Computer Aided Design

Dr. Anoop Chawla
Department of Mechanical Engineering
Indian Institute of Technology, Delhi

Lecture No. # 21

1D-FE Problems: Galerkin's Approach

Today we will see the Galerkin's approach to the one dimensional finite element problems and in the Galerkin's method, we have already seen that this is the equation that we get for the virtual work done. That is this part is the virtual work done against the internal forces and this is the external virtual work done and this two will be equal in magnitude and the total virtual work done will be 0. This we have already seen.

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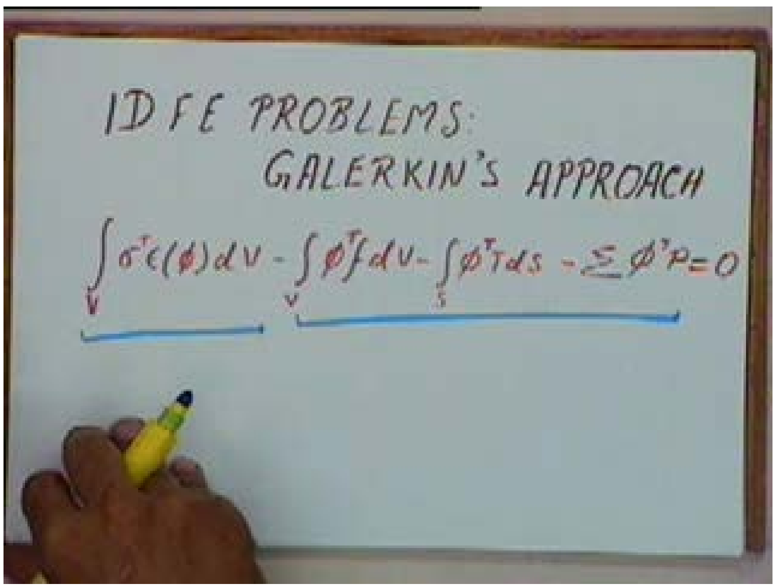

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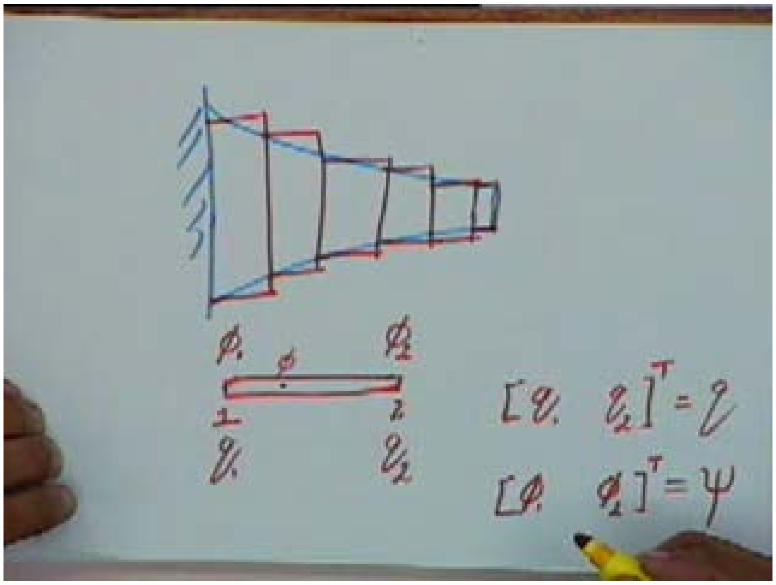

Now if we talk of one dimensional problems, we formulated one dimensional problem like this that if we take a body like this, we are splitting it into a set of 1 D elements like this. And if you take any individual element of uniform cross section having two nodes 1 and 2, we said that this has the displacement given by q_1 q_2 or the element displacement vector we said was q_1 q_2 which we are representing as the vector q or the matrix q.

Now in this Galerkin's approach we are giving it a virtual displacement. So if we give this a virtual displacement, let's say the virtual displacement at this end is phi_1 and the virtual displacement at this end is phi_2. And we will make the virtual displacement vector just like this which will be phi_1, phi_2 transpose and we will call that let's say the matrix psi. If you use this and we want to find out the virtual displacement at any point inside this element, if I take any arbitrary point here, if I want to find out the actual displacement we have written the equations earlier we said u will be equal to N times q where N is nothing but N_1 N_2 where N_1 and N_2 are the shape functions.

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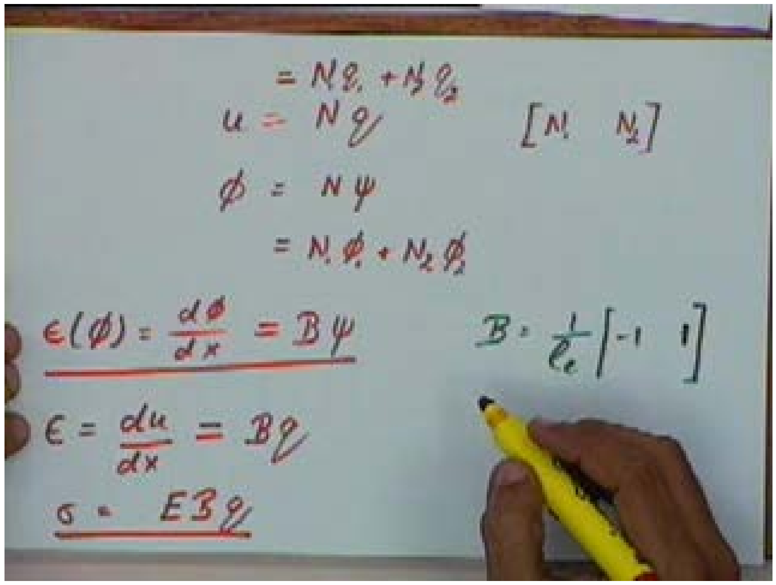

So we have said u is equal to N times q. Similarly, if you want to find out the virtual displacement at any point inside this element, let's say the virtual displacement here is phi. We will again say that this phi will be a linear combination of phi$_1$ and phi$_2$. So we will say phi will be equal to N times psi or this is same as N_1 phi$_1$ plus N_2 phi$_2$. This u equal to Nq that is the same as N_1 q_1 plus N_2 q_2. So we will say phi will be equal to N time's psi. The next step if you notice, the first term that we have here has a term of epsilon of phi and we said that this epsilon, this strain is the strain that will be caused by the virtual displacement that we are giving. So this strain caused by the virtual displacement will be nothing but d phi by dx. That is if you look at the actual strain, this I am saying is epsilon of phi.

