and at a distance z. Let us say there is plain the motion of that we want to analyze, these particular shafts we have various kinds of loading like we can have distributed torque external torque acting. So, that torque will be representing as function of z, the axial position of the shaft also it is time dependent. And if we have some kind of concentrated torque at some location, which is let us say at location z naught that is the location of that is fixed. So, this we will call at as a concentrated torque.

Now, we are interested in a shaft segment which is here. So, if we want to see that particular plain of the shaft due to distortion. So, let us say this is y and this is x axis, this is this basically center of the shaft. So, this will come like this and if we want to study motion of the point p let us say, the coordinate of that is x, y, z. If we have angular twist of this particular plain phi z, which is again function of z and time this particular point p will have displacement in the radial direction, which will come at p prime location due to angular displacement.

In this particular hypothesis we are assuming that what all the loading supplied on this torque, because of this it would not particular plain, which we are considering that will remain a plain, it will not distort during the motion. And a particular particle on this particular plain like p moves in that particular plan. So, we can able to see that in this plain itself it is moving it is not going out of this plain.

Now, if we want to analyze the torque balance for a particular shaft segment, let us say we are taking a very small shaft segment having thickness d z, and if you want to draw the free body diagram of that to obtain the equation of motion of this. Let us take this particular shaft segment, and this particular shaft segment as we have seen the thickness is d z. And once we have taken out from the shaft the reaction torques will be acting on both sides of the, this particular plain.

So, lets us say this particular direction the torque T z is acting, which is reactive torque, which is coming from the other end of the shaft. And we have because inertia property is continuously changing. So, we except that the torque will change at other plain, and this will be given by this expression even the displacements, angular displacement I am representing that let us say positive direction in this form phi z, which is function of z and t. And this direction also I have because from left to right is the positive direction for the z.

So, here we have the displacement as phi z plus d phi z change in the angular displacement because of this inertia property, there will be change in the this particular angular displacement also. Now, these two are the torque, which are coming on this particular, these are the two torque which are coming onto this particular shaft segment. And now, using Newton’s law second law we can able to obtain the governing equation and for that we will be equating the torques. So, I will take in the next side.

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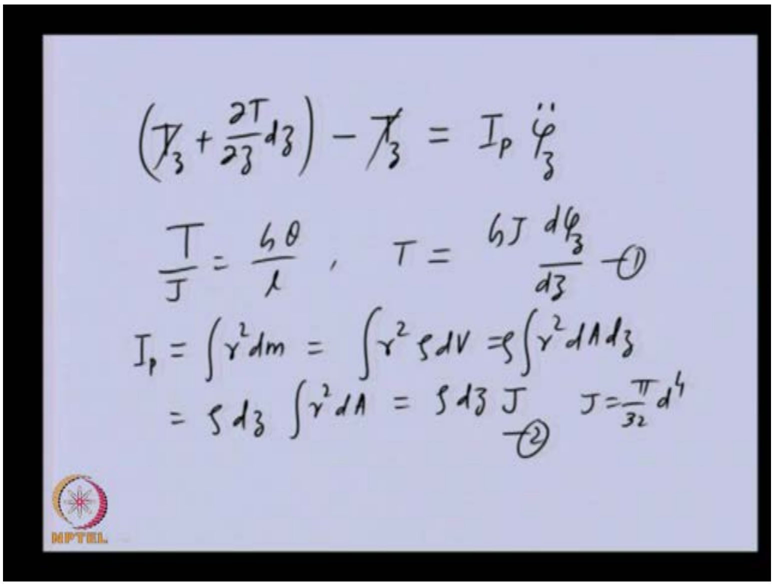

So, T z plus delta t this is the torque, which is acting on the right side on the that shaft segment minus the shaft segment torque and the left plain and that should be equal to the inertia, rotor inertia of the shaft segment this. Now, in this because from torsion theory we know that T by j is equal to g theta by l. So, we can able to get the torque as G J and this theta is the twist of the shaft segment, which is this for this particular case is this much and the length of the shaft segment is d x. So, can able to d z.

So, it can able to see in the previous slide the related twist of the two plains is difference of this and the displacement minus this. So, there is the d phi z and the shaft segment length is d z. So, we got this one and the I p which is mass polar moment of inertia is given as r square d m, d m is the mass of the shaft segment. So, we can able to write this as r square rho d v. So, this is the mass then this we can able to write it as area and d z and rho we can take out outside, and this can be simplified as rho d z because d z will come out r square d a.

And r square d a is nothing but the J, second moment of inertia that is polar moment of inertia of the shaft, which is given as J as pi by 32 d 4 or circular shaft. So, this expression and this expression, we can able to substitute in the, this governing equation. In this governing equation we can be able to see that these two terms are getting cancelled, and here we will substitute torque and here we will substitute I p.

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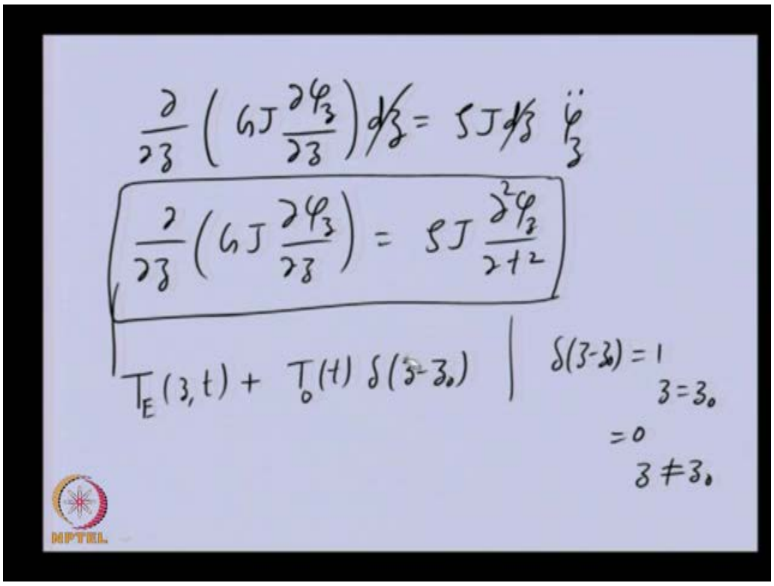

So, d by d z or for torque we will write g j and this for the torque. So, again you can see we have substitute for torque and this one and right side we need to substitute for I p. So, rho J d z and phi double dot, so this is double dot is representing the time derivative with respect to yeah time derivate and this double dot means double derivate with respect to time. So, we can able to see that this will get cancelled and we will be left with equation of motion of the continuous shaft for the vibration which will be given as like this.

So, this is the equation of motion of the continuous system in this particular case we did not consider the external torque, but if you want to consider external torque in the previous slide. Here the we will be having external torques, so we can able to see that if you want to consider the external torque here we will be having distributed external torque and concentrated torque, which was at the location of the z naught. So, we will be using a direct delta function delta to specify that particular torque.

So, we can able to see this direct delta function is having property that when we have this will be equal to be 1 when we have z equal to z naught, this will be 0 when z is not equal to z naught. So, we can able to see that when mentioning when specifying the z is equal to z naught then only this will act otherwise it will not act because this is a concentrated load. This is the external load, which is distributed about certain length that will be specified we have obtained the equation of motion of a continuous shaft for its torque vibration.

We have seen that this particular equation of motion is partial differential equation it is having derivative with respect to z the special derivative. As well as derivate with respect to time and this particular equation of motion represent, because still now we have not considered any boundary condition. So, if we consider boundary condition the solution of this particular differential equation, which we will be calling as boundary value problem will be unique.

So, let us take very simple example of a cantilever shaft for that we will solve this particular differential equation. This particular equation is having similar form, which generally we study in the mathematics that is the, we call it as a wave equation. So, this equation is exactly same in from as wave equation only thing is the variables are different. Now, let us see we will solve this particular wave differential equation for the boundary condition of the cantilever beam.

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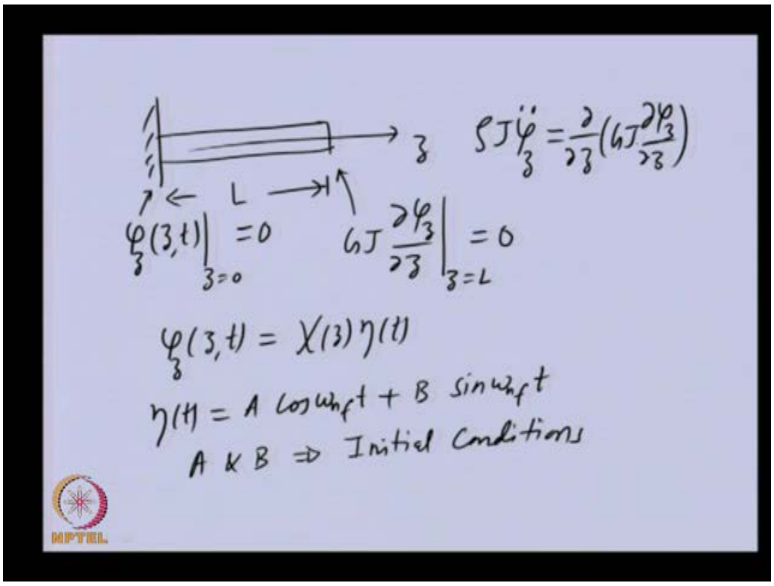

So, we have one shaft is having cantilever condition one can able to see this is the z axis length of the shaft is L and it is undergoing torsional vibration. In this particular case we can able to see that at the fixed end the boundary condition is that this particular displacement, which is function of time and z at z is equal to 0. This displacement is 0 at this fix end on the other end this free end there is no torque and we have represented torque earlier like this.

