Computer Aided Design

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Lecture No. # 32

Modelling of Curves (Contd.)

Last time we are talking of Bezier curves and we have mentioned that these curves are approximating curves which means that if we have a set of control points, the curve will be approximated by a curve between these points.

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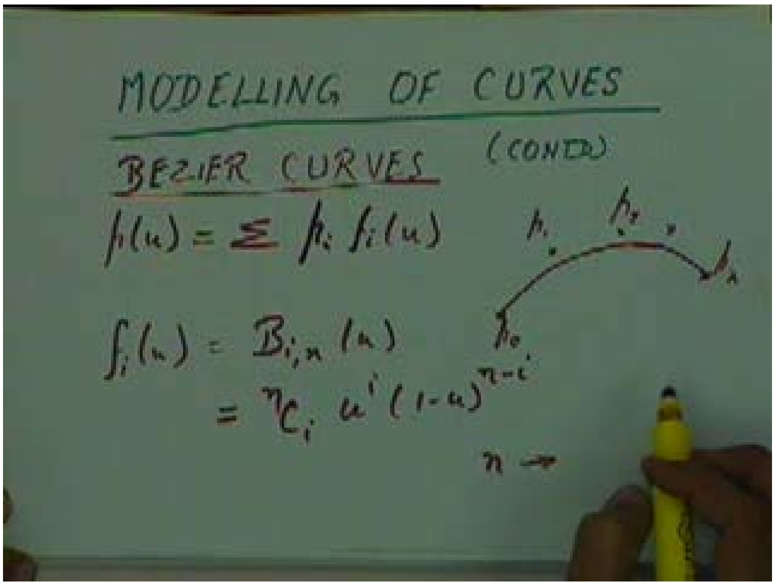

It doesn't interpolate through these points and it's definition, we said is given by this expression where for the case of Bezier curves, these blending functions are essentially given as a Bernstein polynomial and these Bernstein polynomial are given in this form. That means there are combination of u to the power i and 1 minus u to the power i with the coefficient which is a combinatorial term and for these curves n is referred to as a degree of the curve and the total number of points ranges from or the points ranges from p_0 p_1 to p_n that means the total number of points is n plus 1.

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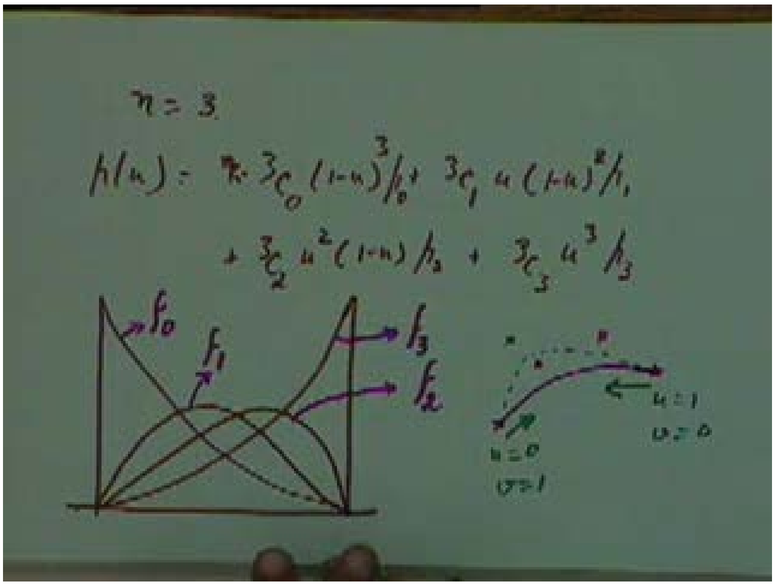

So if we define a cubic curve Bezier curve, for the cubic Bezier curve we say n will be equal to 3 and a curve will be defined as nc, n is say 3 c_0 multiplied by 1 minus u to the power 3 plus 3 c_1, this is multiplied by p_0 plus 3 c_1 multiplied by u multiplied by 1 minus u square multiplied by p_1 plus 3 c_2 multiplied by u square multiplied by 1 minus u into p_2 plus 3 c_3 multiplied by u cube multiplied by p_3. This is for the case where n is equal to 3 and for this case we had drawn the blending functions. These blending functions have a shape like this and where this curve corresponds to f_0, this curve corresponds to f_1, this one corresponds to f_2 and this one corresponds to f_3. These are the 4 blending functions for the points p_0 p_1 p_2 and p_3.

So from these 4 blending functions, the first thing that we observe is that none of these blending functions are 0 at any point, except possibly at the starting or at the end point. So if we have a set of points in this case 4 points, the effect of none of these points will be 0 throughout the curve. So even the effect of this point will be nonzero right throughout which means that I said with these 4 control points, if my curve was defined like this and I move this point to this location, so my curve now will get modified. It will get pulled towards this point and it possibly get modified something like this but the modification will be there right from the beginning of the curve up to the end of the curve. The complete curve will change. So that is one property of this type of curve.

We will see other types of curves were this is not to true. So that is one thing that you observe about these curves. The other thing we can make out from these blending functions is that they are symmetrical with respect to u and 1 minus u. That means if I parameterize from this direction or from this direction, that means in one case let say u equal to 0 and let say u equal to 1 and in other case I defined a parameter let say v equal to 0 here and v equal to 1. My curve will still remain the same whether I define a point in this order or in this order, the same curve is defined. And that you can make out because f_0 and f_3 are symmetric and similarly f_1 and f_2 are also symmetric.

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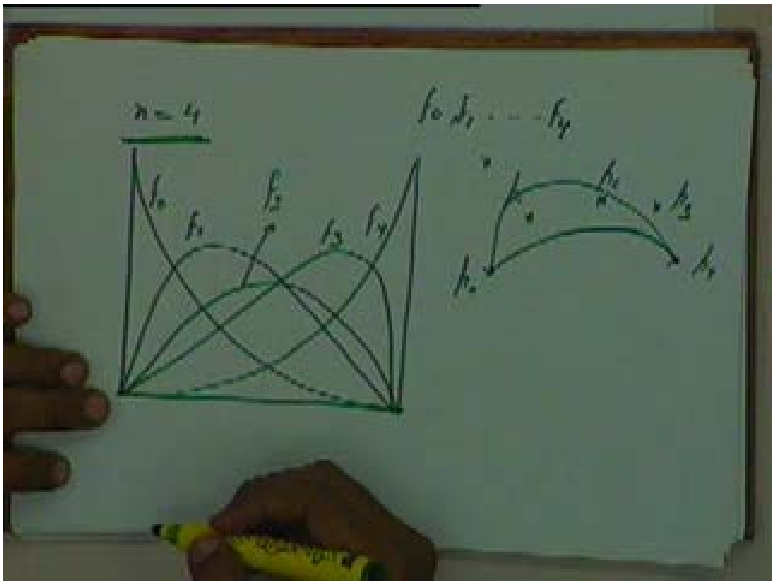

Now if we see some higher order curves, higher order Bezier curves let say if I take the case where n is equal to 4. So, for n is equal to 4, my blending functions will be f_0 to f_4. So I will have 4 blending functions, sorry 5 blending functions and 5 control points. That means if the degree of the curve is 4, my points will be 5 and the shape of these functions would look something like this. This is f_0 and this is f_4, again f_0 and f_4 will be symmetric and if we get the other functions let say this is f_1 and this is f_3. F_1 and f_3 will be symmetric with respect to the center and f_2 will be like this and if this curve is f_2, this is how we will get a curve for degree 4. Again similarly we can define higher order, high degree curves but in all the cases these properties that I have mentioned they will hold true.