If I look at the actual strain epsilon that is equal to du by dx and this we have derived last time, if I differentiate this equation I will say this will be equal to B times q. Similarly if I differentiate this phi equal to N time psi, I will get this will be equal to B times psi and this derivation is straight forward. All that I have to do is differentiate N_1 and N_2 with respect to x and this matrix B is the matrix given by this expression. This matrix B is the same as the matrix here that we had and the derivation for this is also absolutely the same. So epsilon of phi will be equal to B time's psi and epsilon is equal to B time's q and if we say epsilon is equal to B times q, we can say sigma will be equal to E times B times q where E is the Young's modulus, B is the element strain displacement matrix and q is the element displacement matrix. So we have an expression for sigma and we have an expression for epsilon of phi. If I take these two expressions, I can put them into the first term that is this term which is which corresponds to the internal work done.

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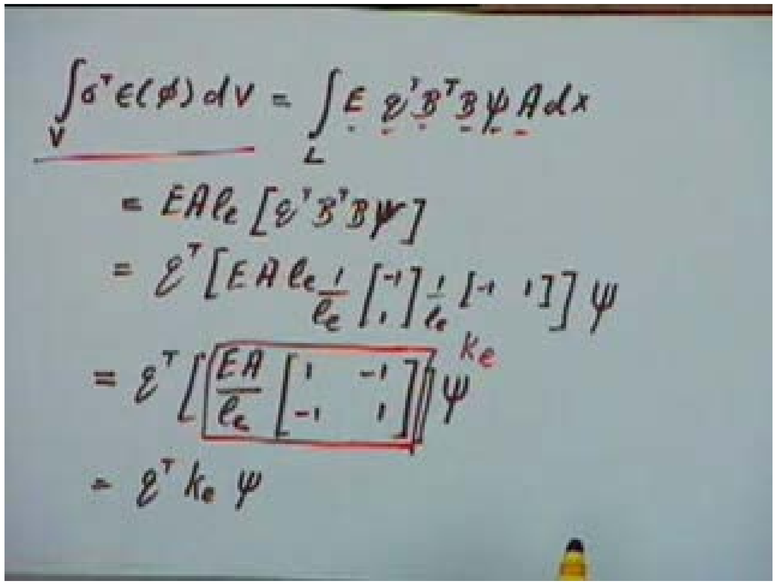

And if I expand this term for the internal work done, my first term here which is the volume integral of sigma transpose epsilon of phi dv, this will be equal to sigma is equal to Ebq. So sigma transpose will be equal to q transpose B transpose multiplied by E which is a constant anyway, it's not a matrix. So I will put that over here, so this will be equal to the integral of E q transpose B transpose multiplied by epsilon of phi. Epsilon of phi is B time's psi, so I will put that over here. B times psi, dv which is the differential volume element that will be equal to area time's dx, so it will be area of the element multiplied by dx.

Now this integral will be integral over the length of the element. And now in this you will notice that E is a constant, q is a constant B transpose B and psi and A they are all constants. They don't change with x and dx, the integral of dx is nothing but the length of the element. So if I put all these values here, this will be equal to E into A multiplied by integral of dx which will give me l_e multiplied by q transpose B transpose B psi. Now again B transpose, B is this expression, so B transpose will be 1 over l_e into minus 1 1, so I will put these values over here. This will be equal to, I will get q transpose into EA l_e into 1 over l_e multiplied by minus 1 1 into 1 over l_e multiplied by minus 1 1 and this whole thing will be multiplied by psi. This will be equal to q transpose into EA over l_e, I have l_e square in the denominator, l_e in the numerator and the product of these two matrices will giving 1 minus 1 minus 1 1. So this thing will be again multiplied by psi. And now this expression is the same as the expression for k_e which is the stiffness matrix. So this will be equal to q transpose into k_e into psi. Sir, what is q and what is psi. What is q and what is psi? Student: This psi we have defined as phi_1 phi_2 virtual display. So q is, q is the actual displacement.

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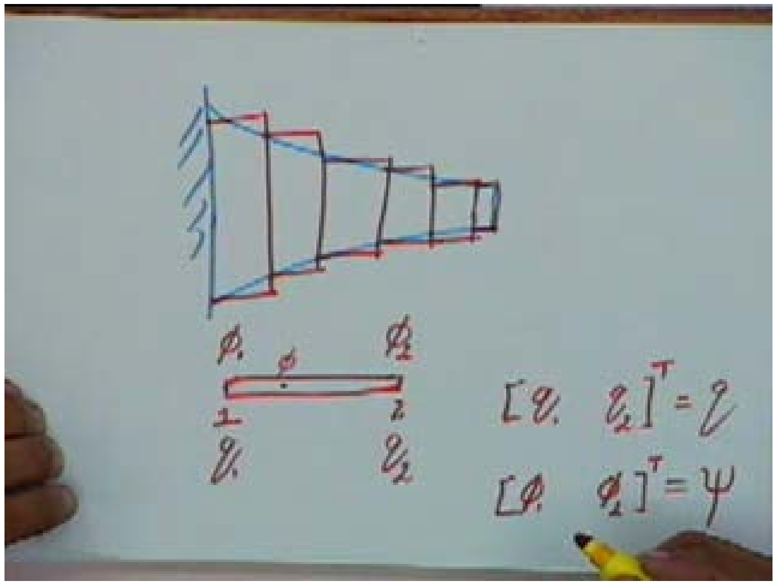

See this is my element, at this point one the actual displacement is q_1, at this point two the actual displacement is q_2. In addition to that I am giving a virtual displacement this phi_1 and phi_2. q_1 q_2 give me a vector which is q, phi_1 phi_2 give me a vector which I am calling a psi. Student: but the strains we have found out using psi. See using psi I followed what, I am calling epsilon of phi, the strains which will be caused only due to the virtual displacement. The actual strains because of the loading, their epsilon which is nothing but du by dx. The strains caused due to the virtual displacement are d phi by dx.

So when I give it a virtual displacement, the effort involved in giving the virtual displacement will be doing some work against the internal stresses. Though internal stresses are actually the stresses caused by the loading. So that internal work done is this term. That is why I have a term of the actual stresses multiplied by the strain caused due to the virtual displacement and the product of the two integrated over the volume will give me the work done. So this internal work done is equal to q transpose k_e into psi. This is the internal work done within the element. Any question up to this point?