So, at z is equal to L we have the condition that torque is 0 because this is a free end. Now, to solve the differential equation, which we had obtained earlier that is I am writing again phi z is equal to G F phi z d z we will be using the separation of variable method to solve this. In this method we assume the solution in such that we have two parts of the solution one function is purely special coordinate dependent that is z. Other is time dependent and this time dependent function is generally harmonic in nature and the form of this particular function harmonic function.

We know is cos omega n F T is B sine omega n F T, where omega is frequency t is time a and B are constants. A and B we will be obtaining based on the initial condition of the problem under initial conditions. We will be obtaining these two constants, now if we take let us because in this particular equation we can able to see that we need to take time derivative of this particular function and also the special derivative here.

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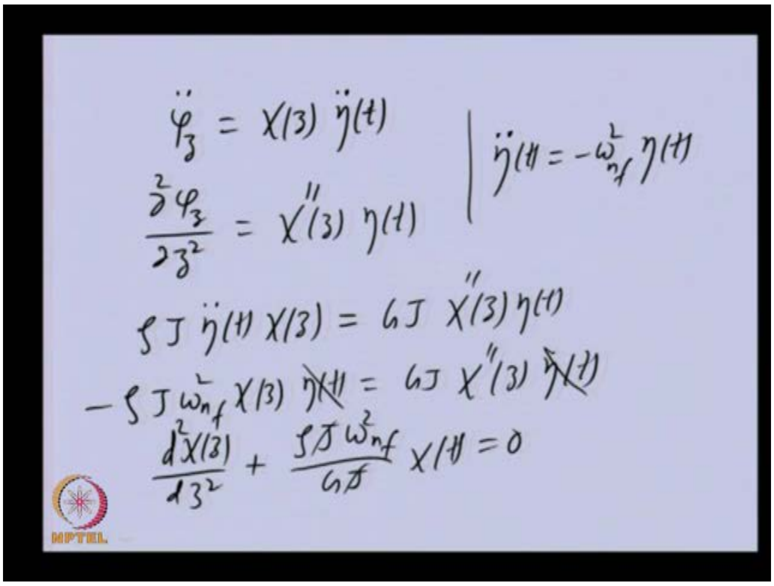

So, now I want to differentiate this twice with respect to time. So, obviously this will not derivate this k I function only it will derivate the eta function, which is time dependent. If I want to differentiate this with respect to z, so that will go into the chi, I am representing that as prime to represent the derivative partial derivative with respect to z and this will be as it is. So, these two we can able to substitute in the equation of motion and if you substitute this in equation of motion, we will in the previous equation here.

So, we will get rho J eta double dot this chi z, so this is in the left side is equal to the right side we have because this particular shaft we have considered uniform.

So, G J will be constant, so it will come out, so we will be having G jJ and double derivative of that function. So, we will be getting chi double derivative with respect to z and eta T. Now, if we see the harmonic function is having property this particular if we take double derivative of this with respect to time basically we will get relation as plus this is the condition of the harmonic motion. If we substitute this here we will get rho J minus omega square n F and chi z eta T is equal to G J chi.

So, we can able to see that this chi will be common and it will may get cancelled. So, we will left with a differential equation, which is now only function of z that means this is a now ordinary differential equation. We could able to convert the partial differential equation to ordinary differential equation. So, this will be having the form of this and here this j will also get cancelled. And this we can able to write it as chi T by del square plus I am calling that constant as alpha square chi T is equal to 0 where alpha square is as we can able to see here is rho omega square n F by capital G is the modulus of rigidity.

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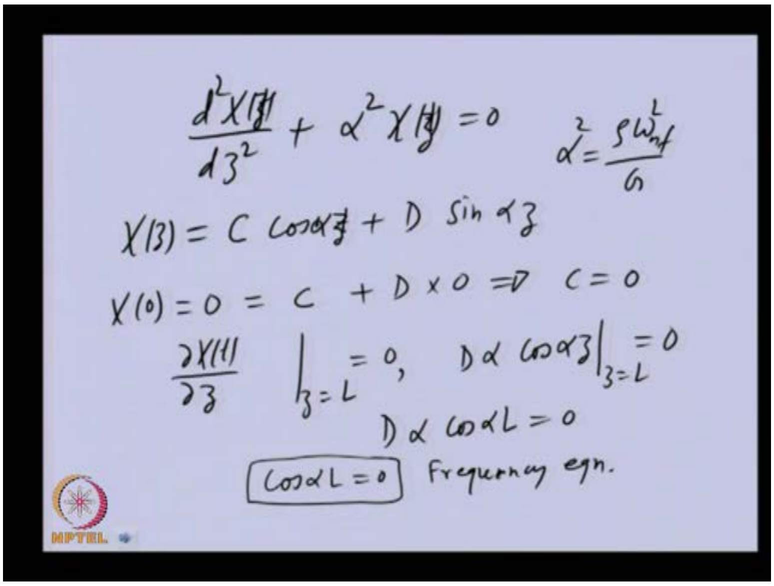

Now, this particular differential equation, which is now ordinary differential equation we can have the solution of this, so this is only z function of z or function of time. So, the function of the solution of this will be of this form C constant cos alpha T alpha z plus D sin alpha z. So, this is the solution, which we are resuming of this differential equation where C and D are constant and that we will we will be obtaining from the boundary condition of the problem.

For cantilever case, which we are dealing with we have seen that these are the two boundary conditions, so from this two boundary conditions we will be obtaining the C and D value. So, let us see the first condition that at z is equal to 0 this displacement is 0. So, we will get this as one plus D into 0, so we can able to see that we are getting C is equal to 0 for first boundary condition is fixed n boundary condition. Second boundary condition we have is the stock is 0 was eta here in place of phi z.

We are writing the assumed solution this solution and because this is a differential differentiation with respect to z, so this will be differentiated. This will be outside this 1, so this is 0. So, that means we need to derivate this with respect to z first and we need to substitute here, so we can able to remove this quantity. Also this is this will not be 0, so I am differentiating this and we know that C is 0 itself. So, only we need to differentiate this D alpha cos this, so this is the derivative of this particular term and at z is equal to 0 z is equal to l this quantity is 0.

So, if you substitute D alpha cos instead of z we will write l is equal to 0, so we can able to see either D if it is 0 then the whole motion will be 0 because C is already 0. If D is 0 motion will not take place, we are not interested in that, so d cannot be 0. So, for that this alpha is general cannot be 0, so we need to have a function cos alpha l is equal to 0. So, this is the frequency equation this is the frequency equation and the root of this will give us the Eigen values and from there we can able to obtain the natural frequencies so we can able to see the solution of this alpha l is equal to pi i by 2, where I can be odd numbers and we have infinite number of solutions.

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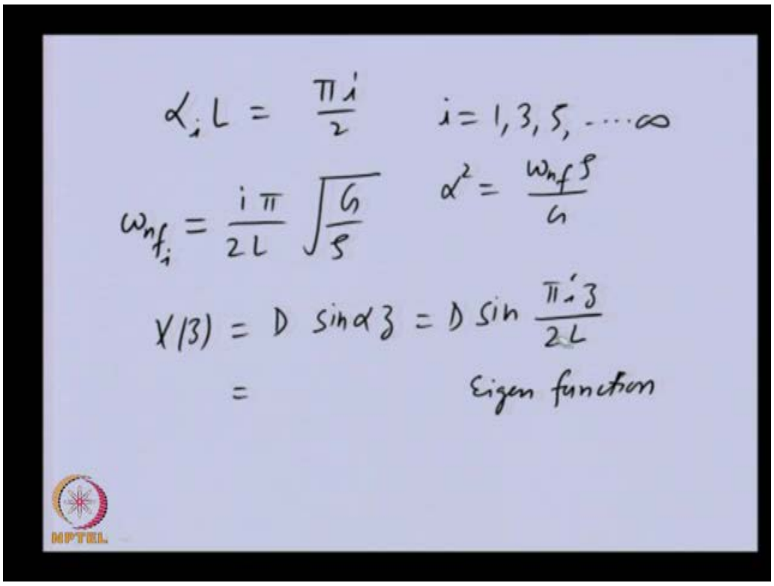

So, infinite number of Eigen values are possible and earlier we have related alpha with natural frequency of the system by this relation. So, we can able to get the natural frequency from this that will be as i pi by 2 L, so after simplification we can able to get the natural frequency from these two expression like this. And here i is again varying from 1 2 3 5 these are the various natural frequency of the cantilever shaft. Now, the Eigen function the assumed solution we had C is equal to 0, so Eigen function is left noting but psi ki is equal to sin alpha z, that again we can able to…

And here some constant we can able to attach and alpha is already given there, so this will be pi i z by 2 L, so this will be the Eigen function. Eigen function will give us how the relative points in the whole shaft will be having displacements because in this particular case each and every particle point on the shaft will be having relative displacement with respect to each other. So, this function will give us the relative displacements of various particles in the shaft system continuous system approach as we obtain the natural frequency and the Eigen function, which represent the mode shape in continuous system.