Firstly all these blending functions will be symmetric with respect to the center, symmetric in the sense f_0 will correspond to f_n, f_1 will correspond to f_n minus 1 and so on. Secondly, none of these blending functions will be 0 anywhere except at the end points which mean that if I move any point the complete curve will get affected. So if I move this point from here to here, initially if my curve looks like this then my curve will get modified to this. So this is the case for Bezier curves were blending functions are defined in this manner.

Now one thing I have already mentioned that if I move one control point, my curve gets pulled towards it or moves towards that point.

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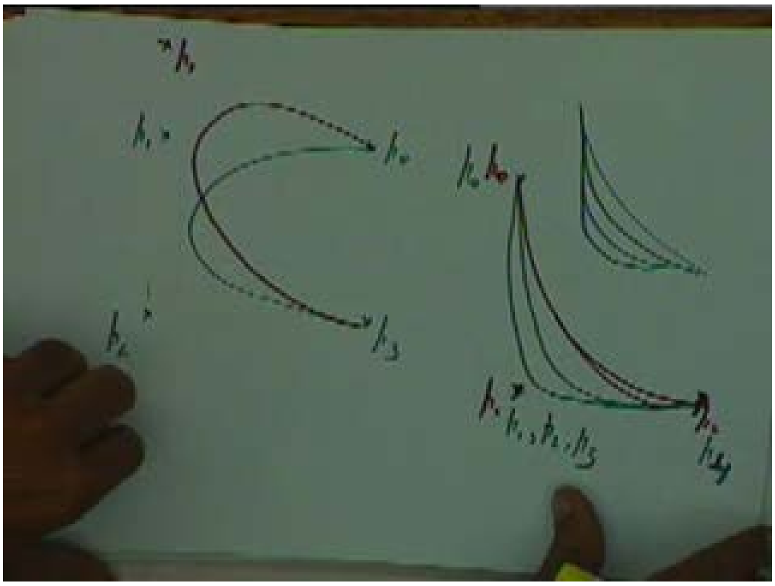

So if I take a simple case, let say these are my 4 points and p_0 p_1 p_2 and p_3. My curve probably looks something like this. Now if I move let say one of the points, let say from here I change it to this place. My curve will get pull towards that point and it will get modified like this. This basically gives us a control over the curve by pulling the curve towards each of the points. So this is p_1, p_1 has been modified to this. In other thing that happens is that let say if I start with 3 points this is p_0, this is p_1 and this is p_2.

My second degree curve will be defined which would probably look like this. Now instead my second degree curve if I define a third degree curve that means now I have to have 4 points p_0 p_1 p_2 and p_3 and I have multiple points at the same place. And in the second point is also at the same place that means this becomes this is p_1, so this is let say p_0, this is p_1, this is also p_2 and this is p_3. This means that now my curve will become closer to this point and will take a shape something like this.

Essentially what I am doing is for the same point I am giving it a greater weightage and this way I can have multiple points at the same place. Let say if I say p_1 p_2 p_3 are here and this is p_4. My curve will get pulled further to this point, this way my curve can become sharper at this point. Initially I have a very smooth curve then may be it became something like this, like this and possibly like this and so on. This way I want to retain the degree of the curve and still want to move points closer to the particular point. This is another way of controlling the shape of the curve. The basic idea is that these Bezier curves give us a good interactive medium for defining these curves, for defining curves in general, for modelling curves. We can define a curve through a set of points and then control the curve by moving the control points.

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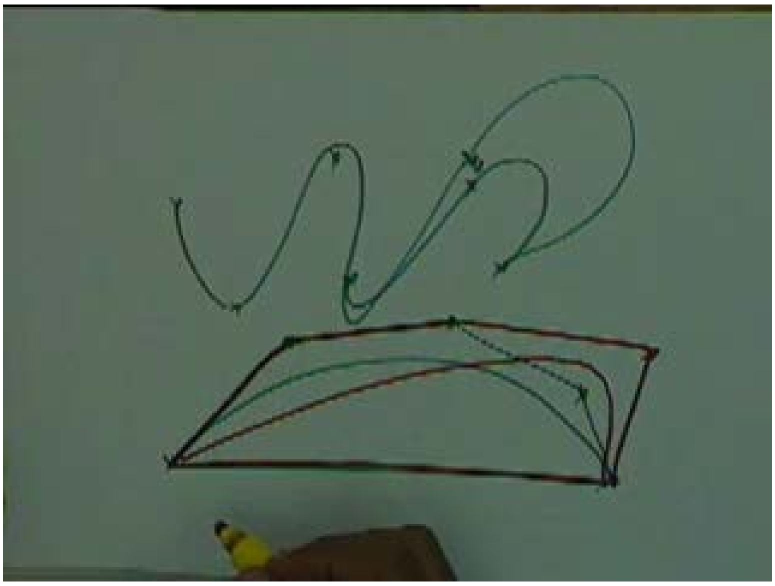

We don’t necessarily have to have the curve passing through them. I mentioned last time that if my curve has to pass through all the points, often what can happen is that let say if I have a set of control points like this, my curve can become something like this. So if I let say, if I move this point slightly, my curve will get modified to something like this. That means the curve can oscillate wildly but in Bezier curves that does not happen because we are just only pulling the points, pulling the curves closer to a particular point. The curve will never pass through that point. And the other property of these curves is that if I consider all these control points, if I have a set of control points like this and I draw a polygon including all these points, this is a polygon of the control points. So my curve will always be contained within this polygon, it will never go outside this polygon.

My curve, it would be something like this. If I move this point to a different location let say move it here then my curve has to remain within this polygon. My curve will get modified something like this, so it is always bounded by the polygon of the control points. Now it is a polygon by a convex polygon formed by the control points.

