

**CAD/CAM**  
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**Lecture No. # 7**  
**B-Spline Curve**

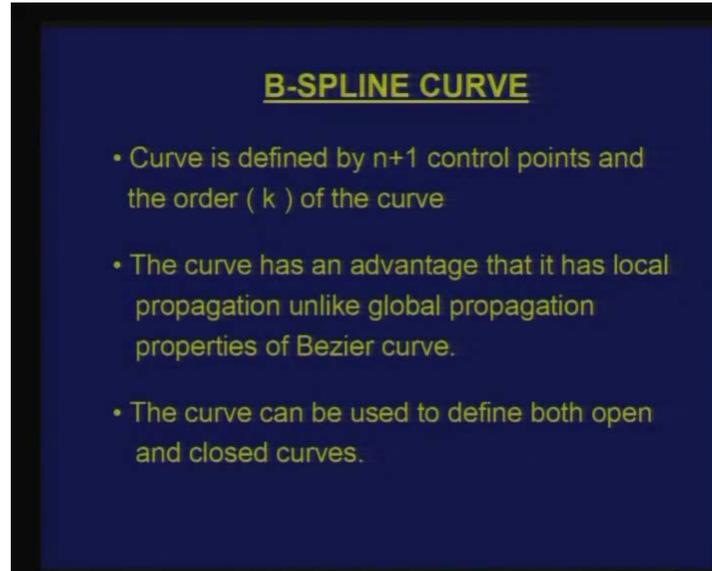
So these days we are discussing the subject of parametric curves. So we started with the Hermit curve which is basically an interpolation curve. Then in the last lecture, we looked at the Bezier curve which is one of the approximation curves. Now in today's class we will take up the subject of B-Spline curves and look at what is the importance of this curve and what advantages this class of curves offer over Bezier curves. And also we will extend the definition of B-Spline curves to define what we call as a non-uniform rational B-Splines or NURBS curve.

Now if you really recall what we discussed in Bezier curve is that one of the problems with the Bezier curves is if I want to define let's say a curve using let's say a certain control points, its order is fixed. Suppose if I take let's say a 4 control points then I have a cubic Bezier curve. If I go for let's say a 5 control points then I have a quadratic curve and so on. Suppose if I have let's say, I have 10 control points which are used to define a Bezier curve then I also have terms like  $1 - u$  to the power of 9 and others. Now this actually poses problems particularly we have also seen that suppose if I want to find out intersections or if I want to do any of the numerical calculation related to the curves, you have to basically deal with polynomials which have degrees like  $u$  to the power of 9 etc which is a disadvantage. Ideally we really do not want the terms like  $u$  to the power of 9 or higher order.

We have also seen that if I have a cubic that is let's say  $u$  to the power of cube that search the purpose of most of my applications because it gives me both slope continuity as well as curvature continuity and beyond that is not of much importance. so there was an attempt to define let's say a class of curves where suppose if I have a points, a large number of points control points but I should be able to still define a curve of whatever order which I want. Let's say if I want a cubic, I should be able to get cubic or if I want let's say one order lower; I have only a quadratic terms related to  $u$  I should be able to define. So B-Spline curves are basically a part of this attempt where given a set of control points, the order which I choose for the curve is independent of the control points.

I may be using let's say 10 points or 100 points or 25 points as my control points but I have a freedom to basically fix whether I want a curve which is quadratic or cubic or higher order as and when they are required.

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Now let's look at how can one go about doing this particular thing like here the curve is again defined by  $n$  plus 1 control points. We have seen this in Bezier curves too like it goes from like point  $p_0$  to  $p_1$  all the way up to  $p_n$  since you are including 0 so these are  $n$  plus one control points. but you don't specify any order which is fixed by the number of control points but in the case of B-Spline curves, I have to separately give what is the value of  $k$ .  $k$  is a common symbol which is used to represent the order of a curve. If I take let's say  $k$  is equal to 4, and then I will have a terms like  $u$  cube. So a cubic basically corresponds to  $k$  is equal to 4. If I choose let's say  $k$  is equal to 3 then that means I have the terms like  $u$  square that is the maximum degree of polynomial which we have or if I choose let's say  $k$  is equal to 2 then this is basically a linear, it's in  $u$  and I can also have  $k$  is equal to 1 which is nothing but the specific points. You do not have any a curve kind of a thing. So I can also go for higher order like if I choose let's say  $k$  is equal to 5, I can also get B-Spline curve which has terms like  $u$  to the power of 4. So it gives you a freedom to choose whatever is the degree of the curve which you want to do that.

Now another advantage of this curve which is shown here is that it has a local propagation like when we discussed about a Bezier curves, we have seen that if I am using let's say  $n$  plus 1 control point, changing any control point will change the entire shape of the curve because every point on the curve is defined by all the control points. So since you have, this is called as a global propagation of the curve whereas here the curve is divided into number of segments. In fact a B-Spline curve is not a single curve, you can call it as a composite curve number of curves which are consisting in terms of segments but they all meet the continuity requirements like suppose if I choose let's say  $k$  is equal to 4 which is my cubic they will all have a continuity which is like  $C_2$  continuity that means slope and curvature will be continuous but still the curve is in terms of segments.