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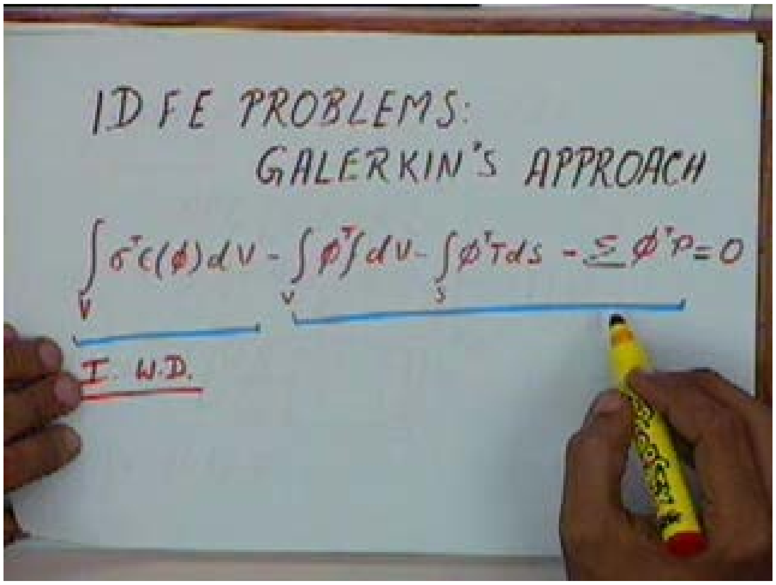

Now if you look at this expression for the Galerkin's method, the first term we have simplified. Similarly you will simplify the other terms and then we will try to get a set of equations for solving for q using this general equation. So let's take up the second term now.

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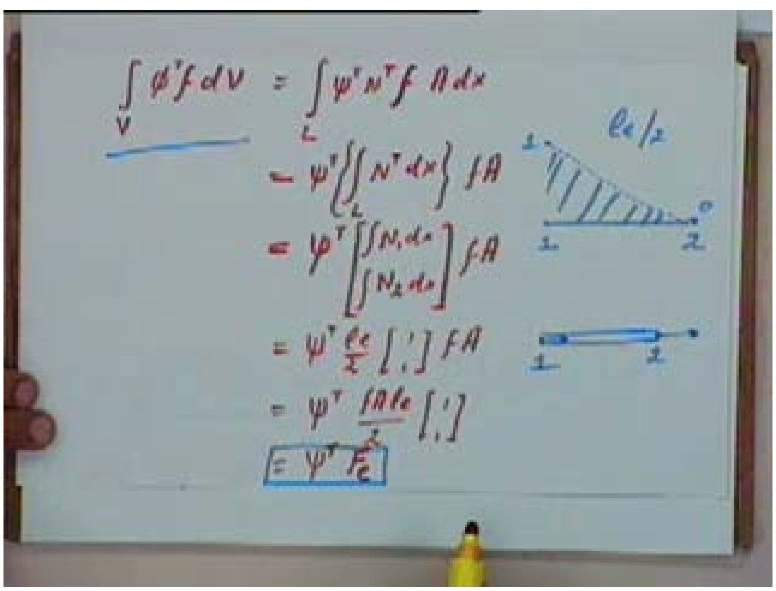

The second term is integral of, volume integral of phi transpose into f into dv. Now if I take up this, again phi is equal to N time's psi, I am writing it directly. So phi transpose will be psi transpose multiplied by N transpose, f is the body force within the element, dv is A times dx where A is the area of cross section in the element. And again now this integral would be over the length of the element.

So this will be equal to, psi transpose is a constant, f is a constant, A is a constant. This integral will give me psi transpose into integral of N transpose dx and this whole thing will be multiplied by f and A. This integral is over the length. And the integral of N transpose dx, we have derived this integral earlier, I will quickly repeat it here. Psi transpose, N transpose will given me a column vector of N_1 N_2 integral of that with dx and this column vector, we have multiplied by fA. And again if you remember integral of N_1 dx, we had said earlier that N_1 is the shape function which along the length of the element is changing from 1 to 0 like this. At the point 1, it is equal to 1, at this point it is equal to z, at the point 2 it is equal to 0. And the integral N_1 dx will be the area under this curve and this integral is equal to l_e by 2 base into height into half.

If we put that over here, this will be equal to psi transpose into l_e by 2 into 1 1 into fA. So this is again equal to psi transpose into fA l_e by 2 into 1 1. Again we get a result which is very similar to what we had last time. That is when you are using the Rayleigh-Ritz method, that fA into l_e is the total body force acting on the element, f is the body force per unit volume and A into l_e is the volume of the element. So this is the total body force acting on the element, so we are saying half of that is acting on the first node, the other half of that is acting on the second node and this we can write this as psi transpose into let's say a body force vector F_e and effectively it means that if you have an element like this. The total body force that is acting is splitted equally into two parts, one acting on the first node, the other acting on the second node. And the virtual work done will be the virtual displacement of the first node into this force plus the virtual displacement of the second node into this force. That is the virtual work done against the body force acting in this element. So this term can be simplified into psi transpose times F_e.

So again if you look at this expression, this term again we have simplified this now. Similarly we can simplify this term also. And I will leave it for you to derive this that the the this term which is integral of phi transpose Tds, the surface integral of this. This will be equal to psi transpose times T_e where T_e will be equal to T times l_e which is the total tractive force acting on the element divided by 2 and that would be split equally between the two nodes.

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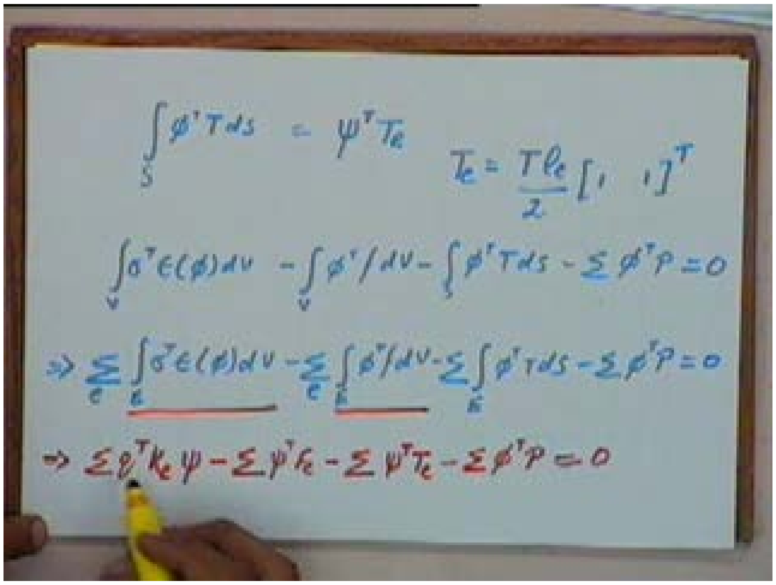

So effectively the total tractive force will also be split equally between the two nodes. So the third term, this term will evaluate to this expression. If you notice the terms that we are getting are quite similar to the terms that we had in the Rayleigh Ritz method except that their instead of psi we were having q, that's the only difference. And now if we take up this complete expression what I will get will be, I will just rewrite the same thing first. Sigma transpose epsilon of phi dv minus volume integral of phi transpose fdv minus surface integral of phi transpose Tds minus sigma phi transpose p is equal to 0.