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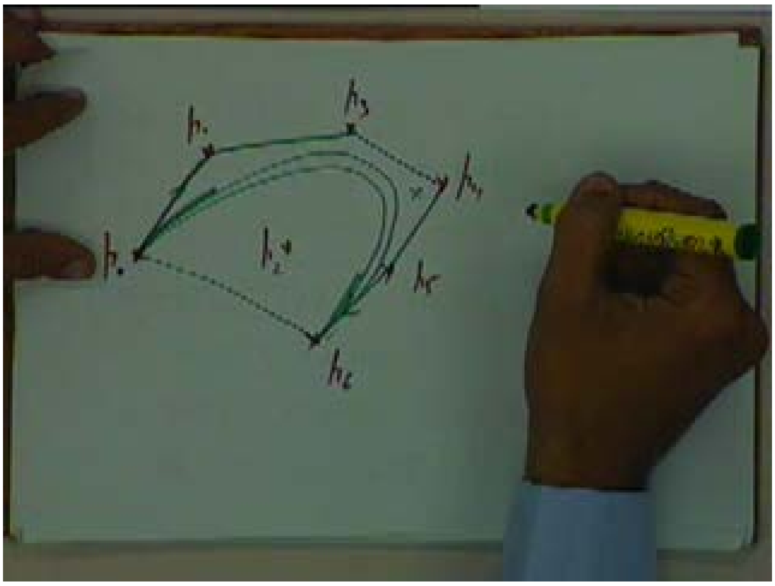

For instance if I have a set of points which are given in this order, this was p_0 p_1 let say this is p_2, this is p_3, p_4, p_5 and p_6. In this case the curve will be bounded within a convex polygon enclosing it which would be like this. The curve will always remain within this polygon. This can be shown mathematically.

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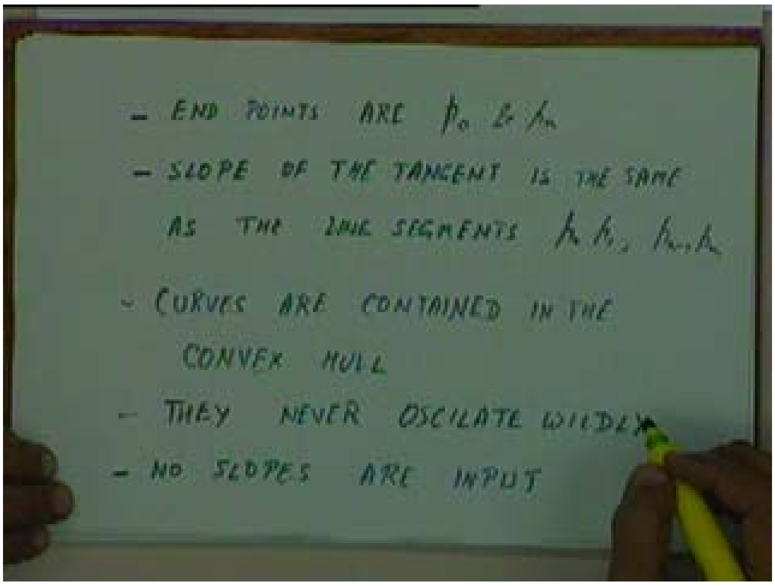

So, some of the properties for these Bezier curves, the first property that we have seen is that the end points of the curve are the p_0 and p_n. Then next property is that the slope of the tangent or the tangent at the end point is the same as the line segments p_0, p_1 and p_{n-1}, p_n. The slope of the tangent at the end points is same as that of the line segments p_0, p_1, p_{n-1}, p_n. That means if I have set of points, let's say if this is my set of points, at the starting point the slope will always be in this direction. The pink this will be the tangent, at the end point the slope will always be in this direction that means it will be like this. My curve will come like this then the curves are contained in the convex hull.

The convex hull is the set of convex points, the set of points is defined a convex polygon around this control point. This is what I have drawn in two dimensions right now. If all these points are 3 dimensional then again we can define a convex 3 polyhedral which will consist of faces and the curve will still remain within that. You think of a ball with faces on the side, so if I have set of control points and I define a let say polygonal ball which defines a convex polygon or a convex polyhedral around those control points. So, I will get a convex polyhedral ball like this. My curve will always remain within this ball. If curves are contained within the convex hull then they never oscillate wildly. If you change some points, the curve will never oscillate wildly. So let say if I change one point from here to here, my change will be perceivable by the pull that is given by this point.

So if initially my curve will like this, it might get modified to something like this. It will never oscillate wildly by changing some points that does not happen and the other thing in this case no slopes are input. The slopes are implicit that means we have only inputting the data points, only inputting these data points. The slopes are implicit that the slope at this point is in this direction is implicit in this. So I move this point, the tangent vector will also change. We are not inputting any slope at any point. In the case of pc curves at least in the one of the formation we were inputting the slope. So these are some of the properties of these Bezier curves.

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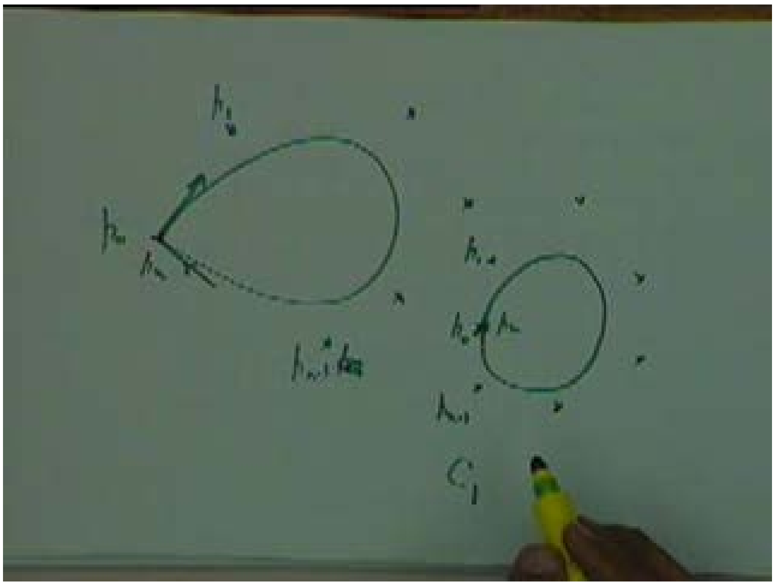

Now if we consider curves like this, we are given a set of points. Let's look at a setup points now. This is p_0, this is p_n. If p_0 and p_n are the same that means my last point is here itself p_x, in that case we will get a closed curve which will look something like this. At this point the tangent will be in this direction. Here the tangent will be in this direction, this is p_{n-1}, this is p_1 and I am making p_0 and p_n the same, we can get a closed curve. Now if I make p_1 p_0 and p_{n-1} sorry p_1, p_0, p_n and p_{n-1}, so if I make these 4 points collinear, I wouldn't get a discontinuity here. We will get a straight, we will get a continuous smooth curve. That means I say this is p_0, this is p_1 and this is p_{n-1} and this is p_n. My curve would now look like this. So at this starting point and the end point, we will have c_1 continuity.