Now the advantage of defining a curve in terms of segments is that a particular segment of a curve is influenced only by a certain control points. So if I change let's say any one of the control points, I may have a B-Spline curve which is defined by 10 control points and k is equal to let's say 4, if I let's say change one of the control point then only a some portion of the curve only a few, you can say pieces of the curve, few segments of the curve they are influenced. That is a major advantage of this particular curve, the one this is used. And as we discussed in the case of Bezier curves that I can define both closed curve as well as open curves. If I have a control point that is starting control point is same as the end control point then I can also use the definition to define the closed curve. This is true with B-Spline curves.

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**BEZIER vs. B-SPLINE CURVE**

**Bezier Curve**

$$x(u) = \sum_{i=0}^{i=n} {}^n C_i (1-u)^{n-i} u^i x_i \quad 0 \leq u \leq 1$$

**B-Spline Curve**

$$x(u) = \sum_{i=0}^{i=n} N_{i,k}(u) x_i \quad 0 \leq u \leq n-k+2$$

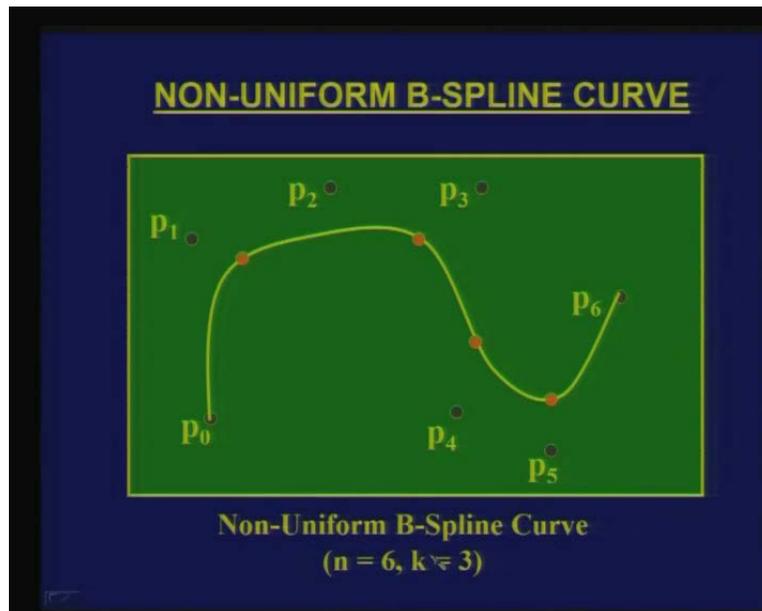
Now let's look at how does this class of curve differs from let's say a Bezier curve. We have already discussed about Bezier curve, this definition we have seen. If I have n control points then let's say an x coordinate of the curve is given by the binomial coefficient. It's a basically summation of number of terms where I have as many terms as there are number of points. Each of these x coordinates will have some influence and this is actually multiplied by the 2 terms, you have 1 minus u to the power of n minus i and u to the power of i and I have a binomial coefficient which is  $nC_i$ .

So a term expression like this suppose if I have a 10 control points, I will have a 10 terms here, they will all participating in the definition of a curve and the parameter range varies from 0 to 1. So if u is equal to 1 corresponds to the last control point or the last point, similarly u is equal to 0 corresponds to the starting point. Now let's come to B-Spline curve. You have a definition if you see here. Now suppose if I have let's say n control points, just like we have the n control points which are participating here in the case of Bezier curve, same thing happens in the case of B-Spline curve two but this function like this particular function which we have seen in the case of Bezier which includes  $nC_i$  1 minus u to the power of n minus i into u to the power of i is called as a Bezier basis

function. That is a name which is given to that, it's called as a Bezier basis function.  $N_{i,k}$  which is shown here which is also a function of  $u$  is called as a B-Spline basis function.

The definition of this is different from that of Bezier B-Spline, a Bezier basis function. Another difference which you will see in the case of the definition of this particular curve is that parameter range is not from 0 to 1. Depending on how many control points which I choose and depending what is the value of  $k$  which I choose, I will have a different parameter range. Take for example, if I have let's say  $n$  is equal to 6 that means I have 7 control points. Let's say I choose  $k$  is equal to 3. So if I choose  $n$  is equal to 6 and  $k$  is equal to 3 what is the range of  $u$ ? So this is  $n$  minus  $k$  plus 2, so this is 6 minus 3 that are plus 2, so that is 5. So there are 5 segments of the curve, the first segment goes from  $u$  is equal to 0 to 1. Second one has a parameter range which is  $u$  is equal to 1 to 2; third as 2 to 3 and so on. We have about 5 segments of the curve which will define the complete curve. And the usually happens is we see here that all the control points are participating. Then how does like the local propagation happens is for a given range of  $u$ , many of these basis functions will become 0. Only some of the functions are, some of the basis functions are nonzero and hence the influence is basically restricted to a local in terms of control points.

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Now let's look at this may be through an example like I have put down a curve here. So what you see here is  $n$  is equal to 6.  $N$  is equal to 6 means 0 to 6, I have 7 control points which are shown here as  $p_0$   $p_1$   $p_2$   $p_3$   $p_4$   $p_5$  and  $p_6$ . The curve also shows like if you see these points which are shown here in the as orange points, they basically represent the curve segments. For example I have the first curve segment which is this, the second curve segment which goes from  $u$  is equal to 1 to  $u$  is equal to 2 is this segment. Then I have the third segment, fourth and five. So when I said  $n$  is equal to 6 and  $k$  is equal to 3  $n$

minus  $k$  plus 2 is basically 5. So I have 5 segments of the curve which are influenced. So that's what is exactly shown here.

