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Nonlinear Adaptive Control
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Week 2
Lecture 9
Convergence of Signals using Barbalat's Lemma Part 1

Hello everyone, welcome to yet another session of our NPTEL on Nonlinear and adaptive control, we are into the second week of this course, I am Srikant Sukumar from Systems and Control, IIT Bombay. As always, we have a nice motivational image right in front of us, which is a rover on Mars and these are the sort of autonomous systems that we hope to drive with algorithms that are developed through the course of what we learn this semester. So, without delaying any further, let me go into the course material.

So, if you remember last time, what we had done was the first thing was that we discussed this corollary of the Barbalat's Lemma, the very famous Barbalat's Lemma and we also of course, have an exercise which requires you to approve the corollary in some sense, using the original Lemma itself.

Then we started to look at how we can use Barbalat's Lemma in convergence analysis in typical adaptive systems. And in order to do that, we had this nice setup where we have a spring mass damper system which is moving on the horizontal plane and we very quickly derived Newton's law second law-based equation of motion for this system and we of course, get a dynamic dynamical system of the form 3.1, which is reduced to the state space form in equation 3.2.

After that we constructed these error variables. So, the aim in most of what we do is to drive systems towards some optimal or some nicely behaved trajectories, trajectories which we are predefined in some sense. One of the things I forgot to mention are that typically these trajectories are assumed to be C infinity. That is infinitely continuously differentiable and of course, separately from C infinity the signals are also assumed to be bounded, bounded with bounded derivatives.

So, these are standard assumptions that we make on all desired trajectories that we choose to work with. Otherwise, your system gets driven to infinity and things like that, which is something we want to avoid. So, once we had these desired trajectories, we defined our error variables using these desired trajectories. And then after that we computed the dynamics of these error variables.

Now, in order to do control design, what we did last time was to create a target system. Now, the idea behind the target system is to have a stable system asymptotically stable atleast, because we want these errors to go to 0. So, we wanted to create this asymptotically stable system in this case of course, exponentially stable because it is linear.

So, we create this target system, which is sort of compatible with the original system. And in the sense that in the original system also, we had e 1 dot equals to e 2 and so, we have the same in the target system. However, the second piece of the dynamics which contained the control in the original system was prescribed to be a nice negative terms. And we could of course, prove that this is in fact and exponentially stable system just by using Root locus sorry, Routh Hurwitz, even root locus and typical Eigen value computations.

And once you understand that this is, in fact, exponentially stable, we want this original system to follow this target system. In order to do that, we compute a control log f of m, of this form. And even this 3.5 that is, if this 3.5 gets plugged here, you are going to get this target system 3.6. So, this is where we were last time. So, this is equation 3.6 is what is called the closed loop error dynamics, why is it closed loop, because we have closed the loop using the control. And this is very standard terminology in control theory.

What is the idea? The idea is that the control depends on the states themselves. So, these terms are all state dependent terms. Therefore, you need some kind of a feedback from the system. So, that is why these are called feedback controllers. So, you need a feedback from the system. That is you need some sensors mounted on the system.

Which will give me this information, without any sensor mounted on the system, it is impossible to actually implement 3.5, and then what do we do, we take the information from the system and feed it back into the system via some kind of a control loop, via some kind of control, hence it is called closing the loop, take information from the system process it in a controller and send it back into the system as a new control signal or an actuation command. In this case, as you can see, the control signal is simply this f of t this kind of a force that I have the ability to exert on this Mass.

Now, let me understand this, now that we understand this quite well. So, this means that this is called closed, do better dynamics. And this is called the closed loop error dynamics. And this is these are the quantities that is e 1 and e 2 are the quantities that we want to drive to 0. Now, what do we want to do? We talked about this last time.

So, let me actually label our lecture. So, this is lecture 3 of week 2, so we did talk about this last time, what we want to do is we want to prove that this system is, in fact asymptotically stable. Using potential functions. Why did we say we were interested in doing it? Because it is obvious that in this case, we have other tools, like, you, your Eigen value analysis, your Routh Hurwitz Criteria, you have so many methods to in fact, conclude exponential convergence.