Similarly we can define geometric conditions to get c_2 continuity and so on. There will be condition between these 3 points and these 3 points. So, this is with respect to the closed curves. Any questions up to this point with respect to Bezier curves?

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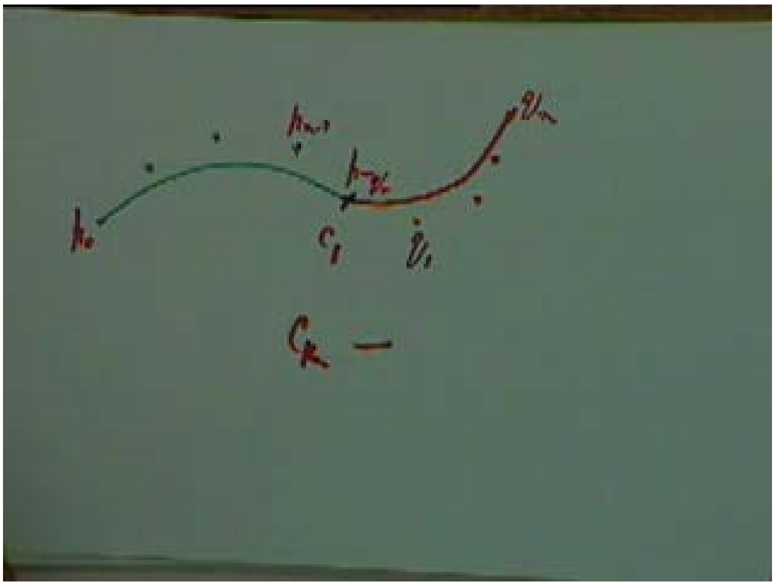

The only thing that remains now is that what I mentioned earlier and if you want to define composite curves, so if I have a set of points like this, the curve between these is like this. If I want to have a second curve starting from there, let's say my second curve would look like this. In order to ensure continuity over here, if I want to ensure c_1 continuity this point, this point and this point should be collinear. This is p_0, this is p_n, this p_{n-1} and this is let say q_0, this is q_1 and this is q_n. If p_{n-1}, p_n, q_0 and q_1 if they are all collinear then my curve will have c_1 continuity here. Again I can define a condition between these 3 points and these 3 points to ensure c_2 continuity. So c_k continuity will be defined by k points from the p curve and k point from the q curve.

It is basically these properties which enable once or which gives or which makes the Bezier curves suitable for interactive curve modeling. So we want to model these curves interactively, we get lot of flexibility using these features. Anything with respect to Bezier curves? Then we will go into next type of curves which are called B-spline curves.

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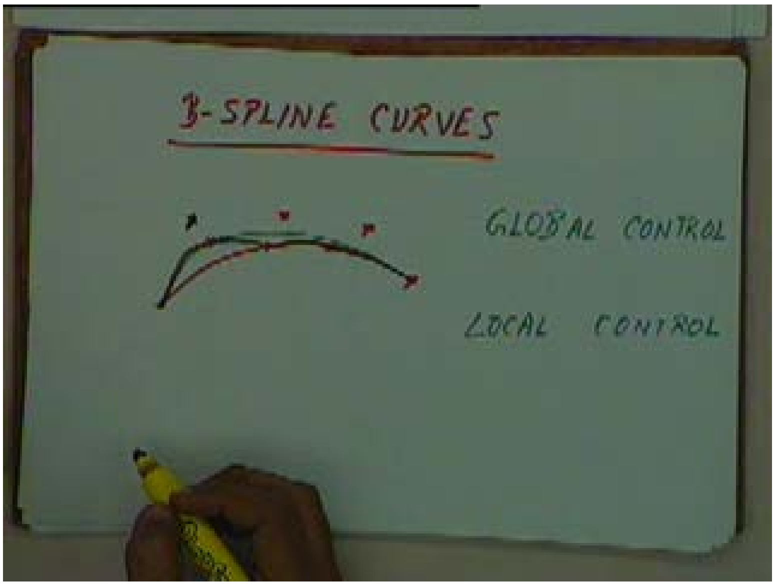

I mention that in the Bezier curves if we have a set of points, I get a curve which is defined within the connection of these points. Now if I move one of these points, my curve will get pulled towards that point but the change will occur throughout the curve. As basically because the blending functions corresponding to this point is not zero anywhere. So it will have an effect at all points of the curve. This is, this property is called global control. If we change, if we move one point the complete curve will get changed. The control offered by this point is global in nature. In contrast to this, what one likes to have is local control. In local what you would like is that if I move this point here, may be my curve will get modified only let's say from here to here.

So my curve will now look like this, the rest of the curve should remain the same. The effect of this point should only be within a certain range beyond that the curve should not get effected.

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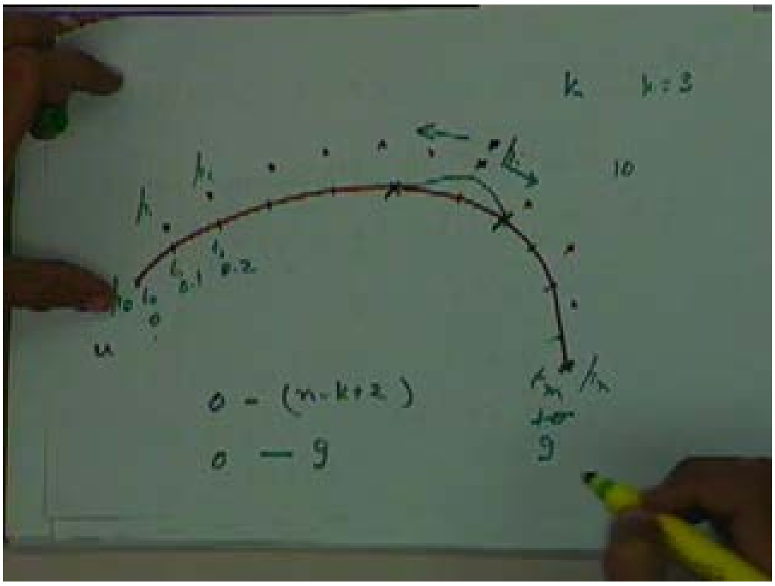

What that essentially means is that if you have a lot of points, I have defined a curve between these points which goes like this. If I want to make some change in this portion, if I move this point here I do not want my complete curve to get changed. I only want that the curve in this range should get modified like this. So if I want the local control then these Bezier curves are not suitable. That is why we go and define B-spline curves and we will see later on that B-splines curves are generalization of the Bezier curves or Bezier curves are specific case of the B-spline curves.