Now coming back to this let's look at how does like what are the points which really influence like if I take a point a curve segment like this, the first segment which goes from  $u$  is equal to 0 to  $u$  is equal to 1, this portion of the curve is influenced by only the  $k$  points, whatever is the  $k$  which I am choosing.  $K$  here is 3, so this portion of the curve, the first portion is completely defined by the point's  $p_0$   $p_1$  and  $p_2$ . Whatever are the coordinates of  $p_3$   $p_4$   $p_5$  and  $p_6$  has no influence on this portion of the curve. Now when I go to the second portion of the curve that is when I have  $u$  which is ranging from 1 to 2, the control points which are affecting are  $p_1$   $p_2$  and  $p_3$  whereas  $p_0$  has no effect on the curve neither the points like  $p_4$   $p_5$  and  $p_6$  have any influence on the curve.

Suppose if I choose let's say  $k$  is equal to 4 then the influence is restricted to 4 control points on any segment of this particular curve. Now because of this, since the definition is in terms of a segments which constitute a composite curve, suppose if I change let's say control point which is like  $p_5$ , it will effect a few segments but it will not affect let's say segment which is going from  $u$  is equal to 0 to  $u$  is equal to 1. You will also see here that I have put the word non-uniform. So there are like when it comes to B-Spline curve, I can always define in two ways, one which I call as a uniform B-Spline curve and another is a non-uniform B-Spline curve.

The most common form is non-uniform. Why the word non-uniform which is coming here. Now this is basically because we know that every segment is influenced by 3 points that is if  $k$  is equal to 3, every segment is influenced by the 3 control points. Now if look from the reverse point of view like if I take a control point, how many segments a control point influences. It is not the same for all the control points. for example if I take  $p_0$ ,  $p_0$  is a control point which influences only the first portion that means it has influence only on between  $u$  is equal to 0,  $u$  is equal to 1. Take for example the control point  $p_1$ .  $P_1$  will influence two segments which go from 0 to 1 and then  $u$  is equal to 1 to 2 these two segments are influenced by  $p_1$ .

Take  $p_2$ ,  $p_2$  has influence on three segments. It has influence on three segments. What about  $p_3$ ?  $P_3$  also has on three segments.  $P_4$ ?  $P_4$  also has influence on three segments whereas  $p_4$  has influence on only two segments and  $p_6$  has influences only one segment. Since the influences which the different control points are not uniform, so you call it as a non-uniform B-Spline curve. Now why this is done is if I choose let's say uniform B-Spline curve, let's say the same type of functions for all the segments then curve doesn't pass through the first and last points. If I want curve to pass through which is a more common option which is required? If I want to force the curve to pass through, let's say first and last control points then I have to go for this non-uniform nature of the basic functions. That is why the word non-uniform which comes here which is likes the most common version of the B-Spline curve which is used.

And as we said I can always choose the value of  $k$  to define whatever the order is. In this entire curve, as I go from let's say the first point, starting point of the curve which is  $p_0$  to

the last point of the curve which is  $p_6$  during the entire curve you have a slope continuity. Every curve has a, we have discussed about the continuity like you have something called as if there is a curvature continuity you call as a  $C_2$  continuous.

If it is slope continuity you call it as  $C_1$  continuity. If all the, there is also a  $C_0$  continuity when all the points are connected with no breaks as far as the curves is concerned.  $k$  itself defines what is the continuity of the curve.  $k$  minus 2 is the continuity which we have. For example here  $k$  is equal to 3, so  $k$  minus 2 will be 1, so the curve is  $C_1$  continuity.  $C_1$  continuous means there is slope continuity during the entire curve. So I cannot expect curvature continuity in this particular curve. If I want curvature continuity then I will go and may be select  $k$  is equal to 4. Then I have a different definition and hence the curve is. So the most common option also which is used as far as B-Spline curves is concerned is also  $k$  is equal to 4. That is like gives you the terms which are maximum degree of polynomial which you have is something like  $u$  cube which is enough to give the curvature continuity.

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**BEZIER vs. B-SPLINE CURVE**

**Bezier Curve (n = 5)**

$$x = (1-u)^5 x_0 + 5 (1-u)^4 u x_1 + 10 (1-u)^3 u^2 x_2 + 10 (1-u)^2 u^3 x_3 + 5 (1-u) u^4 x_4 + u^5 x_5$$

$$0 \leq u \leq 1$$

**Non-Uniform B-Spline Curve (n = 5, k = 3)**

$$x(u) = N_{0,3}(u) x_0 + N_{1,3}(u) x_1 + N_{2,3}(u) x_2 + N_{3,3}(u) x_3 + N_{4,3}(u) x_4 + N_{5,3}(u) x_5$$

$$0 \leq u \leq 4$$

Now let's look at mathematical definition little more closely. Suppose if I choose let's say I am just taking an example of  $n$  is equal to 5. If I take  $n$  is equal to 5 that means I have 6 control points this is true with both Bezier as well as B-Spline. If I take let's say  $n$  is equal to 5, the definition of the Bezier curve which is obtained is defined, say completely defined because I have terms which are starting from 1 minus  $u$  to the power of 5. Next time I have 1 minus  $u$  to the power of 4 which is multiplied by  $u$ . So slowly the 1 minus  $u$ 's power decreases and  $u$ 's power increases from 0 to 5. And I have a binomial coefficients so this is 5  $C_0$  which is 1, 5  $C_1$  is 5, 5  $C_2$  is 10, 5  $C_3$  is also 10, 5  $C_4$  is 5 and then 5  $C_5$  is 1 and these are multiplied by the respective  $x$  coordinates. So this is completely defined between  $u_0$ ,  $u$  is equal to 0 to 1. Let's look at the definition for a non-

uniform B-Spline curve. Here not only I am giving the control points but I have to also supply what is the k which I have selected.