Why do we need potential functions? The is, in a lot of cases, our target system, turns out to be nonlinear you will have to make your target system nonlinear, you have no choice. And if that happens to be the case, then you are forced to use some kind of a potential function analysis, because in those cases, your eigenvalue analysis or your Routh Hurwitz is not possible.

They are all tools that can be used only for linear systems. And since that is the case, we of course, want to see what to do in the more general scenario, how to do potential function analysis for convergence in the more general scenario. So, notice that Barbalat's Lemma is only a tool for proving asymptotic convergence. We already looked at this distinction, Well, I am sorry, we have not yet looked at this distinction. But we will very soon.

So, do not worry if you are confused. The idea is that Barbalat's Lemma proves only convergence. That is the that is it will show that it claims as you can remind yourself from the theorem, that some function goes to 0 as t goes to infinity. It does not say anything about what happens before infinity, it might very well be the case that the function goes to infinity explodes and comes back and converges to 0 as t goes to infinity.

So, you could have something like this. So, I mean, just like I said, when you could essentially have something funny happening with the function, you could have something funny happening to the function like this before infinity, and this is not really predicted by the Barbalat's Lemma. It is not predicted by the Barbalat's Lemma. So, I want you to remember the Barbalat's Lemma only helps you prove asymptotic convergence that is it talks to you it tells you something about the behavior as time goes large.

So, once we understand that, let us try to see what we want to do. So, what we want to do is we want to prove asymptotic convergence like I just said, which is mathematically defined as limit as t goes to infinity e 1 of t and limit as t goes to infinity e 2 of t has to go to 0. So, this is important, so, this is what you have limit as t goes to infinity e 1 of t and limit as t goes to infinity, e 2 of t both of these will go to 0 as t goes to infinity.

So, what we do we as we had planned use an energy functional or we use a potential function, whatever you want to call it, yeah. And what is this? This is very standard for spring mass dampers, this is V of t is half k 1, e 1 square plus half e 2 square. So, these are of course, functions of time. It should be obvious to you that this is non-negative that is it is lower bounded by 0 or because I am simply taking squares of quantities and so it is obviously lower bounded. No problem.

So, now that we understand that this function is lower bounded, we, what do we do, we take the derivative, we very, very carefully take the derivative of this function. So, let me repeat what the function was, it was V is I am, I am sort of ignoring or avoiding writing the time argument here, V is k 1, e 1 square half plus half V 2 square, and then we simply take the derivative, which is k1, e1 dot and e 2, e 2 dot the 2s will cancel out because of the 2 in the derivative.

And what do I do here? Here, Now, I substitute from the dynamics of e 1 and e 2, substitute from the dynamics of e 1 and e 2 and what is this dynamics? It is already written here. This is my closed loop error dynamics. Notice we have assumed all the parameters are known and therefore, we can actually reach here without having any issues if these parameters were not known. This is not possible. But we will worry about that later. That is where adaptive control will work.

So, I substitute for e 1 dot as e 2 and e 2 dot as minus k 1, e 1minus k 2 e 2. And then it should be very clear to you that by virtue of the construction, I did something niche in the construction, which may not have been obvious to you, but I do that. So, what is that I put a scaling k 1, e 1 square and e 1 square. And because of the scaling I get a k 1 e 1 e 2 here. And here of course, I already have minus k 1 e 1 e 2.

And so, what happens is that these 2 terms of course, cancel out, but virtue of my smart construction, if I do not do this, and this is where some kind of an experience or intuition good intuition in constructing Lyapunov functions or these potential functions comes into play.

So, because of my smart choice V, by the way, it is not my smart choice. This choice of potential function for spring mass damper has existed forever, so I am just calling it my smart choice just to tell you that if you are dealt a new problem, it is not a spring mass damper, which is something more unusual and you then will have to be have to make a smart choice.

So, due to this rather nice choice of V, what happens, we get cancellation of these 2 guys here, and then I am left with minus k 2 e 2 square. And then what do I have? This is essentially obvious that V dot is now less than or equal to 0, it can never be positive, because again, I have a square term with a negative term. And k 2 is of course, a positive quantity. And so, k 2 is obviously positive, both k 1 and k 2 are positive quantities.