Because the mass property is continuously distributed, so basically this system can be considered as a infinite number of degree of freedom. Because each and every particle is are independently moving with respect to each other. And we have seen that we are getting infinite number of natural frequencies because we have infinite degree of freedom in the continuous system approach. Now, whatever the Eigen function we obtained let us see the plot of that for first three modes, so this is the plot of the Eigen function.

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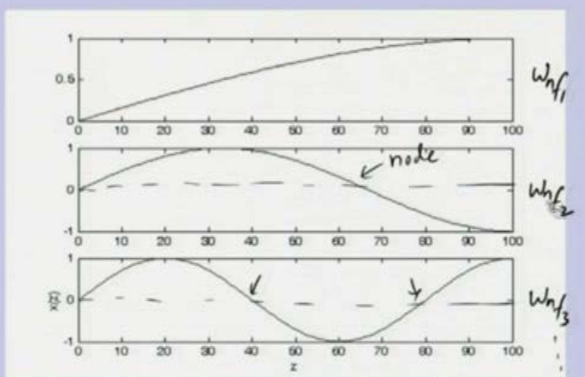

Torsional mode shapes for the first three modes of a cantilever rod

So, you can able to see this is the position of the shaft this is the fixed end of the shaft and this is the free end of the shaft and this is the angular displacement and this is the relative angular displacement. So, we have taken the maximum displacement as unity, which is for this particular case for first mode. So, this is belonging to first natural frequency corresponding to i is equal to one. So, this is the first mode and for second natural frequency the mode shape will look like this is the fixed end maximum is taking place, here also plus minus and here minus 1 this we normalized.

Because this is a relative displacement, so the maximum we are we are taking as 1, but here you can able to see that if we draw the 0 line there is a 1 place where the shaft displacements are 0. So, and either of this particular shaft is having opposite ended displacements, so this particular position is called node. As we have seen earlier also, there will not be any angular displacement of the shaft at this particular place, but either side of these two will be having opposite motion.

Again I am repeating we are talking about the torsional angular displacement, so the shaft at shaft particles, which are left side and right side they will be having opposite motion. Similarly, this is the third one is for third natural frequency and here if we draw the 0 line we will see that there will be two nodes where there will not be any angular displacements. But there will be angular displacement in either side and these two will be having opposite motion and these two will be having opposite motion.

So, we can able to plot this for higher natural frequencies also and we expect for such cases addition nodes will be coming, as we will increase the natural frequencies for the continuous system approach. We have obtained the natural frequency and mode shape for one particular boundary condition, it was cantilever case in continuous system approach. We can able to obtain the, this natural frequencies for simple boundary conditions and few more boundary conditions solutions I am providing here.

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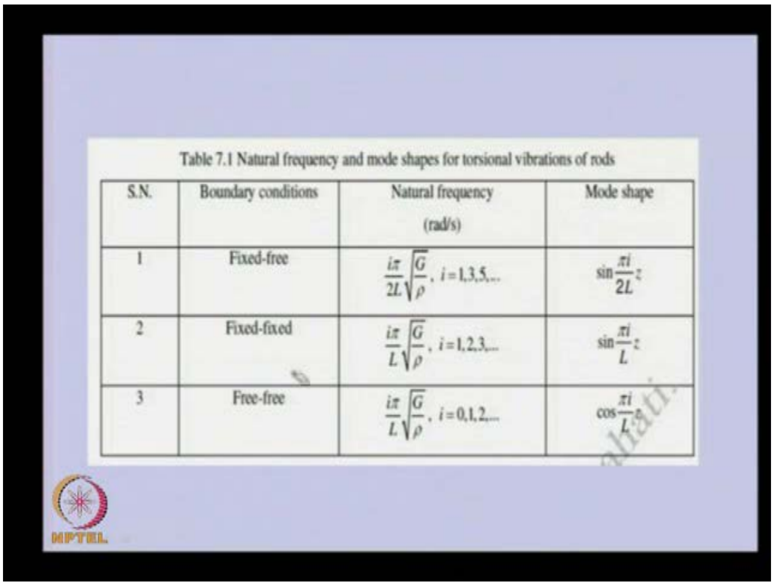

Table 7.1 Natural frequency and mode shapes for torsional vibrations of rods

S.N.	Boundary conditions	Natural frequency (rad/s)	Mode shape
1	Fixed-free	$\frac{i\pi}{2L}\sqrt{\frac{G}{\rho}}$, $i = 1,3,5,...$	$\sin\frac{\pi i}{2L}z$
2	Fixed-fixed	$\frac{i\pi}{L}\sqrt{\frac{G}{\rho}}$, $i = 1,2,3,...$	$\sin\frac{\pi i}{L}z$
3	Free-free	$\frac{i\pi}{L}\sqrt{\frac{G}{\rho}}$, $i = 0,1,2,...$	$\cos\frac{\pi i}{L}z$

So, like we have a free beam shaft is there is no support at both the ends then the natural frequency will be given by this expression here. The Eigen function will be having this function sin function. For another case in which both the end of the shaft is fixed in this particular case natural frequency will be given by this and the mode shape will be like this is also sin function. But there is some difference for this is the fixed free case is fixed case and this free. So, this is fixed free this is similar to the cantilever this is a cantilever case this is fixed and this is free and these are the natural frequency and mode shape.

So, basically we need to satisfy the boundary conditions to get the constant that C and D and depending up that we can able to get the equations. Once we obtain the Eigen function for a particular disturbance if you are talking about free vibration if you are giving a disturbance to the system how the system will vibrate will be given by the expansion theorem in which we mention that any free vibration. We can able to express in terms of these Eigen function contribution from these basic Eigen functions and contribution of various Eigen function will depend upon the initial condition.

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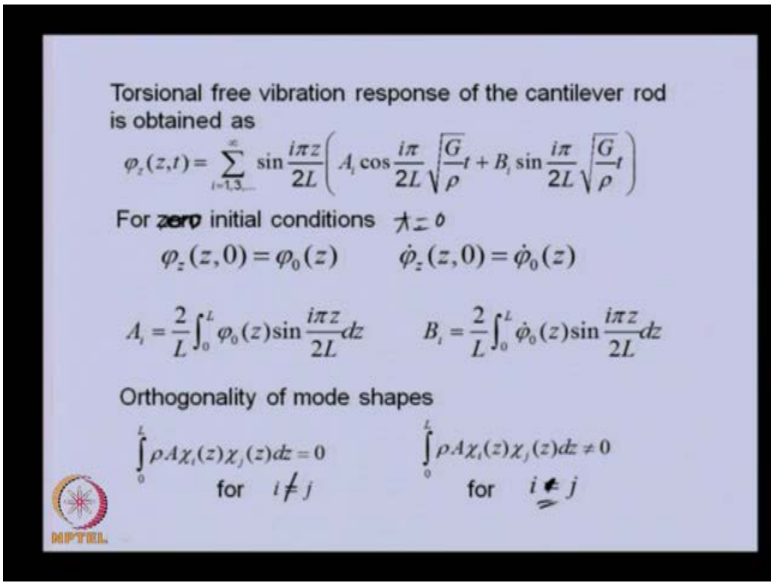

So, let us see this particular torsional free vibration how we can able to express in terms of the Eigen function. So, this is for the case for the simply support for the case of the cantilever beam case, so this was the Eigen function and this is the harmonic function. So, you can able to see that we are expressing any general free vibration in summation of various Eigen function various Eigen function. The harmonic function where A and B is the contribution of various modes and this will depend upon how we are giving the initial conditions and for 0 initial condition like this if our initial conditions is not 0 initial.

Condition at time T is equal to 0 for initial condition T is equal to 0 if we have these angular displacement and angular velocity these constants we can able to get by this. Basically to get this we need to multiply both side of this equation by the Eigen function for one particular mode and we need to integrate over the domain. While integrating we will be using the orthogonality of the mode shape that means when these two mode shapes are same when they are not same then this is 0 and this should be equal to…

So, when they are same then it will not be 0, so when two Eigen functions are same this quantity will not be 0 and when these two Eigen functions are different then this will be

0. So, we can able to see that once we are applying multiply by both sides by Eigen function of the i th mode the terms here will be 0 corresponding to all modes except I and that is the case here. So, these have been obtained using the this orthogonality condition of the mode shapes, so you can able to see for different initial conditions we can able to get these constants and the free vibration can be described. We have analyzed the torsional vibration the analysis for the axial vibration for continuous system is exactly similar to the torsional vibration only thing is some of the variables we need to interchange.

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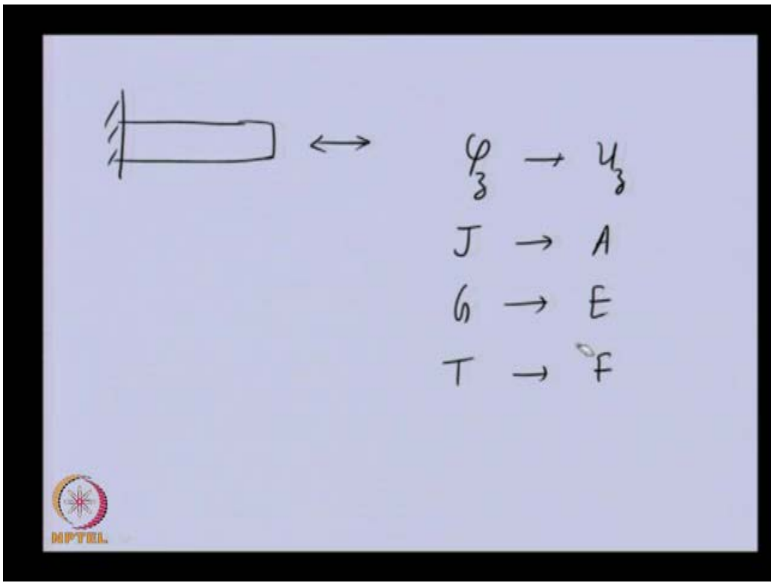

So, for axial vibration so if you are talking about a shaft having axial vibration that means vibration. So, if we want to analyze this system we need to replace the angular displacement which we took in the torsional vibration by axial vibration that is the linear vibration. The J second moment of area A polar moment of area we need to replace with A G we need to replace with E and torque will need to replace by force. So, these will be the changes in the equation of motion and otherwise the analysis is exactly the same differential equation and Eigen function everything is same only we need to replace these variables then we can able to analyze the axial vibration.