Let's say I have selected k is equal to 3 which gives me a slope continuity. Then we have seen in our earlier equation that I have the B-Spline basis function which is multiplied with the coordinates. So if I look at this, since k is equal to 3 this is  $N_{0,3}$   $N_{1,3}$   $N_{2,3}$   $N_{3,3}$  and it goes all the way up to  $N_{5,3}$  which are what you call as a B-Spline basis functions. we will see soon how these, what is the mathematical definition of these B-Spline basis functions but for the time being we can only know that, that you have to evaluate these B-Spline basis functions in order to define the curve completely. And u also we know that is n minus k plus 2, n here is 5, k is 3 so there are curve will consist of four segment 0 to 1, 1 to 2, 2 to 3 and 3 to 4.

Now what happens like when I give a definition like this? For example if I take u is equal to 0 to 1, there is one portion on the curve which has u is equal to 0 to 1. So what are the control points, how many control points will influence? Three control points. What are those?  $x_0$   $x_1$  and  $x_2$ , so naturally for the range u is equal to 0 to 1, all these B-Spline basis functions that is  $N_{3,3}$ ,  $N_{4,3}$ ,  $N_{5,3}$  will automatically become 0. Next time when I select u is equal to 1 to 2,  $N_{0,3}$  will be 0, and this is nonzero, nonzero and nonzero whereas  $N_{1,3}$  and  $N_{2,3}$  will be 0. So these N's what you call as a basis functions are defined in such a manner that the definition itself includes which will be 0 or which will be nonzero depending on the k value which is selected for a particular curve.

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**B-SPLINE CURVE**

**Non-Uniform B-Spline Curve (n = 5, k = 3)**

$$x = (1-u)^2 x_0 + 0.5 u (4-3u) x_1 + 0.5 u^2 x_2$$

$$0 \leq u < 1$$

$$x = 0.5 (2-u)^2 x_1 + 0.5 u (-2u^2 + 6u-3) x_2 + 0.5 (u-1)^2 x_3$$

$$1 \leq u < 2$$

$$x = 0.5 (3-u)^2 x_2 + 0.5 u (-2u^2 + 10u-11) x_3 + 0.5 (u-2)^2 x_4$$

$$2 \leq u < 3$$

$$x = 0.5 (4-u)^2 x_3 + 0.5 u (-3u^2 + 20u-32) x_4 + 0.5 (u-3)^2 x_5$$

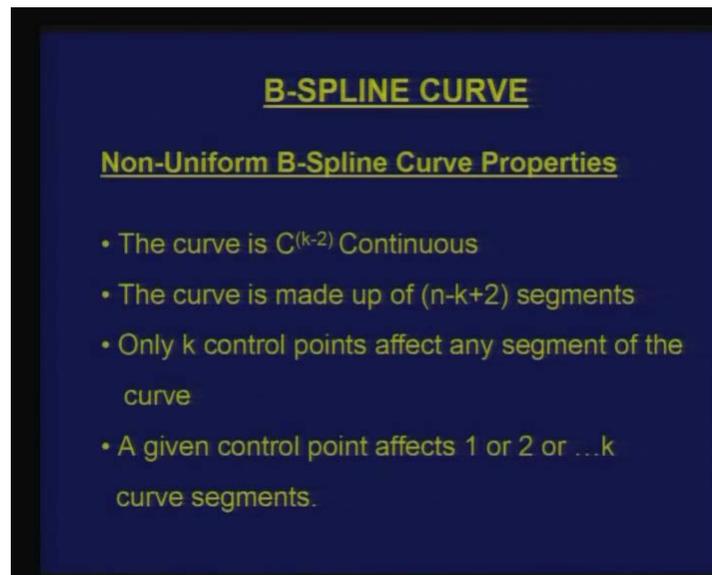
$$3 \leq u < 4$$

Now if I look at let's say, if I try to elaborate for example for the same definition which I have chosen for non-uniform B-Spline curve where N is equal to 5 and k is equal to 3, I have basically written down the equations the parametric equations for the individual segments. Earlier the definition was a common. Now if I see u is equal to 0 to 1, we know that three control points effect others are 0, so I can eliminate them and I have the

coefficients which are like basically the terms like  $1 - u^2$ ,  $0.5u$  etc. so the curve, the 4 segments of the curve which will be obtained can be directly written in terms of equations. These have been basically written based on the definition which will be which we will discuss for the  $N$  value. So I have the four segment 0 to 1, 1 to 2, 2 to 3 and 3 to 4 also tells which portion of the curve is influenced by which of the control points.