Now this is the integral over the complete volume. So integral over the complete volume will be the same as summation over the integral of each of the elements. So this would be, this expression will give us, if I do a summation over each of the elements of the integral for each element that will be the integral over the element sigma transpose epsilon of phi dv minus summation over all the elements into the integral over the element of phi transpose fdv minus summation over all the elements integral over the element surface area phi transpose Tds minus summation of phi transpose p will be equal to 0. Now this term, this term is what we have just shown to be q transpose k_e times psi.

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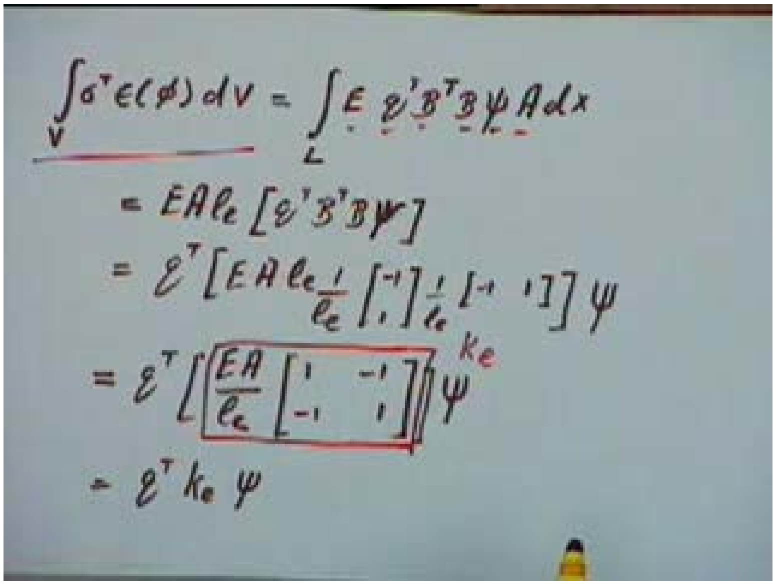

So this will, this will give me summation of q transpose k_e time psi minus summation of this term is what we just derived as psi transpose times F_e. We will say minus summation of psi transpose F_e, this term is what we have just mentioned over here minus summation of psi transpose times T_e minus summation of phi transpose times P is equal to 0. And again k_e that is the stiffness matrix is the symmetric matrix. So just for the sake of convenience, what we will do is this term q transpose k_e psi, I will write that as what I am saying is q transpose k_e times psi.

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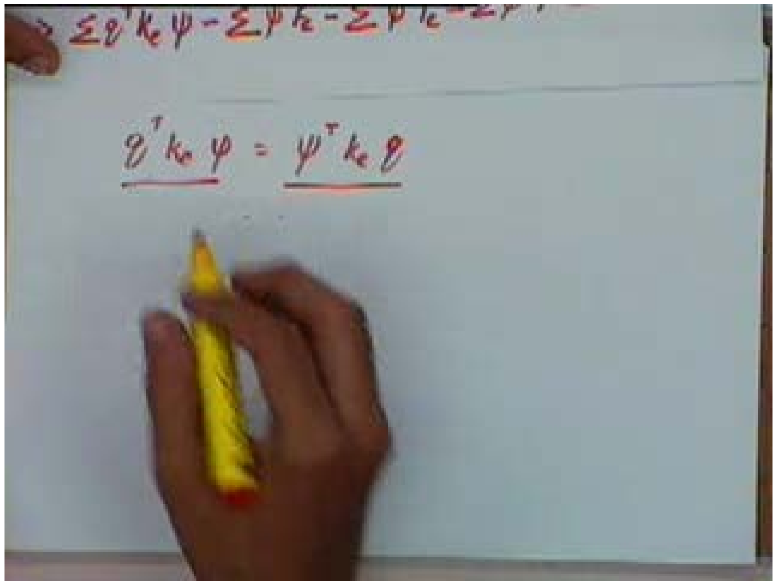

This is the same as psi transpose k_e q because k_e is symmetric and I can take a transpose without changing nothing. The product of these three is just a single number so I take the transpose, I will get this and so this is the same as this. So I will use this expression from now onwards and what we will get, this thing will now become summation of psi transpose k_e times q minus summation of psi transpose F_e minus summation of psi transpose T_e minus summation of sorry phi transpose times P.

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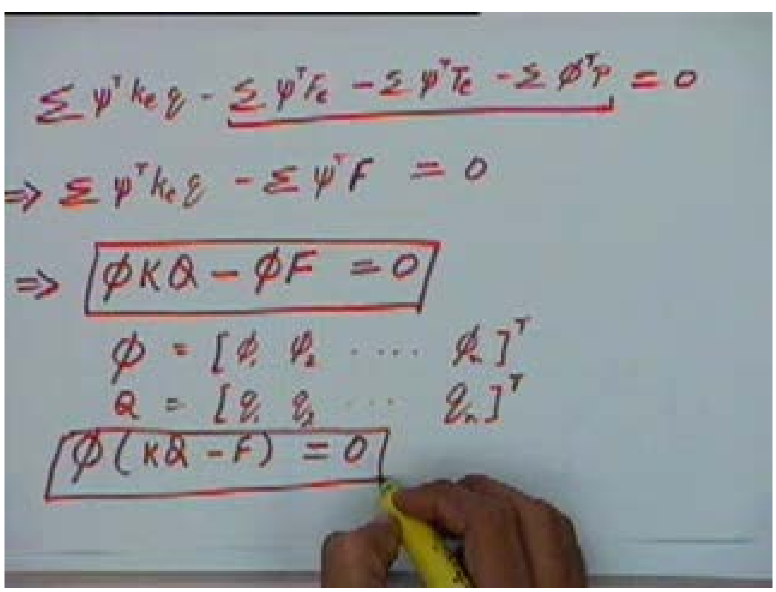

Now if I look at these three terms, in each of the three terms I have a force matrix. This is the force matrix multiplied by in the mutual displacement, this is also a force matrix multiplied by the virtual displacement and here I have got point forces multiplied by the virtual displacement at that point. And just as we did earlier, I can combine these three terms into one single force matrix. So this would be, this is equal to 0. I can rewrite this as summation of psi transpose k_e times q minus summation of psi transpose F where this F is the element body force matrix and I will say this will be equal to 0. And if I follow the same method as I did earlier for converting this element matrices into the global matrices, I can say that this is the same as the writing. I will convert this psi into a global matrix phi, this k_e into a global matrix k and this q into a global matrix q.