So, we will not be repeating the axial vibration analysis here, but just trying to show the analogy here in the continuous system approach as I mentioned for more complex boundary condition for multiple disk or multiple support the obtaining solutions are not easy. I will just show one particular case in which the same cantilever beam in which we have the shaft is having continuous continuously distributed stiffness and mass property. But along with that there is a concentrated disk at the free end and this additional addition of the disk how the complexity increases because of this we will try to see.

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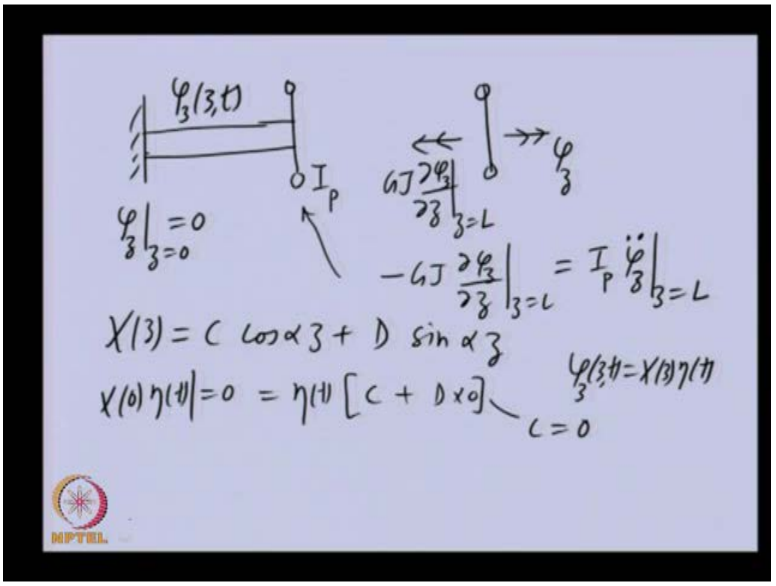

So, we have in this particular case the cantilever shaft having distributed mass and stiffness property. And there is a concentrated disk at the end let us say the polar momentum of inertia of that is I p. So, in this particular case we can able to see the boundary conditions this particular disk either we can take in the boundary condition. So, fixed end the boundary condition is same that is this is at z is equal to 0 is 0 here to obtain boundary condition let us take the free body diagram of the disk. So, in this particular case if this is the positive direction for the angular displacement there will be reactive torque from the shaft on to the, this that will be acting and that will be this at z is equal to L.

So, from this we can able to get the boundary condition here that means if we take the equation of motion of this. Because this particular torque is acting in the negative direction this is the only torque acting on this particular disk should be equal to this is at z is equal to L should be equal to inertia this is the polar moment of inertia of the disk and phi z and z is equal to L. Because this disk location is at z is equal to L this angular displacement which is basically in general representing the displacement angular displacement of the whole shaft at any location of the z.

So, we need to specify that this angular displacement, which we are talking about for this disk is at z is equal to L. So, this is the equation, which we obtain is the boundary condition of the problem earlier because there was no disk. So, right hand side was 0, but because there is a disk this is having right hand side, which is this. Now, let us see how this boundary condition gives the complexity in the solution. So, have the equation like this in which C and D we need to obtain with the boundary condition.

So, for this boundary condition we because this phi z we chose as multiplication of the this and the time function. So, in this particular case time function is not there, so basically we can able to write this as so I am satisfying here, so X i at 0 eta T at this is at z is equal to 0 is 0. So, we can see eta T C plus D into 0, so this we already seen that for this particular case because eta cannot be 0 C this gives C is equal to 0 this was the similar condition like the previous one a complexity is there in the second boundary condition. Now, we need to substitute here the solution, so I am taking this in the second slide.

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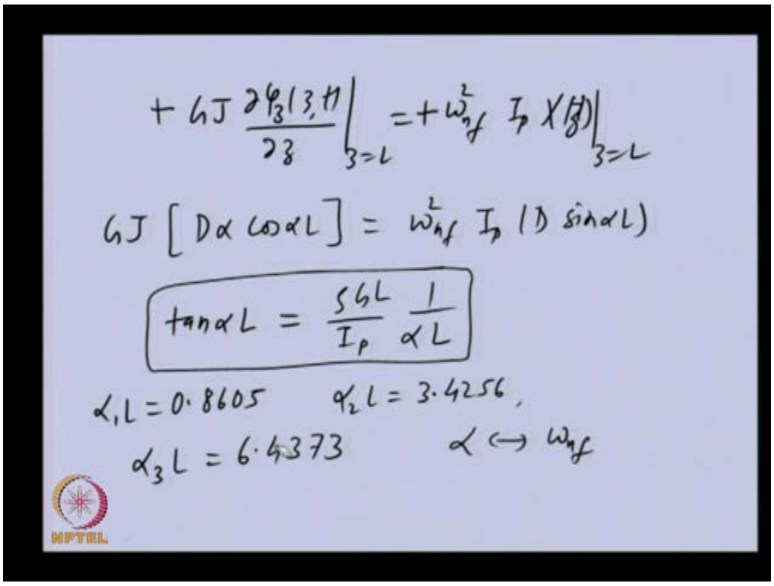

The boundary condition I am rewriting like this and because we have the harmonic function eta. So, we can able to write, so basically here if we substitute this here we will get negative of that will get negative of this. So, they will get cancelled, now we will satisfy the boundary condition. So, we can able to see that if we take the first derivative of this because C is already 0. So, this will be this is similar to the previous one we have substituted for z is equal to L is equal to 0.

So, here also we have substituted for this, so we can able to see that now this equation we can able to simplify as tan alpha L is equal to rho G L by I p 1 by alpha L. So, with some rearrangement we can able to write this equation like this, so here you can able to see that the solution of this is not easy because we need to solve for alpha, which is here also this is a transcendental equation. So, this we need to solve numerically and few solutions of these I am providing here, but we will having infinite number of solutions.

So, first three solutions are these and once we obtain this because we know alpha and omega N F are related. So, natural frequency can be obtained, but we can able to appreciate that with additional of the mass itself the frequency equation, which was earlier simpler. Now, it has become a transcendental equation and solutions of those roots are not easy. Now, so in the present lecture we introduce the continuous system approach for torsional vibration this particular system is most accurately we can able to model this system.

We can able to get the natural frequency and Eigen function, but the difficultly is for very simple boundary condition we can able to get the solutions for more complex conditions we need to for approximate solutions. In the subsequent class we will choose finite element method solution of subsystem in which we will be, because we know that finite element method is having flexibility in incorporating more complex boundary conditions.

So, that particular method we will be exploding in two lectures in this continuous system approach as we have seen that we solve the system equations as a whole. In the subsequent lecture we will see when we will be using finite element method in the in that particular method we break the system in various small segments that we call it as element. We obtain the elemental equation and once we obtain the elemental equation then we obtain the system equation by assembling such elemental equations. Then finally, we apply the boundary condition to get the governing equation for the whole system, so that particular method we will see in the subsequent lecture.

Today we will be doing the torsional vibration using the finite element method. So, basically I will be introducing the finite element method, which will be useful for purpose of the analyzing a torsional vibration. We will not go into more detail on the infinite element method as such, but how it can be used for the for analyzing the rotors for torsional vibration we will try to highlight. Then we will through some examples will try to see how this particular method is having capability to analyze more complex rotor system as compare to either analytical method or Newton's method the transverse matrix method. We already seen it can handle large systems of rotors and…

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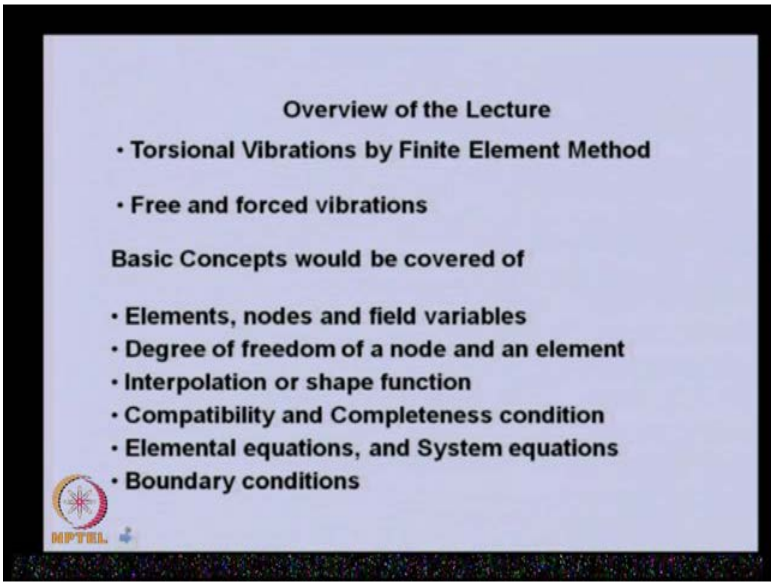

So, let us see what are the things that we will be covering in this particular lecture? So, we will be analyzing the torsional vibration by finite element method free and forced vibration. Both we can able to do it with this method various concepts of the finite element methods like elements nodes field variable degree of freedom of a node or of an element interpolation or shape function. And two conditions which we need to satisfy of the interpolation function that is the compatibility condition and the completeness condition.