Now if I choose for example  $k$  is equal to 4, and then I will have a different equation which we will have terms  $x_0$ ,  $x_1$ ,  $x_2$  as well as  $x_3$  and all these values will change. So I have a different, you can say functions which will define when the  $k$  is equal to 4. And if I want to plot the curve also, suppose if I want to display the curve for display purpose I want to plot the curve or I have to take let's say a cutting tool along this particular curve, if I want to evaluate number of points on the curve then you need to use different equations for the different segments of the curve like when I code it as a program, I have to have a different cases if  $u$  is equal to 0 to 1 then there is a particular equation which is used. If it is  $u$  is equal to 1 to 2 then I have another equation which is defined. Now these equations are derived in such manner that you don't have to worry about the continuity, it is a part of a definition. So the slope continuity will be automatically met, if I go through this kind of a definition.

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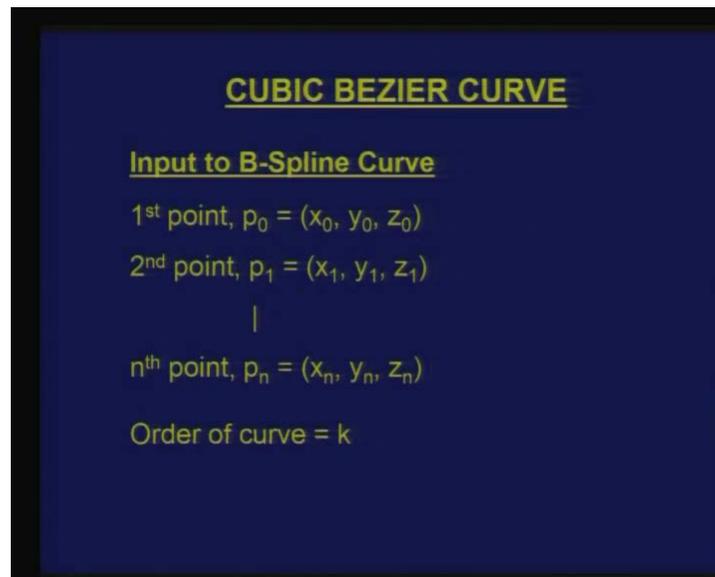


So let's look at some of the properties of this particular curve before going for mathematical definition. The curve is  $C^{k-2}$  continuous, this we have already seen. Whatever is the  $k$ , minus 2 will be the continuity requirements which will be there and the curve is made up of  $n - k + 2$  this also we have seen. Once I know the  $n$  and  $k$ , I know what the segments are.

And only  $k$  control points effect any segment of the curve that also we have seen and a given control point effects 1 2 or  $k$  curve segments depending on where they are located,

towards the ends like suppose if I have a B-Spline curve which is defined by 100 control points which is or let's say 101 control points 0 to 100, your n is equal to 100. What happens is the only influences that means only non-uniformity which you have is only for the first two segments and the last two segments because the first control point effects only the first one, second control point effects first and second. the third will effect the next three, so that continue still you reach the other end where I have again the non-uniformity which is coming into picture in between all the other control points are like have the same influence that means they would be influencing k curve segment if k is equal to 3. They will be influencing the three curve segments which they have. So that is another important you can say a property of this particular curve.

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Now what is the input which I will give in order to define a B-Spline curve is whatever are the control points just like we define the, when I define a Bezier curve this is the same if I have a cubic Bezier curve, I give four control points. here whatever are the n points which you are trying to define and you also define what is the order of the curve which is by giving the value of k which is an additional input given to a B-Spline curve.

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**B-SPLINE BASIS FUNCTION**

$$N_{i,k}(u) = \frac{(u-t_i) N_{i,k-1}(u)}{t_{i+k-1} - t_i} + \frac{(t_{i+k}-u) N_{i+1,k-1}(u)}{t_{i+k} - t_{i+1}}$$

$t_i$  ( $0 \leq i \leq n+k$ ) are called knot values

$t_i = 0$  if  $i < k$ ;  
 $t_i = i - k + 1$  if  $k \leq i \leq n$ ;  
 $t_i = n - k + 2$  if  $i > n$

$N_{i,k}(u) = 1$  if  $t_i \leq u \leq t_{i+1}$   
 $= 0$  otherwise

Now let's look at how are these basis functions are defined. The expression looks little complicated but it is not as complicated as it looks. Any particular value of a basis function  $N_{i,k}$  is given by a mathematical expression which is this. Suppose we have taken earlier an example of  $n$  is equal to 5 and  $k$  is equal to 3. So what are the  $N$  values which I need to evaluate? So I need  $N_{0,3}$ , I need  $N_{1,3}$ , I need  $N_{2,3}$ ,  $N_{4,3}$  all the way up to  $N_{5,3}$ . So I need to evaluate 6  $N$  values in order to completely define the curve. If suppose if I select let's say  $k$  is equal to 4 then it is different. I would be interested in getting let's say  $N_{0,4}$ ,  $N_{1,4}$ ,  $N_{2,4}$  which are different functions.

So given let's say a value of a  $k$  that means which is an input and the number of points I know what are the  $N$  values which I have to find out. And in order to find out let's say a value of  $N_{i,k}$ , I have a expression. This finding  $N_{i,k}$  requires finding the value of  $N_{i,k-1}$ . Suppose if I take for example  $N_{0,3}$ ,  $k$  is equal to 3 if I want to find out  $N_{0,3}$  then I should know what is  $N_{0,2}$ . That should be evaluated first so this is a recursive function. In order to evaluate this I require this, in order to evaluate this I may require some other function. So this is a recursive function and we know that a recursive functions can be very easily implemented particularly when it comes to a programming. So that is one reason you can say that B-Spline curve is a purely you can say evolved out of the availability of computers.