And this again will be minus, this psi I am changing to global matrix phi and this F is changed I am changing that into a global force matrix and I will say this is equal to 0 where this global force matrix phi will be equal to phi_1 phi_2 till phi_n if I have n elements. The definition of K, Q and F will be the same as what we had earlier. Global matrix K will be assembled from the individual stiffness matrices just like we did earlier. The global force matrix will be assembled from the individual force matrices and the point loads just as we did that earlier. So we will get this equation for the Galerkin's approach. And this I can write that as phi into K Q minus F will be equal to 0 where phi is the global, is the global matrix of virtual displacements, Q is the global matrix, Q will be equal to q_1 q_2 till q_n transpose as a global matrix for deformations, F is the global force matrix, K is the global stiffness matrix. So we will get phi into K Q minus F will be equal to 0. And this is the equation that we will be using for the Galerkin's approach.

And we have already said that as for the Galerkin's method, this equation has to be satisfied for all values of phi that means whatever be the virtual displacement I give to this system, this equation should be satisfied that means any virtual displacement which is consistent with the boundary conditions. We have if you remember earlier we said that in the Galerkin's approach the first thing we were saying was that phi has to be has to take the same form as q that is the virtual displacement and the actual displacement should have a similar description.

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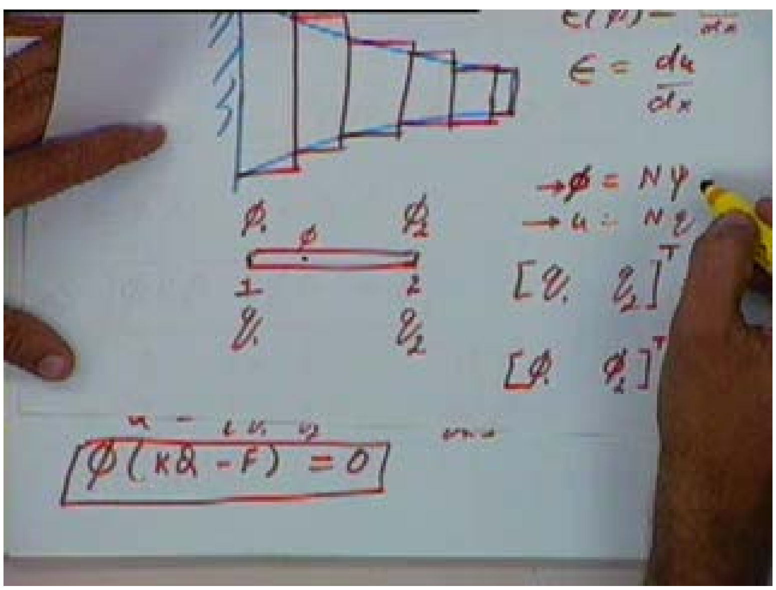

And that we have ensured by saying that q is equal to q_1 q_2 transpose and psi is equal to phi_1 phi_2 transpose and by saying that phi is equal to N time's psi and u is equal to n time's q. So we are taking a similar formulation for phi as well as for u and the other constant is that this phi has to be kinematically admissible. That means this has to satisfy the boundary conditions. We can only give it a virtual displacement which is consistent with the boundary conditions. So if we keep, if we keep this constant in mind then this equation has to be satisfied for all values of phi. So this equation, we can use for solving for Q and if it has to be satisfied for all values of phi, we can essentially say kQ is equal to F and then attach the boundary conditions to that.

We attach the boundary conditions so that we will come back to the same equations as we had for the Rayleigh Ritz method. So essentially we will see that whether we use the Rayleigh Ritz method or the Galerkin's approach, the final equation that we are getting, they will be the same sorry. The final expression for the stiffness matrix, force matrix and the resulting values that we will get for Q they will be the same irrespective of whether we use the Rayleigh Ritz's method or the Galerkin's approach. Any question on the Galerkin's approach? So essentially if you have if you look at the method that we are using for the one dimensional problems, let's say in the Galerkin's approach we started with this equation.

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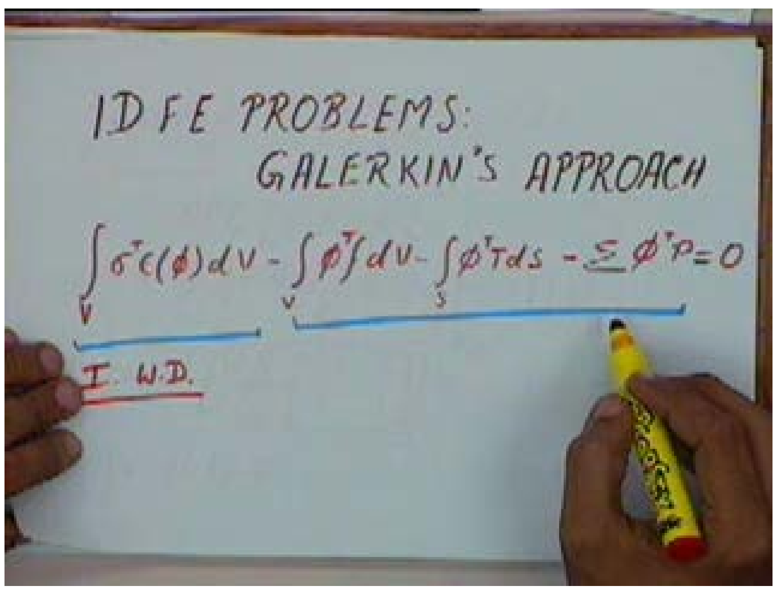

Then we simplified this first term and we got an expression like this, an expression like this.