Based on these we will be choosing the interpolation function, so we will introduce these terminologies and then once we have discretized the system into various elements. How to obtain the elemental equation and form those elemental equations? How to get system equation and application of the boundary condition in the system equation? So, that either we can able to solve for free vibration or forced vibration of the torsional vibration, now with a very simple case we will try to see how the finite element method works in the continuous system approach.

We have seen that the governing equation defines for each and every particle all to the system. So, that that particular equation represent the governing equation for the whole system, but in finite element method we will see that we will first discretize the system will break the system into various pieces. Then we will develop the elemental equation for each of them and then system equation can be built.

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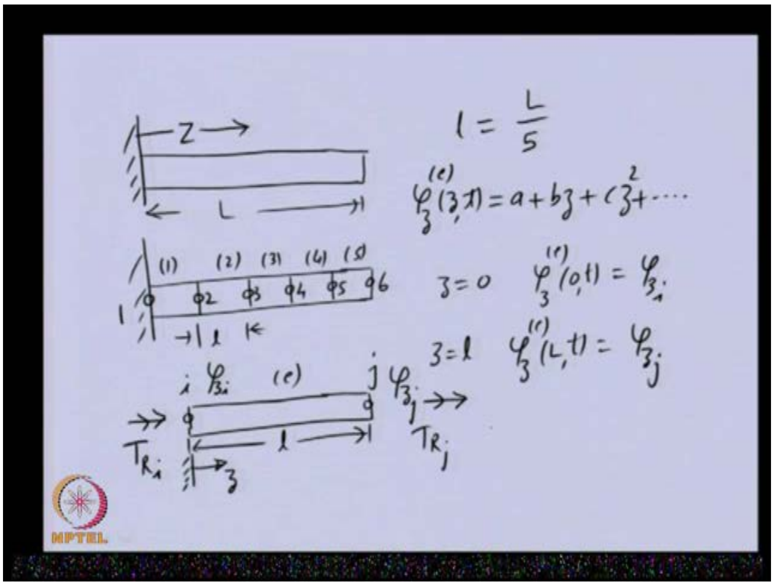

So, let us take one case in which we have a cantilever shaft like this one end of the shaft fixed another is free for finite element method we need to break this particular shaft into number of pieces depending upon the accuracy. Required these size of the each of the shaft segment can vary here I have taken uniform, so if total length of the shaft is L, each ((Refer Time: 53:49)), because we have divided into five element length if it is uniformly divided. In that particular case we will be having length is equal to L by 5 and each of these shaft segment we will call these as element.

So, element 1 element 2 element 3 element 4 element 5 and between the 2 elements there is a common interface and those we call it as a node. So, we have node 1 here node 2 node 3 node 4 node 5 and node 6, so these are the node positions and in finite element method we will try to find out the solution at these node positions and we will see. That will be if we want the solution in between we will be using some kind of interpolation function to obtain that and if we want to develop the elemental equation.

So, we can take one of the representative element and we can draw the free body diagram of that, so if we draw the free diagram of one of the element. So, let us say this is the one of the element I will large it let us say this is the E th element and to define the location of this element I am attaching a coordinate system, which we call as local coordinate system in the actual system. We can have a coordinate system which is I should call this as capital z.

So, this is defining a axis system, which is global axis system this is local to the this element. Now, the length of this element is L small L node here is let us say I th node and this is J th node and at each of these we now want to specify the field variables. So, obviously field variable is at I th node, we have angular displacement phi z I. Here angular displacement phi z J and the directions we are taking in left to right as positive. So, both are having the same direction at two nodes at J n I n J apart from this we will be having the torque associated with this displacement.

So, I am writing here torque as T this is the reaction torque from the other element. So, T R I and here T R J the direction of this torque is also same as the displacements, we are taking same as the displacement and depending upon the their actual direction the positive or negative we will take care of the actual direction. So, this is a an element of the shaft system for torsional vibration and you can able to see that at each node one field variable that is angular displacement is there.

So, this particular mode is defined by this variable a single field variable for this particular problem this is another node there also there is another field variable. So, for one element which we have considered here there are two nodes and the elemental degree of freedom is two. So, 1 is phi z z 1 phi z 1 and phi another is phi z J, now we because whatever the governing equation we derived for continuous system which was valid for each and every particle in the whole system.

So, obviously that governing equation will be valid for this element also, but in this particular case as we I as I mention will be seeking the solution at the nodes only. If you want the solution in between solution means the angular displacements in between then we will be using some kind of shape function. So, that means in this particular case this particular shape function in finite element method is generally a polynomial. So, this is the angular displacement of an element, so E is representing this is the angular displacement for one element, which is function of both axial position and time we will be representing this as a polynomial.

So, polynomial will be having some constants like A B z plus C z square we can take this as several terms depending upon the problem we need to truncate this polynomial up to certain degree of the polynomial. So, you can able to see that in this interpolation function some unknowns are appearing and these unknowns we will be finding from the boundary condition of the element. Because this solution, which we are assuming is for is element, so we need to find out the, these constants from the boundary condition of the element.

Now, once we have chosen the solution in this from, so you can able to see that in this particular case boundary conditions is z is equal to 0. Here we have angular displacement of the element at z is equal to 0 is phi z I and at z is equal to L other end of the element we have this angular displacement at L is phi z J. So, we expect that these constants will be in terms of these field variables at node I and J, so that means this particular solution, which we is in the more general form can be written like this phi z element z T is equal to…

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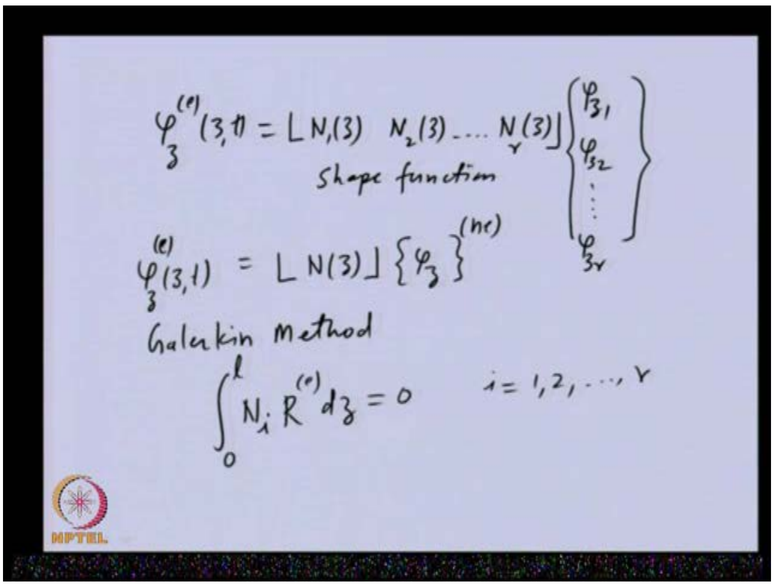

Now, here we can able to write in terms of some functions which is function of z there could be several functions number of functions will grown by the type of the problem we are dealing with. But, in general we can able to write like this and here we can have various field variables. So, this is the more general representation of the assumed solution in which we have taken R number of field variable for the present problem R is 2. So, we except only two terms here and two functions here and this in more compact form we can able to write this as this is for the nodes. So, you can able to see these field variables are defined at nodes node 1 2 r th node for present case R is 2.

So, up to second will be coming so n E is representing that these field variables are at the nodal position these are the function of z, which will be of the form of polynomial. And actual from of this will be obtaining depending up on the boundary condition of the present problem. So, this is the solution for the element and we can able to appreciate here because we are using polynomial and we need to truncate the polynomial at some degree.

So, this solution is approximate in nature, so once the solution is approximate in nature if we satisfy try to satisfy this solution into the equation of motion. So, obviously that equation of motion will not satisfy completely there will be some residual and now we will be, we will be trying to see how these assumed solution approximate assumed solution can be substitute in the governing equation.

How we can able to derive the elemental equation for the element, which we have chosen, so we will be using Galerkin method for this in this particular method whatever the residual comes after substitution of this assumed solution. In the governing equation we minimize that particular residual, so let us say R is the residual for that particular element. So, we multiply these by some function, so in Galerkin method we sub multiply by the these functions which are we call as shape functions and we integrate these quantity over a domain or whole length of the element and we equate it to 0.

So, that whatever the residual is there, that is distributed over the length and because we can have this type functions R in number depending up on the degree of freedom of the element. So, that many equations we will be getting from this method, now let us see what is the form of the residual the shape function will at present keep in the general from. Subsequently we will be obtaining the explicit form of this particular shape function.