Otherwise the mathematical calculation or manual calculations would have been more difficult for different situations. And another thing which uses is all these are functions of  $u$ , this is a function of  $u$  and you also see that there is another term which is coming in the equation which is  $t$ 's. These  $t$ 's are called as a basically the not vectors which are shown here and the number of  $t$ 's which one has to evaluate is 0 to  $n$  plus  $k$  like if I take an example of  $n$  is equal to 5 and  $k$  is equal to 3 which i have we have been using, so I have to evaluate all the  $t$  values first which goes from 0 to 8 and that means I have  $t_0$   $t_1$   $t_2$   $t_3$  all the way up to  $t_8$ . And in order to define the  $t$ 's I have three expressions whether a given

value of  $i$  is less than  $k$ , I have one definition. If it is between  $k$  and  $n$  I have another definition and if it is greater than  $n$  then I have a third expression. So  $n$  is an input,  $k$  is an input. Given the  $n$  and  $k$  I can always calculate what the  $t_i$ 's are using an expression like this. So this is not a difficult thing. So once I know the  $t_i$ 's, whenever a  $t_i$  is required I can always use that in an expression which is there.

Now whenever I am using evaluating this recursive expression, as I am going let's say one down below like in order to evaluate let's say  $N_{0,3}$  I require  $N_{0,2}$  and similarly at some stage I should also know where the recursion should terminate. So in order to terminate you have an equation  $k$ . So this is basically the equation which is used to define or to terminate this recursive function that means if  $N_{i,k}$  is equal to 1 if it is between this otherwise it is 0. Now the whole definition is given in such a manner like we were discussing about what we call as a non-uniformity of B-Spline. In order to define, make it non-uniform the definition becomes little more complex because you have to choose the knot vectors or the knot vectors have to be defined in such a manner that the influence is different at the ends than those at the middle. In order to basically do that we have a situation or we end up with an expression which is this is basically given by DeBoor's person, so this is also called as a DeBoor's expression for a B-Spline which is this thing.

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**B-SPLINE BASIS FUNCTION**

$n = 5$  and  $k = 3$

$t_i$  ( $0 \leq i \leq 8$ ) are knot values

$t_i = \{0, 0, 0, 1, 2, 3, 4, 4, 4\}$

$N_{0,3}(u) = (1-u)^2 N_{2,1}(u)$

$N_{1,3}(u) = 0.5 u (4-3u) N_{2,1}(u) + 0.5 (2-u)^2 N_{3,1}(u)$

Now let's just look at a simple example. For example, if I have  $n$  is equal to 5 and  $k$  is equal to 3 which is this thing. So we know that the knot values go from 0 to  $n$  plus  $k$ , plus  $k$  here is 8. If I substitute in the previous expression which we have seen, I can obtain all the knot values which are this. If you see here these are zeros at the ends because  $k$  is equal to 3 and these are 4 values, all are fours the other three end points and in between you have it is increasing in the ascending order. So the knot vector automatically comes up in an order which is required for a non-uniform B-Spline curve definition. And if I go back and try to evaluate those recursive expressions, I can always

say suppose if I want my first expression  $N_{0,3}$  then I should know what is  $N_{2,1}$  like for example this automatically becomes 0 for that particular thing.

And you have what is called as  $N_{2,1}$  which is required which is what is shown here. So I need to evaluate  $N_{2,1}$  again going back recursively. If I want to evaluate  $N_{1,3}$  then I have to evaluate  $N_{2,1}$  and  $N_{3,1}$  which are the expressions. So given this mathematical definition it's possible to completely define the B-Spline curve satisfying all the requirements which we have seen in the properties which is there. Now I will not go into derivation of this expression, it is actually its quite lengthy. May be i require about three classes to just derive this particular expression, if I want to go for this.

And most of the CAD/CAM books also they don't do it. If you probably look into books like theory of splines then probably you may find basically a derivation for this but the definition basically evolves out of the properties which we have been discussing. So this is just an example of how to evaluate a B-Spline curve and one can, a better way to do is to write a program for a B-Spline curve rather than sitting and doing the calculations. if you really see why this is also required suppose if I am plotting a B-Spline curve, I have to evaluate let's say if I am plotting a curve between let's say  $u$  is equal to 2 and  $u$  is equal to 3, I may take small increments of  $u$  and for every increment I may evaluate the expression. So it requires lot of manual calculation, if I don't use let's say programming in order to plot or use this.