So, I know that V dot is less than or equal to 0. So, what do you want to do? We are now claiming that as t goes to infinity, both e 1 and e 2 will go to 0, this is our claim. And of course, we want to see how to prove this claim. And this is where our Barbalat's Lemma will come to the forth. So, the analysis that we do subsequently is called a signal chasing analysis.

Which is through a lot of segments. And this is very, very standard steps. And so, the important thing for all of you, who are taking this course, is to almost memorize these steps, because every time we tried to use Barbalat's Lemma, these will be the standard steps that need to be implemented. Every time, every time these will be the standard steps. So, I want all of you to sort of memorize almost, you know, what the steps are now.

So, let us see. So, the first step is to note that V is bounded, what does it mean? It means that V is lower bounded? We already said that, because it is composed of squares, so it is a non-negative quantity, so it is lower bounded by 0. That is obvious. And further, it is not increasing. Why is it not increasing? Because the derivative is less than equal to 0. And you will know very well, that for any function of time, for example, if the derivative is less than equal to 0, then the function cannot increase, it can stay constant or go down.

The function is lower bounded and non-increasing. And this immediately lets us use Lemma 1.1. What was lemma 1.1? There is 1of the first Lemmas we did. It said that if a function is bounded and non-increasing, and so then the function has a finite limit as t goes to infinity, then the function has a finite limit as t goes to infinity. And that is what we use Lemma 1.1. That is was the use of all these very, very nice Lemmas. So, since we is lower bounded, and it is not increasing using lemma 1, we have that there exists a finite limit. Yes, V infinity, let us call it V infinity, we are calling it V infinity.

The second point is that both e 1 and e 2 are bounded. Why? Because, again, V dot is less than or equal to 0. And this implies that V t for any time t greater than equal to 0, is less than or equal to 0, obvious again, because it is a function of time derivative is negative. It is nice and continuous also differentiable and everything because of how it is constructed. And therefore, whenever you have, the derivative to be less than 0 less than or equal to 0, then V of t has to be less than equal to V 0.

What does it mean? It means that V itself is bounded, because whatever value I started it, at it is always going to remain below that value, so if V is bounded. And now notice that V is composed of quadratics and e 1, e is that as, V is actually e 1 square and e 2 square. So, it should be obvious to you that if V is bounded, then all of these have to be bounded, because there is no subtraction happening anywhere. It is easy to argue, if e 1 was unbounded, then V will become unbounded or if e 2 is unbounded, V will again become unbounded.

So, the only way for V to remain bounded is that both e 1and e 2 remain bounded. This is rather critical. Rather critical. So, V is bounded. Both e 1 and 2 are bounded, purely because of a quadratic nice construction, which is squares, they are all getting added nothing is getting subtracted. Instead if V was of the form something ridiculous like e 1 minus e 2 square and I said V is bounded, bounded is same as L infinity.

Remember, I cannot guarantee even e 2 is not necessarily bound I cannot guarantee that they are bounded why because it is e1 can become really large and e 2 can become really large, but they can remain same and they can go to infinity together, but this difference will always remain bounded. So, if this so, this is sort of a counter example remember, so V bounded implies e 1 e 2, bounded because of nice construction of V, if I make some arbitrary and what I would call ridiculous choices for me, this will not be true.

So, always be very careful whenever you claim there is not that I just wrote a V there for you if V is bounded e 1 and e 2 are bounded, no, it is not a guarantee and I gave you a counter example, if I choose V as e 1 minus e 2 square, this still is greater than equal to 0 remember, this still has a lower bound, but you know V being bounded in this case will not imply that the errors are bounded. And of course, we are using boundedness in L infinity interchangeably, we have already proved this in class that L infinity is same as boundedness of the function.

The next and a rather critical step a rather critical step is that e 2 is an L 2 function. So, we are coming back to everything we have learned until now of these signal norms and everything function non-vector norm signal norms. So, keep these definitions in your mind.