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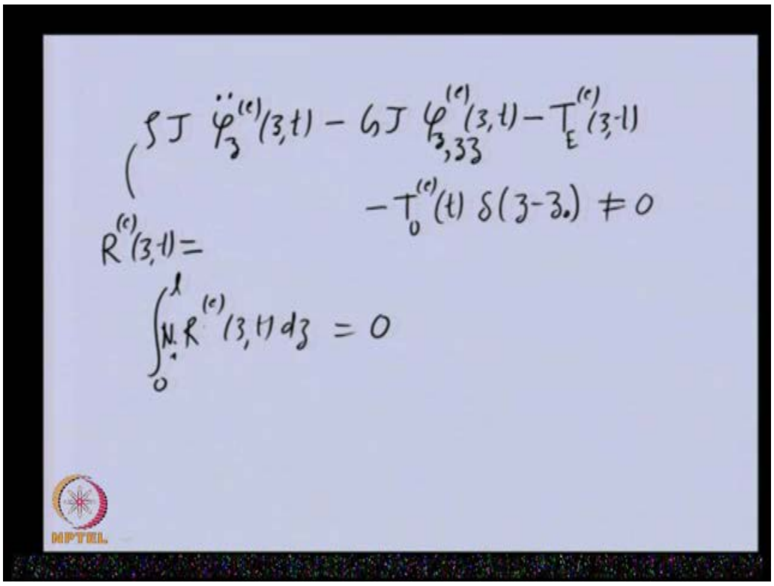

So, let us see the residual because we earlier derived the governing equation, which was having this form for continuous system. So, this is double dot derivative with respect to time this is function of both z and T then I am representing the double derivate with respect to z as comma double z. So, that means phi z this is same as like this, so this representation is more compact. So, I am using this representation and this phi is function of z and T and if we have external torque which is function of which is distributed torque.

So, this one or if we have concentrated torque at particular location in this particular element, so this will be function of time only and we will be using a direct delta function z minus z naught location. So, here if the torque is within the element then only it will contribute otherwise it will not contribute. So, if z naught is within the element then only it will be appearing similarly, the external torque which is distributed if it is there within the element then it will be having contribution in the equation of motion.

So, this equation of motion which we written earlier only difference here is you can see instead of only phi z I am writing the superscript E. So, that means we have written this for element because this governing equation is valid for each and every point on the system. So, it is valid for this element also, you can able to see this is, this is representing for the element and because these solutions are approximate so this will not be perfectly 0. This quantity we will call it as residual, which is function of z and T so the whole quantity is residual. Now, we want to minimize this particular residual as we using the Galerkin method by multiplying by the shape function and equating it to 0. So, this is the Galerkin method, now I am substituting the residual, which is here and this in the next slide.

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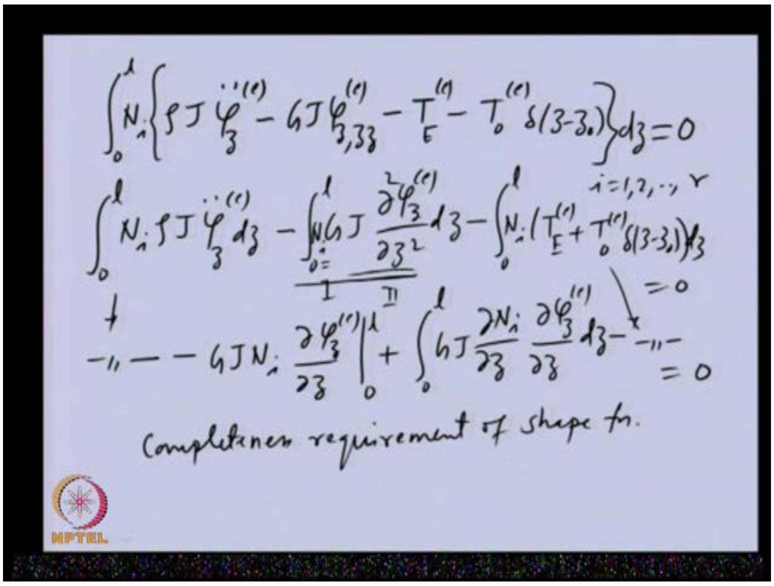

So, I will get A terms like this, so within the bracket the residual I am expressing. So, this is the residual term basically this is the inertia term and this is elastic term comma z z means double derivative with respect to z then external torque, which is distributed and concentrated. Torque which is located at one location at z is equal to z naught. So, this was the residual we are integrating this and over the domain and equating it to 0 where this I is from 1 to R depending up on the degree of the freedom of the element.

Now, in this particular equation because we are using the assumed solution in the form of polynomial we can able to see that here we have second derivative involved with respect to z. So, whatever the polynomial which we need to choose that should have at least quadratic term, so that means because if it if you are taking up to only the linear term A and B the second derivative of this term will give this quantity 0. So, here this term will be 0 so this is not a complete representation of the polynomial of the assumed solution, so for this particular case you can able to see at least we need to go up to the quadratic polynomial we could we can take more also.

But at least quadratic is required, but by some mathematical manipulation we can able to reduce this particular differential order that we will see how we can able to reduce the differentiation order. So, that we can able to use lesser degree polynomial that we will be having a lesser requirement in the computation, so let us see how we can able to so what we will be doing we will be doing basically integration by part of this term with respect to z.

So, first term is as it is there is no change in the first term only terms containing the double derivative with respect to z we are doing integration by part once. So, here I am writing this in more explicit from 0 to L G J d z and other terms m I is there T external distributed and T naught, which is concentrated torque. Now, I want to do the integration by part of this term I am doing this separately here. So, this term will be as it is minus in this let us say N I is the first part and remaining is the second part.

So, we can write this as J G J N I is the first part integration of the second part that means phi z 1 derivate will reduce here and limit is 0 to L. Because it is negative here 0 to L we can have differentiation at the first part and the second part integration D z and the torque term this external torque term will remain as it is equal to 0 so main term which we have manipulated is the second term. So, you can able to see this is the first part integration at the second part limit and then differentiation of the first part and integration of the first part.

Now, we can able to see in the whole expression the phi z is having derivative of single. So, we could able to now use lesser a degree polynomial without compromising on the terms. So, you can able to see even if you are taking a linear polynomial the terms will not vanish and this particular requirement is called the completeness requirement of shape function. So, we have seen that how we evolve the requirement of the shape function one of the requirement that is a completeness requirement that…

So, we should choose the polynomial such that when we substitute in the formulation which I have shown it should not vanish. So, in the formulation we after the integration by part we could able to reduce the order. That is why this particular step is called weak formulation infinite element method in which from polynomial requirement to the linear requirement of the shape function, we could able to reduce.

So, this requirement we could, we could able to regain it, so that is why this particular step is called weak formulation and we can able to see now that now we can able to have the explicit form of the shape function to get the explicit form of the shape function. Obviously now we have we have seen that the linear polynomial is good enough for this particular problem from the weak formulation now we will obtain the explicit form of the, this shape function using the boundary condition of the element.

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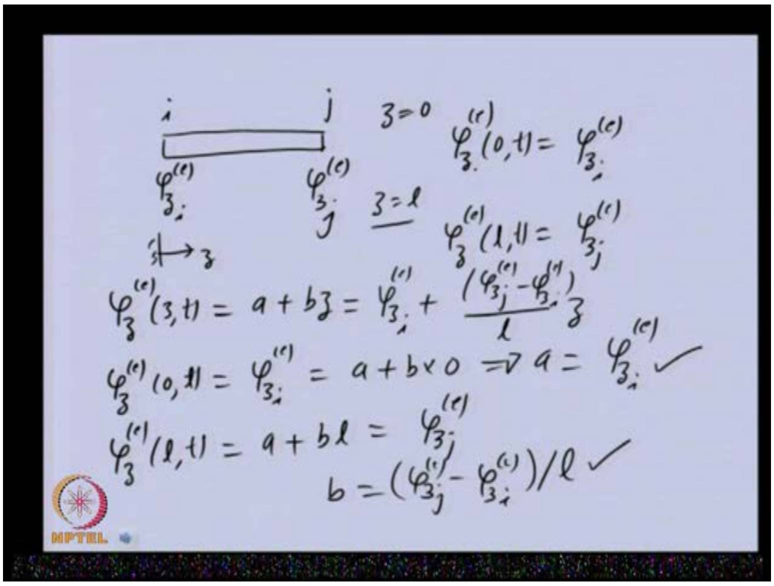

So, for this case we chose one element and we had I th and J th node and there the field variables were these two phi z I and phi z J and local coordinate system was z. So, we have boundary condition that at z is equal to 0 phi z I is sorry phi z is element phi z E all are element because we are representing for this particular element only. Similarly, at z is equal to L we have another boundary condition that is phi z J. Now, once we have the form of this elemental field variable in the form of polynomial as we have seen in the front of weak formulation [FL] this is good enough and you can able to see that there are two constants.

We have two boundary condition, so in that also in that way also we can able to obtain the explicit form of A and B in terms of the this nodal displacements. So, first condition if we satisfy here, so we will get this as phi z I R element. So, A plus B into 0, so from here we can see a is phi z I and from the second boundary condition we have this A plus B L and that is equal to phi z J this one.

Now, we have we already know a is this 1, so B can be written as phi z J minus phi z I for element divided by L. So, you can able to see that we could able to obtain the A and B and this we can able to substitute in this equation phi z I plus phi z J is for all for element phi z I divided by L and z is here. So, the assumed solution, now contain the field variables and the z which is polynomial. So, this equation we can able to rearrange we can able to collect the terms of the phi z I which are here and phi z is single term, so we can able to write this as phi.