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**NURBS CURVE**

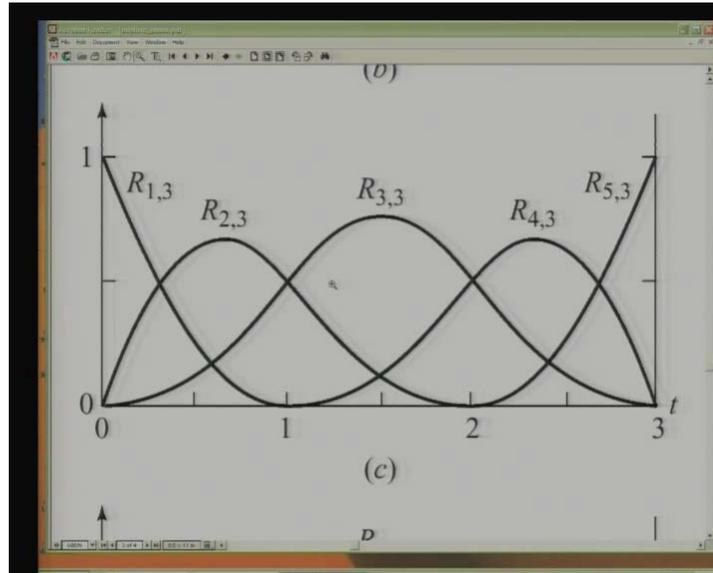
**Non-Uniform B-Spline Curve**

$$x(u) = \sum_{i=0}^{i=n} N_{i,k}(u) x_i \quad 0 \leq u \leq n-k+2$$

$$x(u) = \frac{\sum_{i=0}^{i=n} h_i N_{i,k}(u) x_i}{\sum_{i=0}^{i=n} h_i N_{i,k}(u)}$$

Now before let's say I go to the next part of the curve, let me just go back and look at the basis functions like we have been discussing about the basic functions which we have used, what we call as  $N_{0,3}$  or  $N_{1,3}$  or  $N_{2,3}$  which we have been using, the same things have been plotted.

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Don't worry about R, R is another this thing variable which is used to define this. Let's look at the first basis function that is how does this look. so what is shown here which goes from let's say which goes all the way up to a value which is  $u$  is equal to 1 up to here is the first basis function. the second basis function which is shown here is this which goes between the parameters range  $u$  is equal to 0 to 2 and then this is the a third basis function which you are seeing here. Now how do I interpret like these are the basis functions which are actually plotted?

Now how do I interpret here is suppose if I take  $u$  is equal to 0, if I take  $u$  is equal to 0 then what the value is. I have only this which is nonzero, only that means the entire thing is defined by the first basis function which becomes value equal to 1 that means curve passes through the first point. Now if I take let's say any value between  $u$  is equal to 0, let's say I take  $u$  is equal to 0.5 what are the curve segments which are influencing. So I have 3 which are coming into this thing and if I really add up all of them like this value which is shown here and this value and the three values they all are equal to one. Like, if you remember we have seen this property as far as Bezier functions is concerned, if I add up all the basis function values at any particular  $u$  value, they add to one the same property is valid for B-Splines too.

At any particular 0.5, if you see that I have 1 2 3 which are influencing so that means the corresponding  $x$  values are multiplied by a factor which is different at different values of  $u$ . Once I go to let's say closer to  $u$  is equal to 1, let's say if I take  $u$  is equal to 0.9 what you see is that the influence of this is become much less whereas the influence of other functions have increased. And I have a value which is like basically the weightages of these values which are used  $x_0 x_1 x_2$  is changing continuously as I go from one end to another.

Let's say I come back to somewhere here, what happens. Some of the earlier functions have no influence like the first basis function which I am using has no influence on this, second also has no influence. Only thing is the values like third, fourth and fifth have some influence when it comes to here. As I move the same curve continuously, the same curve continues to the other end. Here you have only taken a few points that means  $n$  is which has been taken is only for a few points, you see like this otherwise the curve continuously moves along this. So what you see here this one, the middle one which, what is the significance of the middle one is that this is basically influencing the three segments. Isn't it? What are those?  $u$  is equal to 0 to 1,  $u$  is equal to 1 to 2 and  $u$  is equal to 2 to 3 whereas if I look at this an earlier segment it is influencing only two and this one is influencing only one. So this particular function which is influencing the three continuously repeat.

Suppose if I take let's say 100 points, how does the like all the basis functions look is that first and the next one will be different. All other will be a uniform functions till I reach the other end where I have again which are basically a mirror of this. So I have these functions, this is basically mirror of this; this is some kind of a mirror. This is actually symmetric about the  $u$  and this function actually repeats.

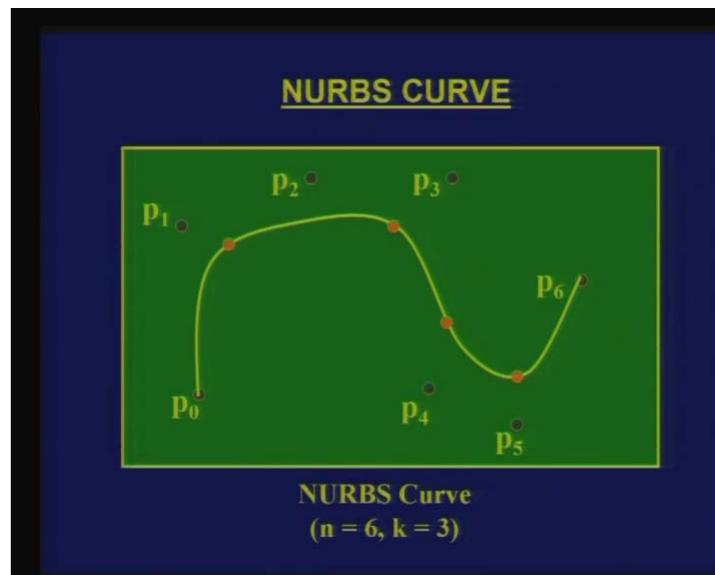
If I use, if suppose if I take let's say B-Spline definition where I forget about these end basis functions and use only the common B-Spline basis function which is defined in the middle that is the common which is repeated for most of the points. If I choose that and try to define a B-Spline curve that is called as a uniform B-Spline curve like, we define we said there is a non-uniform B-Spline curve that is because if I look at these functions they are different at the ends compare to the middle.