So for defining these B-spline curves, what we will do is in order to define and let's say this is my point p_i. What range or in what portion of the curve is this point p_i effective, in order to define that we will define a certain set of marks on this. And these marks let's say we call them... So this is p_0, p_1, p_2, this is p_n. We will define a setup naught value let's say, let's call them t, t_0, t_1, t_2 and so on. Let say this final is called t_m, m need not be equal to n. We will define a set of naught values and we said that this point p_i that will have its control within some naught value here and some naught value here. Beyond this, this point will not have any effect.

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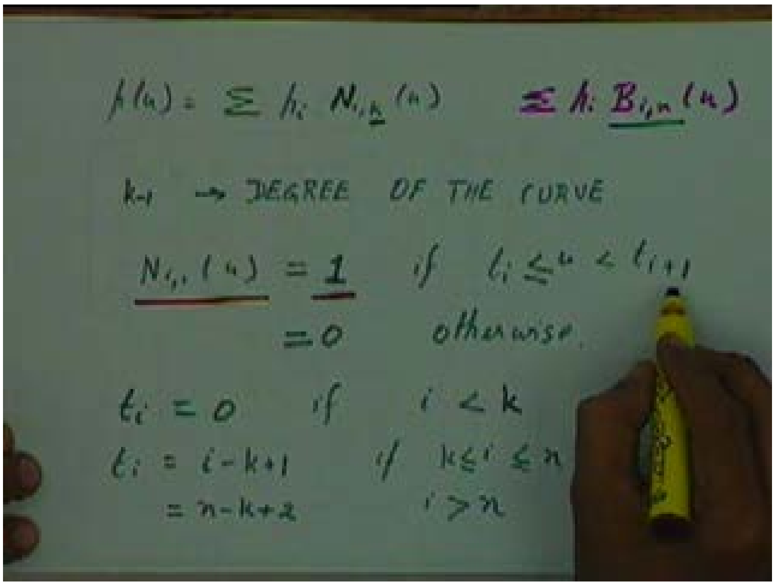

And in order to mathematically define these kind of curves, we will say firstly p of u is, we will say will be equal to sum of all p_i's multiplied by $N_{i,k}$ of u. If you remember what we did in the case of Bezier curves, this was sigma p_i multiplied by the Bernstein polynomial which is i, n of u that means this Bernstein polynomial in the case of Bezier curves dependent on n and it dependent on i that means for different points, we have a different blending function and that blending function dependent on n and we say n was a degree of the Bezier curve.

Now this blending function $N_{i,k}$ doesn't depend on n, it depends on k where the k depends on the degree of the curve. If this is k, we say k minus 1 is called the degree of the curve. So if you want to define curves with degree 2 and we will say k is equal to 3 and these blending functions will of course depends on every point but will depend on k and not on n. What that means is that let's say these are the different naught values I have. The blending functions for this point, it does not depend on n, even if I have hundreds of more points, the blending functions for this will not depend on that. It will depend only on the degree of the curve which depends on k and that will control the range in both the directions, the value of this naught value and the value of this naught value and the kind of control between this.

If a k is 3, I will have a second order control over here. If a k is 4 that is I am talking of third degrees curves and I will have third control between these region but beyond this, there will be no affect irrespective of the number of points. So if we define these blending functions, now these functions we will say $N_{i,1}$ of u will be defined as 1. If u is between t_i and t_{i+1} where t_i have different naught values. I will give a definition for these curves and then we will see where the definition slightly complex mathematically and then we will see how the curves get defined using this definition and equal to 0 otherwise. So this is the definition of $N_{i,1}$. The t's are defined in terms of the parameter, I will just give the definition of t's also.

So $N_{i,1}$ will be 1 if u is within this range of values, t_i's are basically the naught values, these naught values $t_{0,}$ $t_{1,}$ t_2 and so on. It has a t_0 is equal to 0, t_1 is equal to 0.1, t_2 is 0.2 and t_m is 1.0 something like that. These naught values are defined in terms of this parameter.

So if my u is between t_i and t_{i+1} we define $N_{i,1}$ to be equal to 1. Mind you I am defining $N_{i,1}$ and not $N_{i,k}$ right now irrespective of the degrees of the curve k, k_{-1}, the definition of $N_{i,1}$ is like this. The definition of the naught values we will say t_i is equal to 0, if i is less than k. And we will say t_i is equal to i minus k plus 1, if i is between k and n and it will be equal to n minus k plus 2, if i is greater than n and for the time being this curve, this value of this parameter is 0, values of parameter of u will not vary from 0 to 1 but it will vary from 0 to n minus k plus 2. Let's say if I have 10 points and k is equal to 3 then the last naught value or the range of this parameter will be from 0 to 10 minus 3, 7 plus 2 is 9. So here my parameter is 0, here my parameter won't be 1 but it will be 9.

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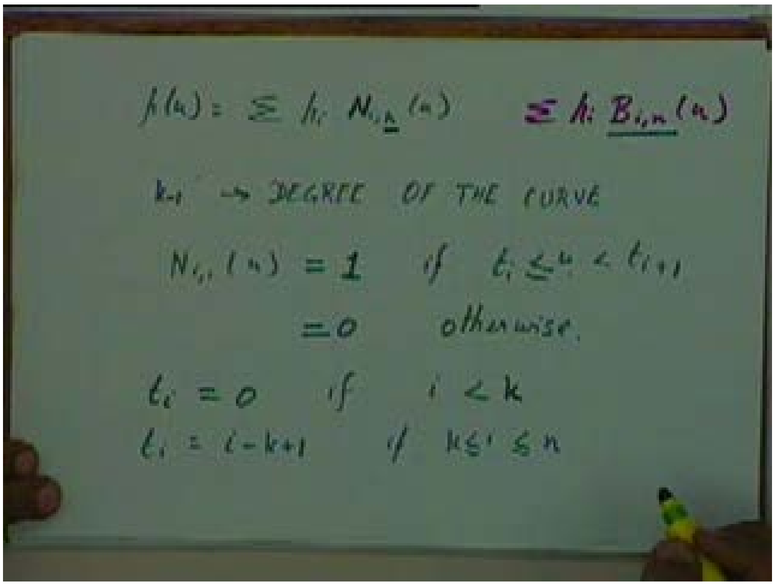

So the parameter will say u will vary between 0 and n minus k plus 2 and in this range of u will… again? Student: every point of the curve doesn't have unique q (Refer Slide Time: 00:31:53 min) every point of the curve will have a unique q. What that means is that $t_{0,}$ t_1 and may be t_2 will all be coincident at this point. So t_0 is equal to t_1 is equal to t_2 which is equal to 0, so the multiple naughts defined at the same point. Similarly multiple naughts are defined at this point. Again we want symmetric definition. I have multiple points but the u value for this point is unique. So we will define a naughts in this manner, t_i will be 0 if i is less than k. It will be i minus k plus 1, if i is between k and n and it will be n minus k plus 2 if i is greater than n. Let's take a simple example and see how these naught values will get defined. Let's say if I take n equal to 5 and k equal to 1 that means I am talking of zero degree curves.