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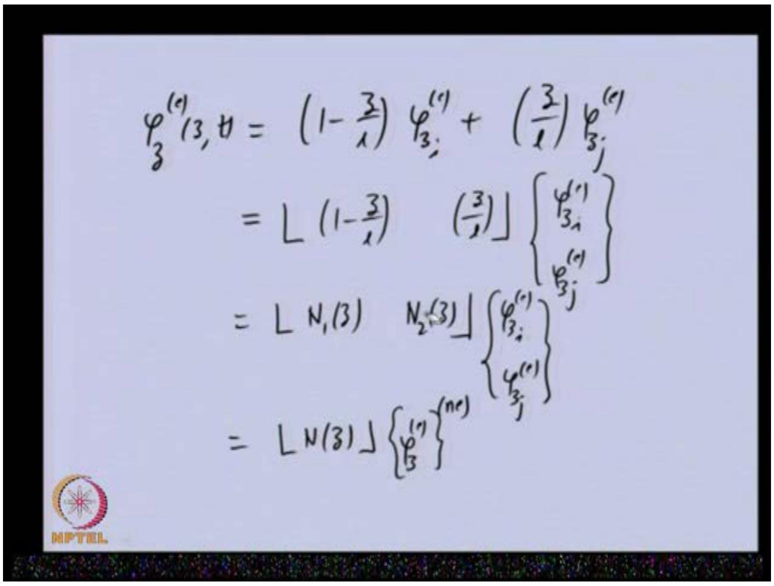

Assumed solution in the form of polynomial can be written as 1 minus z by L phi z I plus z by I phi z J. Now, with some manipulation we can able to express this as 1 minus z by L z by L this is representing a rho vector this particular symbol is representing a rho vector and this is representing a column vector. So, we can able to see if we multiply these two vectors we will get the same expression as the previous one this is a column vector this is A row vector.

Now, if you recall we wrote this as earlier and 1 z and 2 z and here we have the same thing and this these n 1 and n 2 are the shape functions. These are the nodal displacements and we wrote this in the more compact form as shape function row vector and the field variable at nodal location. Now, we will see the property of these two shape function.

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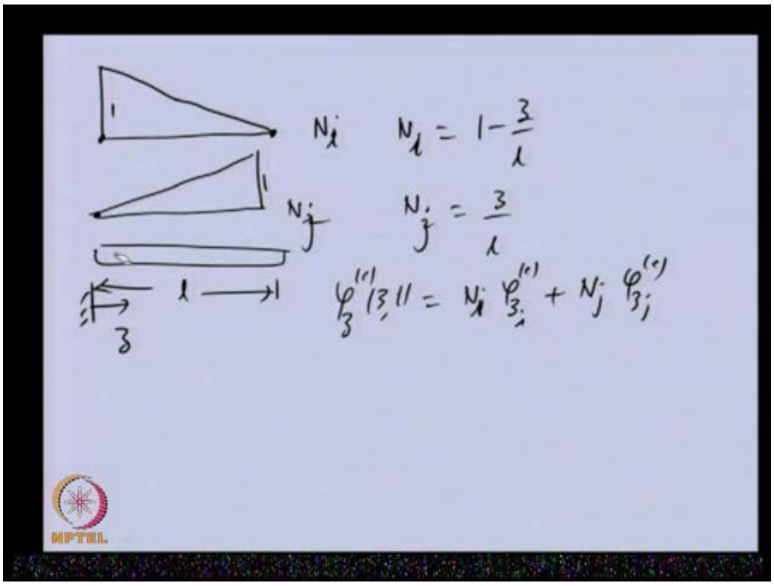

If we see N 1 is 1 minus z by L and N 2 is z by L if we see the element having this axis total length is L. So, if we want to plot the N L. So, let us say if you want to plot N 1 here, so at z is equal to 0 which is this location the value of N y is 1. So, this is 1 N at z is equal to L this quantity becomes 0. So, L is linearly varying in between, so this will vary like this and 2 at z is equal to 0. It is 0 here at z is equal to 0 and at z is equal L is having 1 value. So, is 1 value and in between is varying linearly.

So, you can able to see that how this interpretation function changes and from this we can able to obtain the… So, this is N I and this is corresponding to node I and this is corresponding to node J. So, we can able to see that if we want the one way of this field variable at z is equal to 0 this will be 1 this will be 0 and that will be giving as the field variable at I th node if it is z is L N I will be 0 this will be 1. So, we will be getting the field variable at the j th node, but if you put z in between then it will interpret the field variable from this field of this nodal displacements. We can able to get the field variable at any other or the angular displacement at any other location.

So, basically here all these corresponding to the I and j th node are there and the previous case also, now we have see the property of the shape function we have obtained the explicit form of the shape function for this particular element in which only we had two nodes. We had two degree of freedom of the element, now we will see another property of the shape function, which we need to satisfy that is the compatibility condition the compatibility condition is between two nodes or two elements. We have seen the completeness condition we satisfy within a particular element, but this compatibility condition we need to satisfy two elements at the common node.

So, if we have two degree elements and there is a common node in between then let us see what is this compatibility condition we need to satisfy? So, basically we have divided the shaft into various elements and we have drawn the free body diagram of that and during assembly obviously we need to join all those elements. During joining there has to be some compatibility that the displacement predicted by one element at one particular common node. Displacement predicted by another element at the same node should match that is the compatibility condition.

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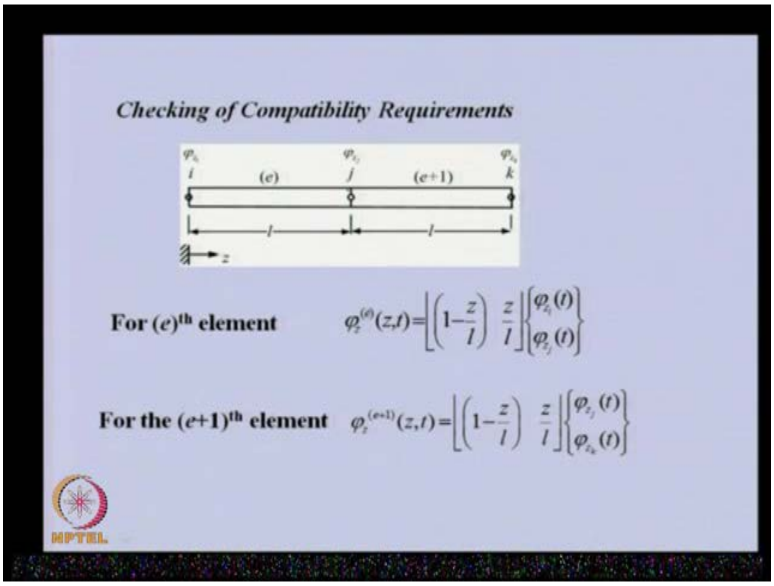

So, let us see through how we should whether this particular shape function, which we have chosen is compatible up to the requirement for us. So, in this particular case basically if we go back to our equation a weak formulation equation, so in this within the integral whatever the highest derivative of the field variable is there we need to satisfy the compatibility condition one order less. So, this is the, I should say thumb rule that within the integral in this weak formulation we have first derivative of the field variable.

So, we need to satisfy the compatibility condition one order less in transverse vibration we will see that these derivatives will be having higher order. We need to satisfy even higher order field variable derivatives but for this torsional case only we need to satisfy the compatibility of the field variable itself. There is no need to satisfy for the its derivative with respect to z, so with this rule of the compatibility condition let us see how we can able to check this condition for the chosen shape function.

So, we have two neighboring elements, element e and element e plus 1 this common node is J this 1 is I node J and K J is the common node between these two elements. Now, for element E we can able to write the field variable using the shape function, here you can able to see they are corresponding to node I and J field variable similar for E plus 1 the expression this is the field variable for E plus 1 shape function remains the same.

Here the field variable at the nodes will be J and K. So, using these two we can able to predict so from first we can able to predict the displacement angular displacement in the E element and from second we can able to predict from the e th element. But at common node we can able to predict from this end element also and from this element also and we need to see that this prediction should be same or not, so let us see for e th element. If we substitute J is equal to L, which is corresponding to the J th node. So, if we see back for this element for J is equal to L this is the node this is the local coordinate system.

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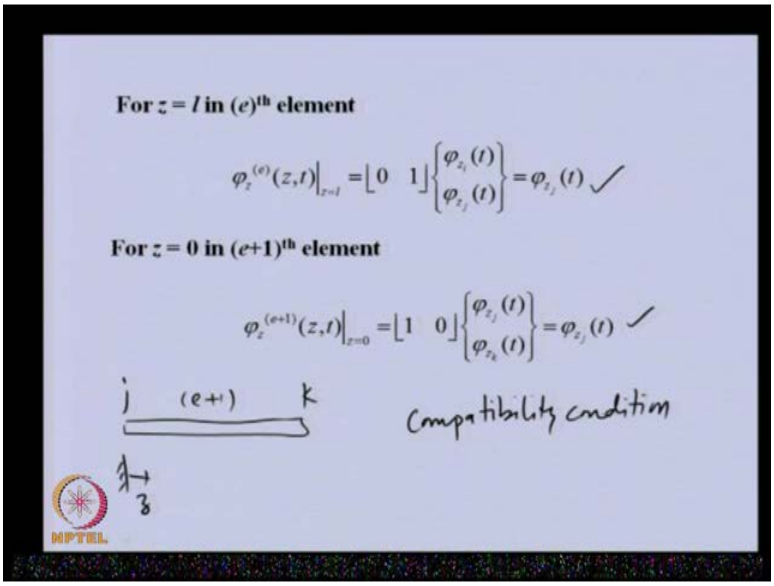

So, if we put z is equal to L this N 1 will be 0 N I will be 0 and J will be 1 and if we multiply this we are getting the field variable at the J th node. Now, if we take element J plus E plus 1 then the J th node location is the E plus 1 element local coordinate is again is staring from the same point here, but node is J here and K here. So, you can able to see that for element E plus 1 the J th node is at z is equal to 0. So, that is why we are trying to predict the field variable at J th node by putting z is equal to 0 in the E plus L th elemental equation.