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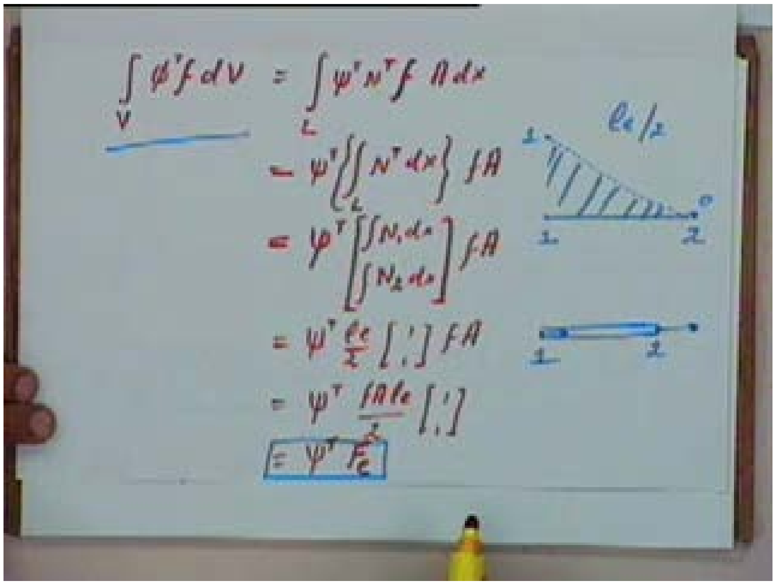

Sorry, this is the second term. For the first term we got an expression like, we got this expression.

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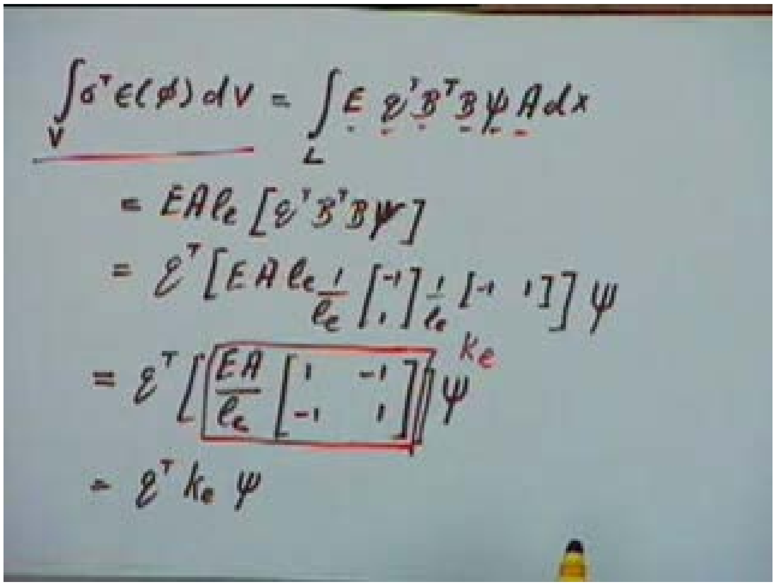

And for the second term we got this expression and a similar expression we got for the third term that is for this term. Now whether you have one dimensional problem or two dimensional problems or three dimensional problems, our aim will always be the same. We will take expressions like this, simplify them using a set of shape functions. What we have done is we took a set of shape functions as we had an element like this which are the one dimensional element and we said that we will take two nodes on this element 1 and 2.

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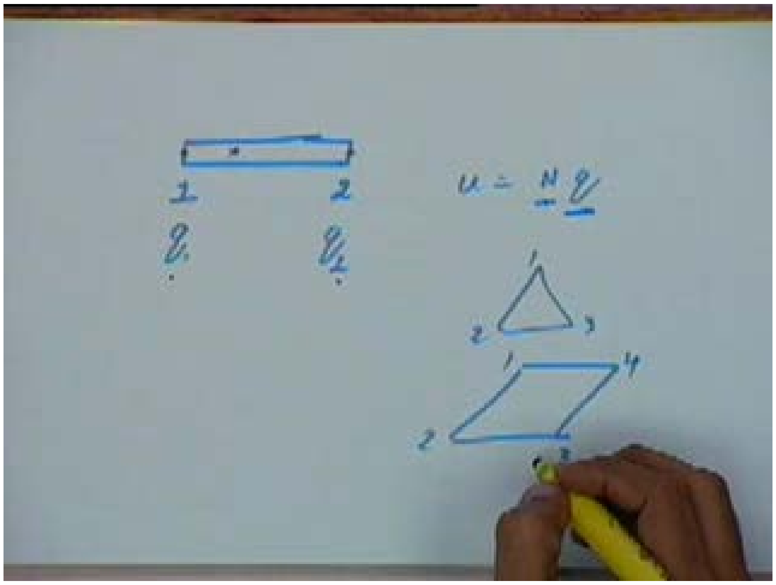

If you specify the displacements at these two q_1 and q_2, we will find out the displacement at any arbitrary point by taking two shape functions N_1 and N_2. So we say u will be equal to N time's q when N_1 and N_2 are two shape functions. Those shape functions are basically deciding the contribution of q_1 and q_2 at this point. And then we are using this formulation to simplify this expression, this expression as well as this expression. And we finally simplify these expressions to simple matrix products and those matrix products are again in terms of q's.

Now whether we have one dimensional element, 2 D elements or 3 D elements, my approach is going to be the same. Let's say in 2 D, we will try to take higher triangular elements or quadrilateral elements maybe with 3 nodes, 4 nodes or higher number of nodes. I have shape functions for each of these nodes and using that we will simplify this. And finally the expression that we will get will be, expression will be the same expressions, psi transpose times F_e or this expression or there will be expressions of this type. The final expression will always be the same irrespective of what is the shape of the element we are choosing. Our aim will be that the final expression should be the same.