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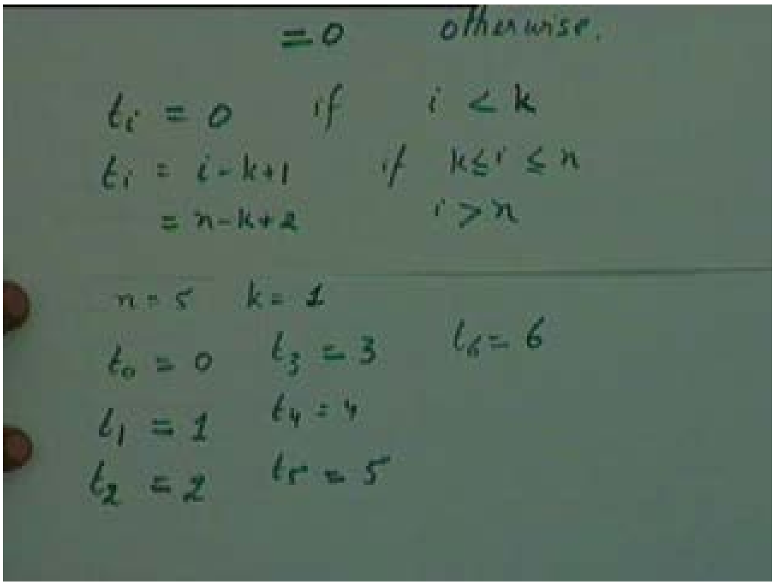

So let's starts with t_0. Now t_0 if i is less than k, it will be equal to 0, for t_0 is equal to 0. t_1, it doesn't come in this, it will come in this range, it will be i minus k plus 1, i is 1, k is 1, so this will equal to 1, t_2 will again come in this range so i is 2, k is 1, so this will become 2, t_3 will be 3, t_4 will be 4, t_5 will again come in this region because n is equal to 5 so that will be equal to 5 minus 1 plus 1, so it will be equal to 5. And t_6 will be equal to n minus k plus 2. So n minus k plus 2 will give us 5 plus 2, 7 minus 1 which is 6. So for k equal to 1 and naughts will get defined as 0 1 2 3 4 5 6 which means n equal to 5 that means I have 6 control points.

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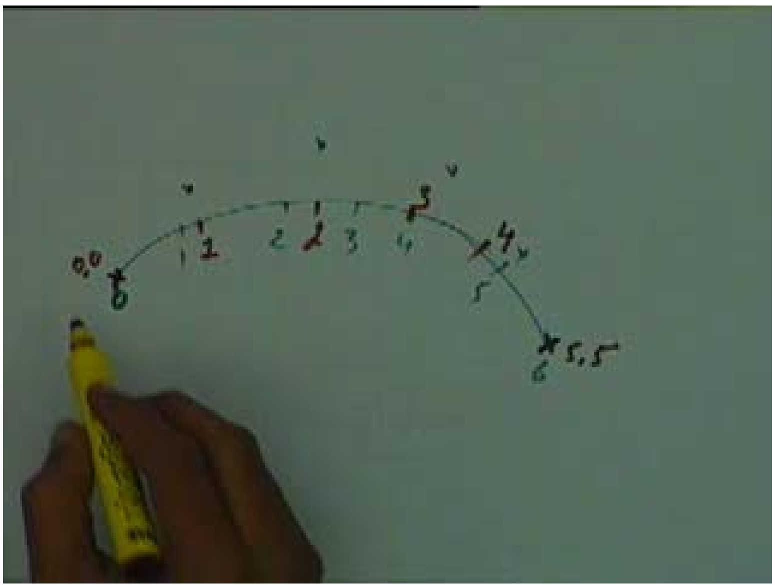

Whatever be the curves that we derive between these points, it will be defined as let say from 0 to 6 and my naughts will be something like this, 1 2 3 4 5 and 6 and the control of each point will be defined in terms of these naughts. We will just see how that is defined, will come back to that, so this is how the naughts are defined.

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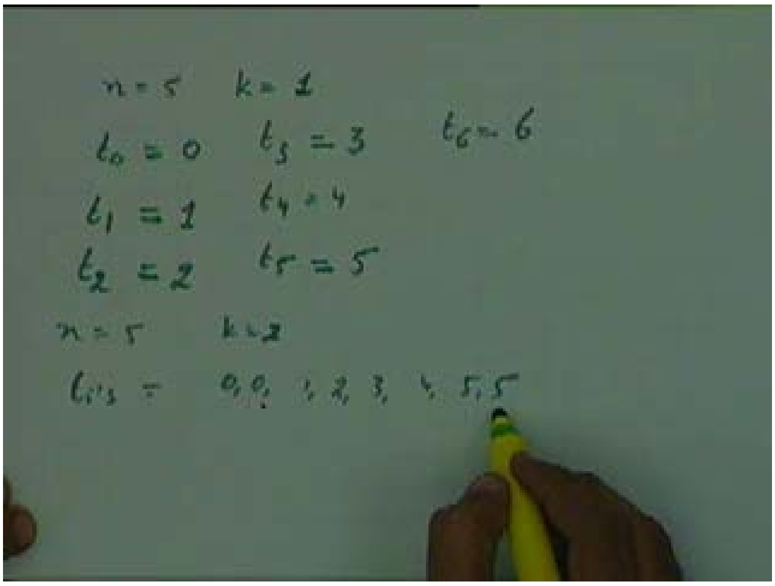

If I take let say n equal to 5 and k equal to 2, in that case my naughts values will get changed and we will get t_i's, we will get them 0 0 1 2 3 4 5 and 5. That means they are multiple naughts at 0 and multiple naughts at 5. That means that this curve, now my naughts I will have 2 naughts here so that will 0 and 0 1 2 3 4 and again two naughts here which is 5 and 5, this is 1 2 3 and 4, this is how my naughts will get defined. If I take a higher degree that means if I take k equal to 3, I will get 3 dots here and 3 naughts here and a number of naughts in between will go down. So after I mean define the naughts, we look at the definition of the B-spline curve and in this blending function $N_{i,1}$ I said is 1, if u is between t_i and t_{i+1} and is 0 otherwise. So we have only defining $N_{i,1}$ and yet to define $N_{i,k}$. So we will define $N_{i.k}$ again recursively in terms of lower values of the parameter.