So, in this particular case J will be 1 and this will be 0 and we can able to see after multiplication we are getting the field variable at the J th node. So both elemental… Both are giving, both elements are giving the angular displacement at common node as same. So, we call these as they are compatible, so this is the compatibility condition satisfaction. So, whatever the polynomial we have chosen in the form of shape function they are compatible for the present problem.

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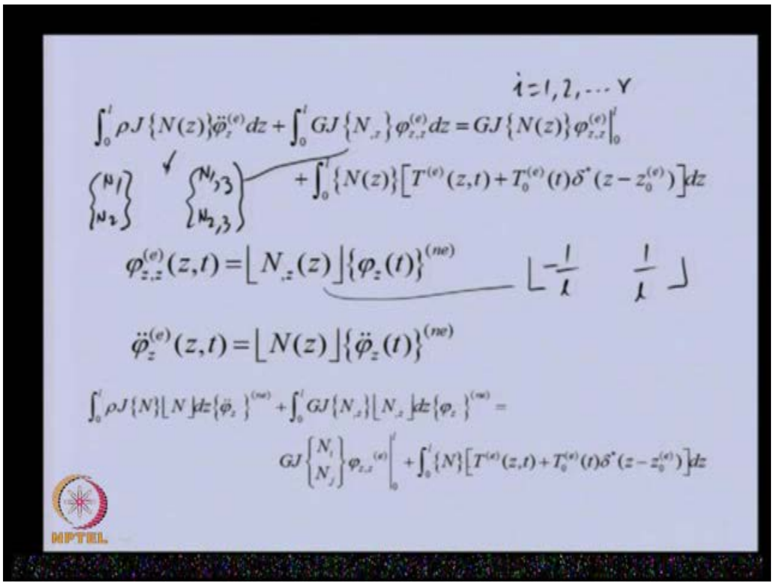

Now, once we have chosen the explicit form of the shape function we can able to see the previous equation, which we had we had for I is equal to 1 to R. We had R such equations because N I was there in each of the expressions and those equation we can able to put in a single equation in the vector form. So, here now R is 2 for present case, so this is the say weak formulation equation, which we are putting N I in a form of vector, so you can able to see the form of n will be for present case N 1 N 2. So, basically if you expand again you will get two equations out of this, similarly here because there was derivative with respect to z.

So, I am writing this as basically this expression is N 1 comma z N 2 comma z comma z is representing that we need to derivate the shape function with respect to z. Once both the shape functions and here this is the other term, which is after integration by part we got an addition term and this is the external torque terms distributed torque and the concentrated torque. So, this is this is the term which we got after integration by parts, so this shows that particular term, here also we have put the all those N I in a vector form. So, that we can able to represent this in a more compact form, now having obtained the shape function and in these particular equations we need the derivative of this with respect to z of shape function.

So, the shape function we can able to derivate once and because only the shape function will get differentiated this is time dependent. So, this will not get differentiated with respect to z, so this is the field variable, which is function of both z and T this is the field variable at node, which is function of only time. So, this is the difference, so this is for the whole element this is for the element that is why it is function of time only and in between we interpret using this interpolation function this comma z is representing derivative with respect to z. So, because we know the form of this, so we can able to differentiate this to get 1 by L and 1 by L because we know once we difference will we will get this quantity because z was there 1 is constant that will vanish.

Similarly, if we differentiate with respect to time this term will get derivated and will be so these two terms we can able to substitute in the above expressions. So, this I am substituting so you can able to see I am getting a row a column vector of N and row vector of N from this and because this is time dependent so it can come out from the integral. Similarly, from second expression this is a row vector column vector row vector, which is coming from here and this time dependent term I am keeping outside. And here I am writing in a more expanded from like this and this is a torque external torque.

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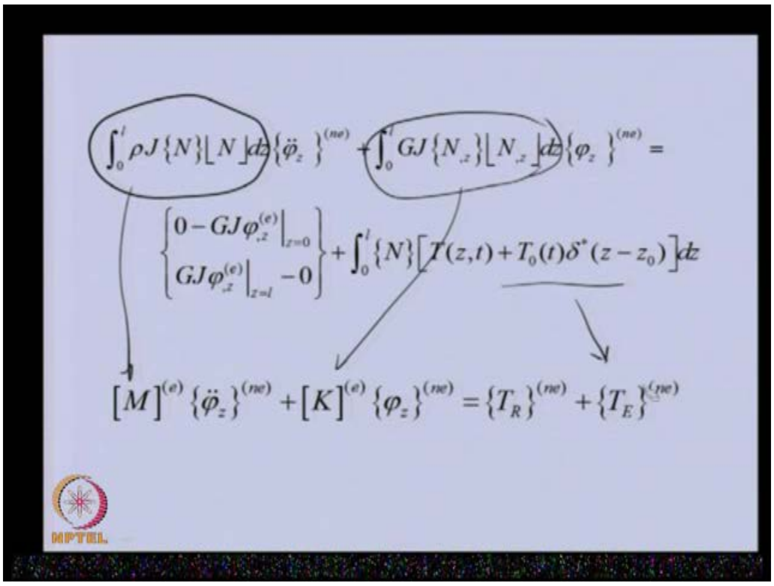

So, I am keeping this as it is now you can able to see the same equation I have written in more explicit from is exactly same equation as previous one only thing here I am substituting for the limit. So, in previous case we had this one I am, I am substituting the limit, so in this you can able to see for z is equal to N this will be 0 N 1 will be 0 and 2 will be 1. So, this will give me this and similarly, for z is equal to L N 1 will be 1 and n 2 will be 0, so will get this. So, basically these are the reaction torques at the two ends of the, which I am representing as a this particular vector and this term I am calling as a mass matrix. This term I am calling as a stiffness matrix and remaining this term, I am expressing as a external torque term. So, this is the reactions, because we have divided into element and this is A element this is a external torque.

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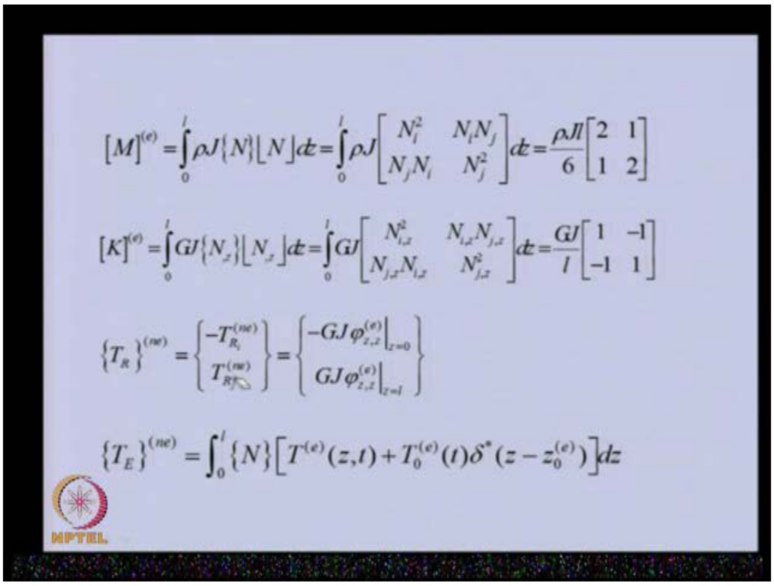

The explicit form of this mass matrix will be like this, so if we multiply this will be the explicit form of the mass matrix. Similarly, the stiffness matrix we can able to multiply, so this will be the explicit form of the stiffness matrix. This we already expanded the reaction torque terms. So, this will be reaction torque term at one of the i th node and this for the j th node and this is the external torque. So in the present lecture, we have seen how to formulate the elemental equation or torsional vibration, we started with very basic concept of the finite element method.

We first, we need to discretize the system into various part that we call it as element and we took one of the element and we obtained the governing equation that we call it as a finite element equation for that. And this particular finite element equation, the form is similar as we have standard equations in the vibration, which contains the mass matrix stiffness matrix reaction torques. External torque we have not considered at this particular case, and once we obtain this elemental equation. In the next lecture, we will see how these elemental equations can be used to analyze more complex system or how we can able to solve a system equation, because this equation is representing for one of the element.

So, similar equation we need to write for all the element in the system, and then we can combine them and then boundary condition can be applied to get the system equation and that can be solved for either free vibration or forced vibration. So, you can able to appreciate once we take up with some example, how this particular method is can tackle various kind of boundary conditions and even for large system, the formulation is easy. So, is more once we have these elemental equations ready, we can able to make a computer program for analyzing more complex system that we will see in the subsequent class over all procedure we can able to solve a legal system.