Middle one is suppose if the middle is repeated for all the cases from let's say starting point to the end point. Take the same function as this then I also get a curve which is uniform B-Spline curve and in that case the curve doesn't pass through the starting and end points. So I get a curve which start somewhere and end somewhere doesn't pass through the thing. So since it doesn't pass through usually the starting and end point passing through a specific point is a more common requirement for design and manufacturing applications. People like non-uniform B-Spline version is more commonly used than the uniform one.

Now let's come back to this. If I have understood B-Spline curves then I have understood NURBS too. What is put here is basically a definition of the B-Spline curve which we have been discussing all the time. So I have 0 to  $N$ , I know how to evaluate  $N_i$  values using a recursive function and I have the values which are  $x_i$  and I also know what is the range. Now in NURBS curves is basically a modification, rational version of a B-Spline curve. So the NURBS actually stands for non-uniform rational B-Splines. You know what non-uniform B-Splines are. So if I make it rational, I have a non-uniform rational B-Spline version. What is done in this is if I compare this expression with this, I have the same  $N_{i,k}$  basis function which is used here too. Same points are also used; you are actually multiplying with another scalar value a number.

So how many h values are there? As many control points, so every control point is given a weightage which is basically what you call as a h. So if I give a different weightages presently in a non-uniform B-Spline curve, what you do is that you give equal weightage to all the points but suppose if I want to influence, I want that let's say third control point should have maximum influence on the curve. I will give a very high value for the associate a weight with this particular control point and multiply by them. What you see here is basically a fraction of that in the sense suppose if I am giving let's say weightages like 1 2 3 4. Then you are actually dividing one with the summation of all the 1 2 3 4 so you have a larger fraction which is coming up as a part of this. Now if give all the h values are one then this is nothing but a non-uniform B-Spline curve. If I give different values of h then you have non-uniform rational B-Spline curve which comes.

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Now what is the effect of that? suppose instead of giving let's say weightage of one which is to that, suppose I give let's say weightage of 1 here and weightage of 100 to the second control point then the curve will be pulled more towards that particular point. So curve instead of let's say being away from this, you will see that the curve has very like it has, it is passing through a point which is very close to this particular thing. If I give let's say weightage as a infinity, the curve must pass through that particular point two it can be made to do that. So by influencing one can one of the ways to basically say this is like you have a something like a spring which is attached to this. You are only varying let's say stiffness of that particular spring. Stiffer the spring the curve will be pulled more towards that particular point. If I have a very stiff spring then it passes through almost close to that particular point which is a case with that.

Now what is the basic let's say a beauty of this particular curve is we can see later that if I want to we have seen the example of approximating let's say a Bezier curve approximating a circular arc using a Bezier curve it's not possible to do that. Whereas if I take a NURBS and vary the weightages I can make, I can define NURBS curve which is

an exact we can say there is a, which can be exactly used to model a circular arc. Not only circular arc it can be used to model a straight line when the points are collinear, when it can be made to model conics, parabola, hyperbola and ellipse by changing the weightages. Since, and we know that B-Spline is a subset of this. If I choose  $k$  is equal to  $n$ , I can also reduce B-Spline curve to a Bezier curve.

Since all the curves which we have and we know that any of the Bezier curves can be represented as a Hermit curve and accordingly that also we have seen. So a NURBS curve is enough to represent most of the other versions because of its, you can say a universality or it is some kind of a superset of all the curves. For most of the CAD CAM applications this has been selected as a standard curve like when the standards were developed for representing curves and surfaces, if different people use different forms you may have a problems. So if everybody uses let's say a NURBS form, I have some kind of a universality so that is the reason why NURBS is so popular which is shown here.

So what we will see next is basically take up this NURBS curve and try to see how this can be used to represent other curves and also extend it to surfaces. The same definitions which we have seen the Hermit curves, B-Spline and the NURBS can be extended to the surfaces which is very trivial in fact, you don't require much of discussion as far as this.

So I will just stop it here and if you have any questions ask me. No here, additional weightage is by giving additional weightages. You are actually getting a family of curves. For example if I take a B-Spline curve, if I give the control points and the value of  $k$  the curve is uniquely defined. there is no multiplicity as far as that is concerned but if I take for example NURBS by default by giving the weightages as same that is one then you are getting the B-Spline curve but if I by changing the weightages, I am getting a different curve so you are actually representing a family of curves using the NURBS.

And this variation can be used to represent let's say class of curves like there are many situations where I want a constrain. For example I want to define a B-Spline curve with a given control points and also  $k$  is defined but I want this curve to pass through a specific point. How do you do that? Once I give a control points and  $k$  the curve cannot be made to pass through unless it is by default it is passing through. But now if I have a weightages, I will try to manipulate these weightages such that the curve can be made to pass through this. So even if I have additional constrains, I can impose and satisfy more requirements which are used. So that is why when we say that it can be used to represent or it can be used to represent most of the known forms like circle etc. That is because you are making use of, making use of this option which is given by the weightages of a NURBS curve. Any other question? I think the best way to learn NURBS curve is to write a program to plot it. So if I have written let's say a program to plot a NURBS curve that means I have understood most of it. Otherwise theoretically it looks simple but once you start writing a program then you will know that it requires quite a lengthy program in order to do that. Then I will stop it here and we will take up some aspects of NURBS curve followed by the surfaces in our next lecture.