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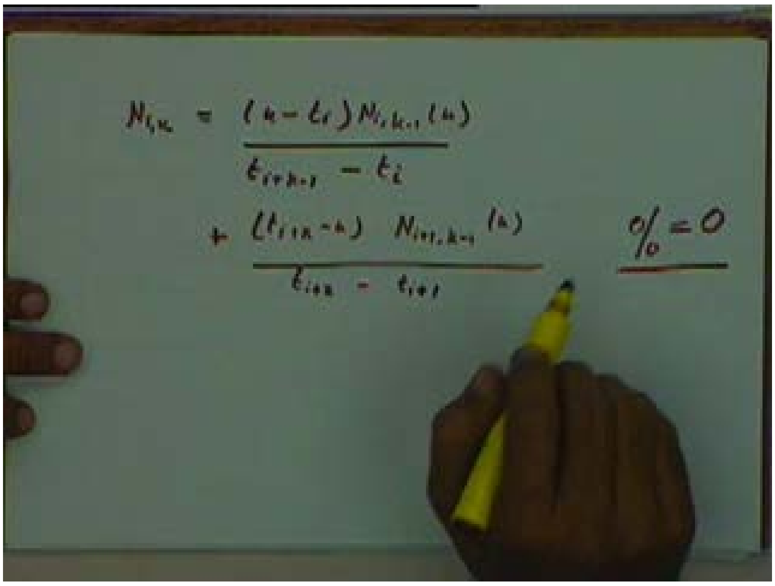

we will say $N_{i,k}$, we say this is equal to u minus t_i multiplied by $N_{i,\ k-1}$ of u divided by t_{i+k-1} minus t_i plus t_i plus k minus u times $N_{i+1,\ k+1}$ of u divided by t_{i+k} minus t_{i+1}. So this is how we will define $N_{i,k}$ in terms of $N_{i,k-1}$ and $N_{i+1,k-1}$ and since t_i and t_{i+k-1} can be the same. As we have seen some of the naught values are the same, so this denominator can become 0 in some cases. So if I am taking the limits we will say 0 by 0, whenever we have a term of this type we will take that to be 0. We will just see, we leave this because u minus t_i and t_{i+k-1} minus t_i both of them can become zero sometimes. So whenever that happens we will take this term to be equal to 0, that doesn't happen in this definition so that has been defined in that manner. Actually this is basically recursive function which has been derived in a certain manner. Now this formula is also bit complex so just I will see what effect this formula has on different kinds of B-spline curves. Let's start with the case, the first case that I have taken for the naught values that is where n is equal to 5 and k is equal to 1.

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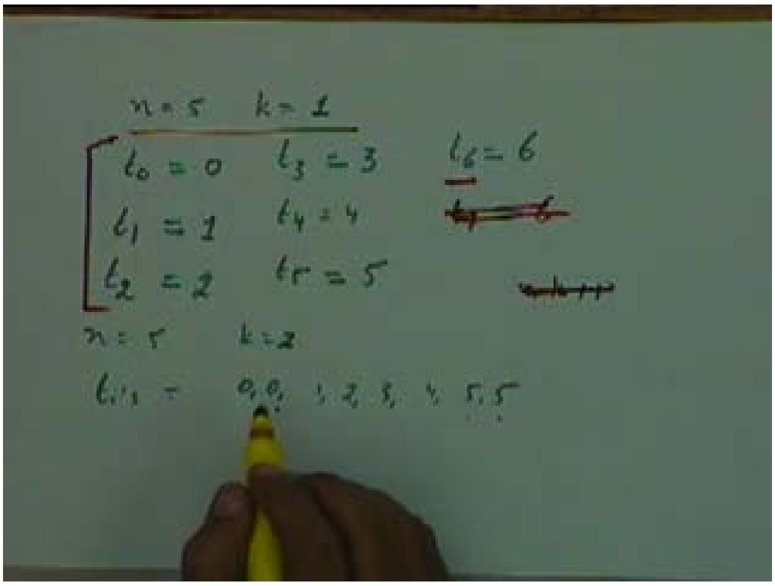

So when is equal to 5 and k is equal to 1, my t_i's are defined in this manner and for the definition of the B-spline curve, I have mentioned $N_{i,1}$ of u is 1, if u is in this range and is equal to 0 otherwise. So if i… Student: where we define this t's from t_0 to t_6… Only from t_0 to t_6 because we need only those values, we will just see that. In this case it comes to upto n minus k plus 1, I think, I will just confirm it. The number of naughts that have to be defined. Student: after that all of them are the same. now after that you won't need that. See because what happens is, just it. See these blending functions are defined for the particular point and the control of every point will be defined in a range upto k_0 from either side. So of the last point, we will define on k naughts from this side and k naughts from this side, there is no point defining naughts beyond that. Do you get that?

If I take let say p_i, the control of this will be between t_{i+k} and t_{i-k}. Yeah, between i plus k and i minus k, so there is no point defining t_i is beyond n plus k, t_{n+k} is the last naught that you will get, there is no point defining naughts beyond that. We can define multiple naughts in the same point that doesn't make any difference. If we look at the definition, even if you want to define t_7 that will be equal to 6. We can define multiple points but you don't get anything than that. We will just see you how that happens. So we have defined naught values that we have defined and for k equal to 1, we are interested in defining $N_{0,1}$, $N_{1,1}$, $N_{2,1}$, $N_{3,1}$, $N_{4,1}$ and $N_{5,1}$.

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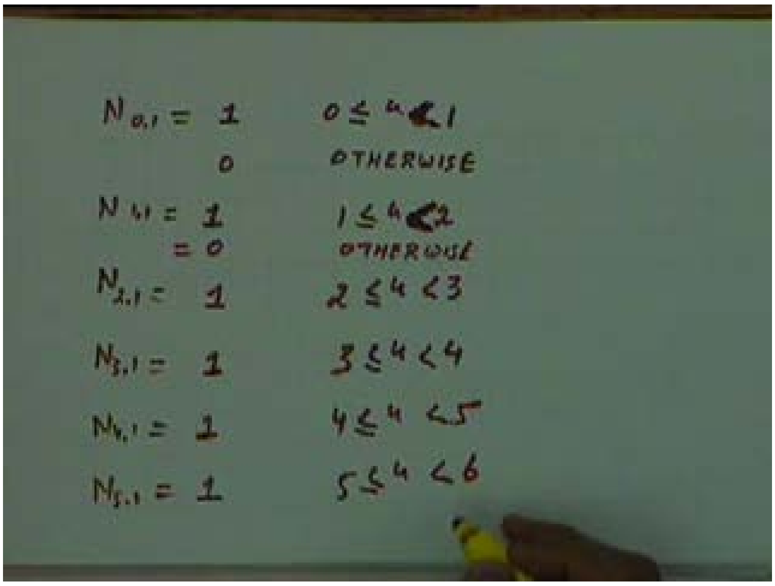