So, we are claiming that e 2 is an L 2 function. How do we claim that? So, what do we do? We integrate V dot from 0 to infinity. So, we are here. So, this is where we integrate both sides. Now, I know. So, the right-hand side is of course minus k 2, 0 to infinity, e 2 square t dt. Now, from the left-hand side, I know is just, this becomes if you may this is just dV dt, dt, so, this just cancels out. So, this this becomes an integral of dV and so, that is simply V infinity minus V 0.

Now, amazing thing, this can be evaluated only because we have step one. If the V infinity was not defined, or if it was not finite, this cannot be evaluated. So, because of the step 1, I can make this evaluation on the left-hand side, V infinity minus V 0 on the right-hand side, I do not touch I do not do anything to the right-hand side it is simply e 2 square dt.

Now, what what do I know, I know that N 2 norm of this e 2 signal is actually this guy is exactly this guy, and it was the scalar quantity I can I can put an absolute value but it is irrelevant this is irrelevant, I do not have to, that should be clear to you. So, now that I know this, so, this is mine to normal of e. So, what what do we know now that this is the 2 norm of e 2, what do we have essentially and this quantity here is what you have here.

So, from 3 12 and 3 13, what can I say? I can say that the V infinity minus V 0 is nothing but minus k 2 times square of 2 norm. And I also know V infinity minus V 0 is bounded, because V infinity is a finite quantity, V 0 is a finite constant, because I initialize it at a finite quantity and it would be ridiculous otherwise.

And of course, so what do I have here? So here I have the Norm e 2, the 2 norm of e 2 is actually square root of V 0 minus V infinity divided by 2. Now, notice very carefully that this guy on the top, so, this quantity on the top that is this guy is in fact greater than or equal to 0.

So, we do not have to worry about imaginary numbers coming out of this this is greater than or equal to 0, why? Because V is a non-increasing function by our step number 2, V is non-increasing, we just said it here. Therefore, V infinity, which is the value of V, when t becomes really large has to be less than equal to its value, when V is when time is 0, therefore, this is non-negative.

So, you have a non-imaginary outcome here, so it is not so it is a sanity check. So, what do we have because of this expression, I know that this right-hand side is finite and this is precisely what it means for the signal to be an l 2 signal it is precisely what it means for a signal to be an l 2 signal and therefore, we say that as per definition, e 2 is an l 2 signal. So, I want to go ahead and summarize what we did today.

And we will continue of course, with more of our discussion next time. What we looked at today, is we started off our analysis or Signal Chasing analysis. We saw that Barbalat's Lemma can only to prove asymptotic convergence and cannot tell us anything about the behavior of the function as for time reaches, which are less than infinity.

So, nothing about the transients only about the steady state in typical first level control systems language and then we looked at what is the asymptotic convergence and then, we saw a few steps of the signal chasing analysis and this analysis in which we proved signals go to 0 is called the signal chasing analysis.

In order to do this, we chose a very nice very smart potential function, and this helped us to show that V is strictly greater than equal to 0 and V is non-increasing. And this is where we started our analysis. And this is how we started with proving a few of the steps of the signal chasing analysis, what we will do next time is of course, complete at least 1 step of 1 piece of the signal chasing analysis, and we will see how far we get into the rest of the proof of asymptotic convergence for signals e 1 and e 2.

So, the important things to remember here are the very, very nice smart choice of potential function was critical. We saw that if an unusual choice would not let you prove your asymptotic convergence as you require. And this is rather critical thing to remember. And the other thing to remember is that all our knowledge of our what do you say, l 2 functions, L p spaces, l infinity spaces, boundedness, norms, and all the Lemmas that we learned, they start to get used here.

And these, these sequence of steps we do remains almost identical for the entire course, whenever we do Barbalat's Lemma based analysis and this is why it is rather critical that we almost memorize, this set of steps. So, I would strongly urge you to come into memory, the set of steps that we do, because they remain more or less identical, I mean, of course, the expressions are different. The steps themselves remain identical. This is where we stop today. Thank you very much.