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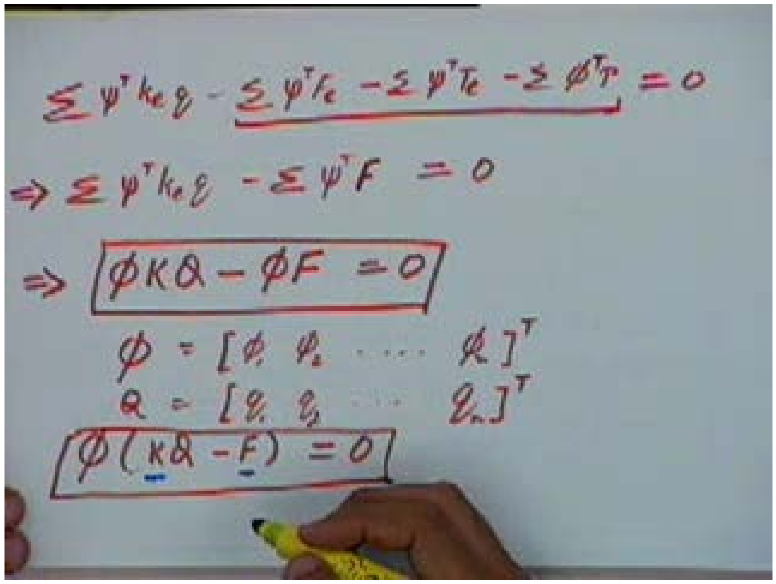

So basically when we say that we will take a particular type of element that is triangular or quadrilateral with 3 nodes, 4 nodes or 6 nodes or 10 nodes or whatever we will finally come to the same set of equations. The only thing that will be different will be how to get K and how to get F. We will take different shape functions with every element and we will get different methods of assembling the K matrix and assembling the F matrix but once this K and the F matrices have been assembled, our method of solution is going to remain the same. And the method of solution is one method that we have seen that is elimination approach and we have one more approach that is called the penalty approach. So let's briefly see the penalty approach and then we will see an example of the penalty approach may be in the next class.

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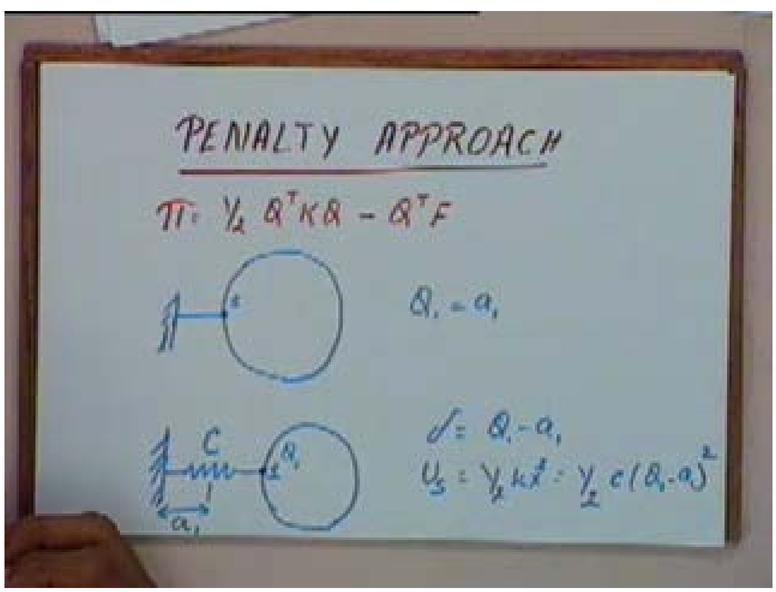

Now now when we were talking of the penalty approach, this is for solving the system of equations. If you look at the potential energy expression that we have had earlier in the Rayleigh Ritz method, we had said that the potential energy pi will be equal to half of Q transpose kQ minus Q transpose F. And for this we had a let's say some body, around this body we were considering some constraint that some node over here is fixed or has a fixed displacement. Let's say this is node number 1 and again we consider the constraint that Q_1 is equal to a_1. So what we will do is we will approximate this body by a body like this. We will add a spring at this location. This spring node has a spring constant of C and this node is 1, the rest of the body is the same.

Now if I give this end a fixed displacement equal to a_1 and the displacement of this end of this node that is node number 1 that is Q_1. The deformation in the spring delta will be how much. Q_1 minus a_1, because a_1 and Q_1 will be in the same direction. The actual deformation of this spring will be Q_1 minus a_1 and the potential energy stored in the spring that will be equal to let's say the u of the spring will be equal to half kx squared which is equal to half c into Q_1 minus a_1 whole squared. If now I look at this complete system, the potential energy in this complete system would be this potential energy plus the potential energy of the spring.

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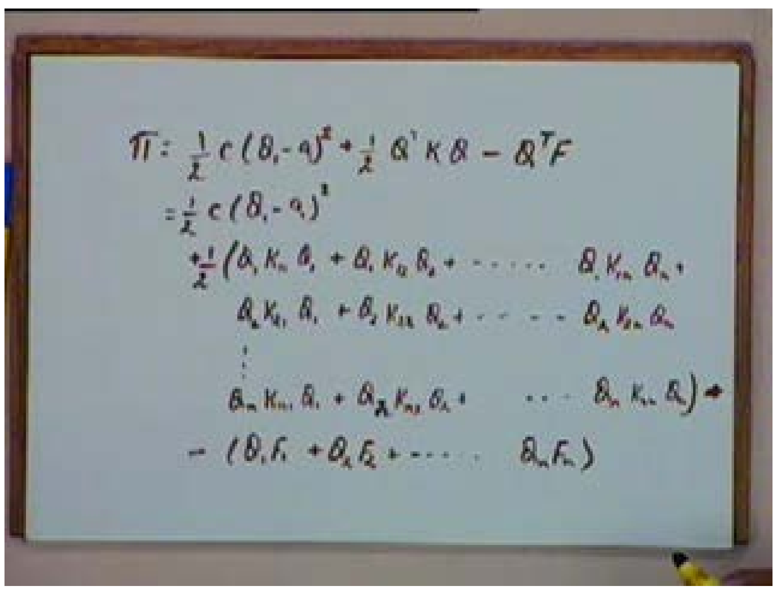

So now my complete potential energy, I will write that as pi which will be half c into Q_1 minus a_1 whole squared plus half Q transpose KQ minus Q transpose F. And essentially what we will say is that if I take this spring and I make the spring very stiff, as C approaches infinity this system would approach this system. As this spring becomes very stiff, this system will be almost the same as this system because then I will essentially get Q_1 equal to a_1 because this spring is so stiff and whatever deformation I giving to this end Q_1 will also go through the same deformation. So we will now solve for this system and then take the limit as c tends to infinity. So I have this expression, again I will write this as half of c into Q_1 minus a_1 whole squared plus I expand this term which is half of the same as what I have written yesterday Q_1 K_{11} Q_1 plus Q_1 K_{12} Q_2 plus so on up to Q_1 K_{1n} Q_n plus Q_2 K_{21} Q_1 plus Q_2 K_{22} Q_2 Q_2 K_{2n} Q_n minus Q transpose time F which will give me Q_1 F_1 plus Q_2 F_2. So this is the expression we have for the potential energy.