We are interested in defining these 6 values where k is equal to 1. If you look at this, $N_{i,1}$ is 1 if u will be between t_i and t_{i+1}. So $N_{0,1}$ will be 1 if u is between t_0 and t_1. T_0 is 0, t_1 is 1, so this will be equal to 1 when u is between 0 and 1, it will be equal to 0 otherwise. $N_{1,1}$, if you look at this $N_{1,1}$ will be 1 if u is between t_1 and t_2. From this t_1 and t_2 are 1 and t_2, so it will be equal to 1 and when u is between 1 and 2, sorry equality only on this side, the equality not on both sides. So we have only this, it will be equal to 0 otherwise. Similarly $N_{2,1}$ will be 1 between 2 and 3 and 3,1 will be 1 between 3 and 4 and 4, 1 will be 1 between 4 and 5 and $N_{5,1}$ will be 1 between 5 and 6. Now the last blending function we have to define is $N_{5,}$1. You don't need to define blending function beyond this. So, the last naught value that we need in this is going to be n plus 1, so there is no point defining the seven after this. So this is how the blending function get defined.

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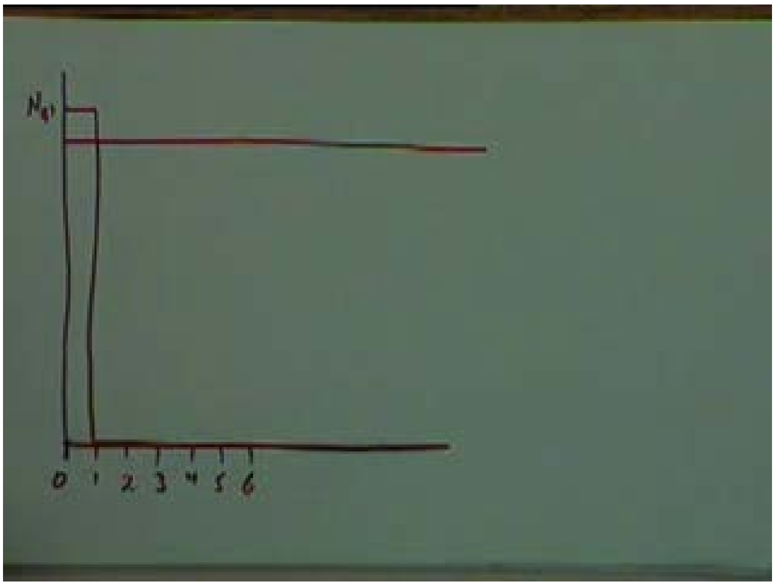

if I plot these blending functions, on the x axis I draw my u values 0 1 2 3 4 5 and 6 and let say start with $N_{0,1}$, so $N_{0, 1}$ is 1 between 0 and 1, so it is 1 here and is 0 elsewhere. This is $N_{0, 1}$ actually anyway, this is $N_{1,0}$, this is level of 1, this is a level of 0. I think we will draw the figure.

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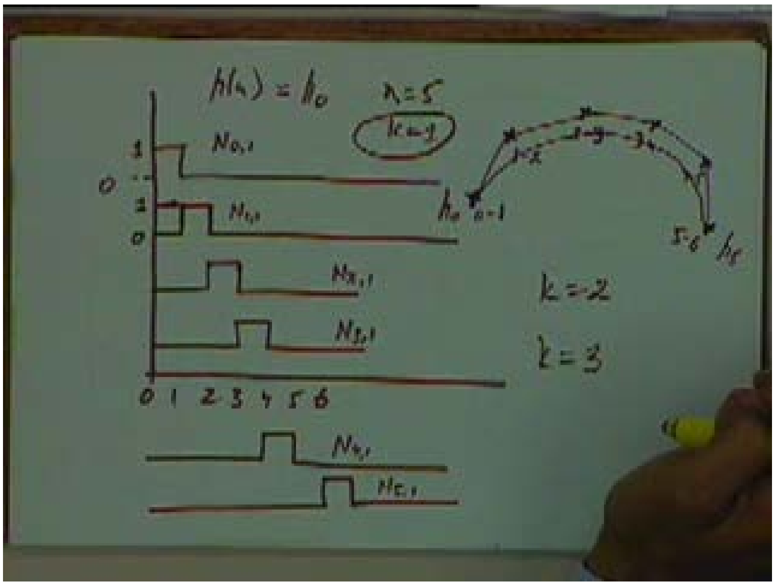

This is a plot for $N_{0,1}$, this is 1 and this level is 0. Similarly if I take up $N_{1,1}$, $N_{1,1}$ is 1 between 1 and 2, so $N_{1,1}$ would look something like sorry, this is a level of 1, this is a level of 0, this would be $N_{1,1}$. This is 1 between 1 and 2. Similarly $N_{2,1}$ would look like this, $N_{3,1}$ would like this, so I have to draw it here $N_{4,1}$ would look like this and finally $N_{5,1}$ would look this. $N_{1,1}$, this is $N_{2,1}$, this is $N_{3,1}$, $N_{4,1}$ and $N_{5,1}$. Effect of point 0 is only between this range, effect of point 1 is between this range, effect of point 2 is between this range and so on. So if I have a setup points from p_0 to p_5 how does my curve look like? These are different blending functions. Anyone? Between 0 and 1 only one point is effective, so between 0 and 1 what will be the value of the curve? p of u between, where u is between 0 and 1 will be equal to t_0 between 0 and 1 only one point has any effect. So between 0 and 1, p of u will be equal to p_0 that means we will have a point.

Similarly between 1 and 2 only one point will be effective which will be p_1. So from 0 to 1 we have only one point here, from 1 to 2 we have only one point here, u from 2 to 3 will be here, 3 to 4 will be here, 4 to 5 will be here and 5 to 6 will be here. So in this case when n is equal to 5 and k is equal to 1, we get a discontinuous curve which is just a set of points. We are not getting a continuous curves in this case. In fact even if I increase the value of n, it won't have any effect because all my blending functions will have the same shape, there will be a shift after this. So for the case 1, when k is equal to 1 that means curve of degree is 0, we get a discontinuous curve.

In fact we will see in next time and if I take k equal to 2, I will get a set of lines like this and if I take k equal to 3, I will get a curve like this. So when k is equal to 2, throughout the curve we will have c_0 continuity. At k equal to 1 we have a discontinuous curve, at k equal to 3 we will have c_1 continuity and so on. We will see these details in the next class.