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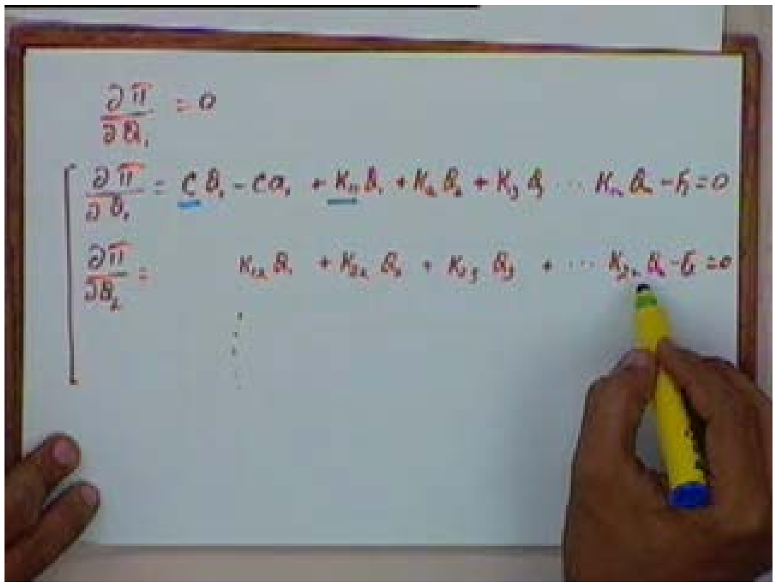

If I take this expression and again apply the constraint that del pi by del Q_i is equal 0. So first I will differentiate with respect to Q_1 and I will get del pi by del Q_1 and what will that be equal to. When I differentiate this with respective Q_1, I will get this term multiplied by 2, I will get this term so on up to this term and this column I will get this term so one up to this term. And just as plus I will get a differential with, when I differentiate with respect to Q_1 I will get a term from here. So the term from here will give me c times Q_1 minus c times a_1. The term from here will give me this plus K_{11} Q_1 plus K_{12} Q_2 plus K_{13} Q_3 so on up to K_{1n} Q_n and then this term will give me minus F_1 so this minus F_1 will be equal to 0.

Similarly when I differentiate with respect Q_2 I will get this term, I will get two times this term, I will get this term and so on and on this side I will get this term so on and this term, just like we differentiated last time. And this is going to give me expressions like K_{12} Q_1 plus K_{22} Q_2 plus K_{23} Q_3 so on till K_{2n} Q_n minus F_2 is equal to 0 and so on. This way we will get a system of n equations. Now here n equations, not n minus 1 equations because I am also differentiating with respect to Q_1. Last time I had put Q_1 equal to a_1 and I had eliminated the variable Q_1, I am not doing that now. I will now have a system of n equations and if I write them in a matrix form in this equation, in the first equation the coefficient of Q_1 is c plus K_{11} and all the other terms have coefficients which are the same as the stiffness matrix K_{12} K_{13} K_{1n} and so on. Similarly in this equation all the coefficients are K_{12} K_{22} K_{23} and so on. So when I write it in the matrix form, it will become the coefficient of Q_1 is K_{11} plus c.

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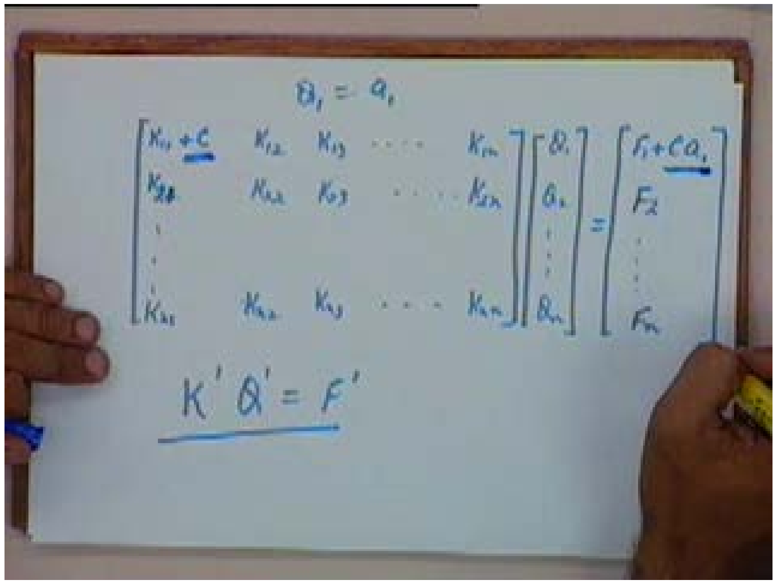

So the first term here will become K_{11} plus c. The coefficient of Q_2 is K_{12}. The other terms will remain the same K_{12} K_{13} so on till K_{1n}. If I take out the second equation, I will get K_{12} K_{22} K_{23} and so on. Now this will become K_{12} sorry K_{21} K_{22} K_{23} and so on up to K_{2n} and this way we will continue we will get K_{n1} K_{n2} K_{n3} up to K_{nn} and this thing is multiplied by Q_1 Q_2 up to Q_n and this will be equal to, the force term here is this F_1 and c times a_1. So the first term here will become F_1 plus c times a_1. The force term here is only F_2, the term second term will be F_2 and all the other terms will be the same. So this is a system of equation that will I now get. This again we will write in the similar manner K prime Q prime will be equal to F prime but now K prime is obtained by taking the same n by n matrix but in the diagonal element I am just adding the stiffness of the spring that I have taken c. And F prime I am getting by just adding c times a_1 in the force term in the first location. So this is the system of equations that we will have and in this system of equation, as I take the value of c to be very large, if I take c to be very large I will finally approach the case where Q_1 will be equal to a_1. We can see that mathematically also.

So in this penalty approach, what we are doing is we are taking the stiffness matrix and in the diagonal element for the constraint which is, for the node which is constrained we will add a very high stiffness, very high value c and corresponding force term we will modify it by adding c times a_1. Mind you we are talking of a constraint which is Q_1 is equal to a_1. So I will I am taking a very high stiffness c, adding that in the stiffness matrix and adding c times a_1 in the force matrix. And then I can solve the system of n equations and get the different values of Q_1 Q_2 till Q_n, Q prime is the same Q. So this is the penalty approach. Next time I will take a small example of this penalty approach and then see how it can be used.