Nonlinear Adaptive Control
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Week 10
Lecture No: 60
Sigma- Modification and Epsilon-Modification in Adaptive Control

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Hello, welcome to yet another session of our NPTEL on Nonlinear and Adaptive Control. I am Srikant Sukumar from systems and control, IIT Bombay, we are almost at the end of the 10th week of lectures on Nonlinear and Adaptive Control. And of late, we have been focusing on robustness in adaptive control. It is actually a property that is rather critical for real world implementations such as the Space X satellite orbiting the Earth that you see in my back.

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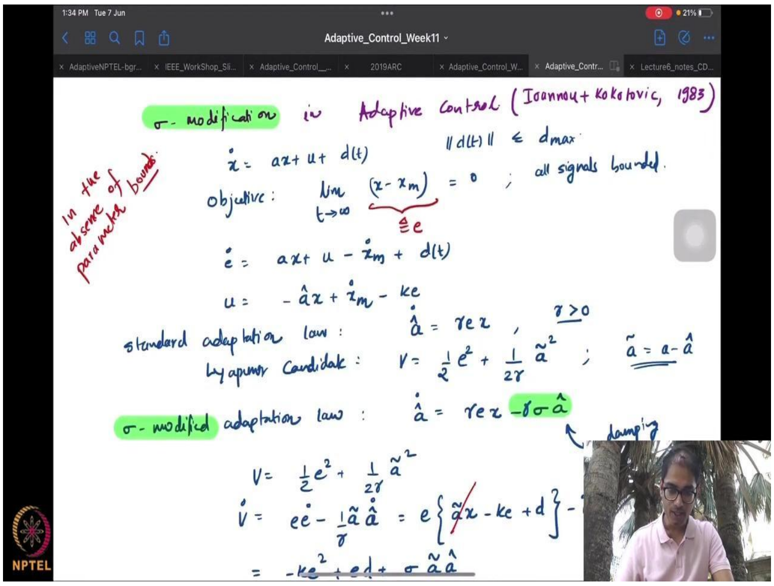

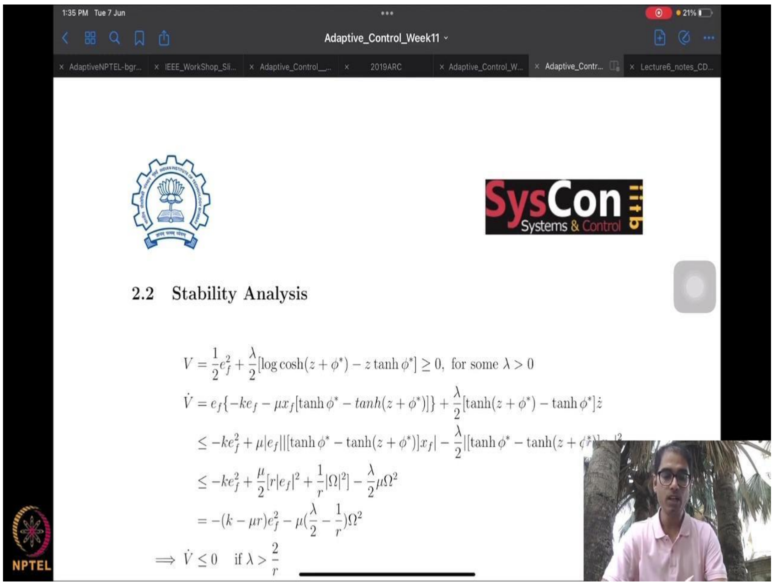

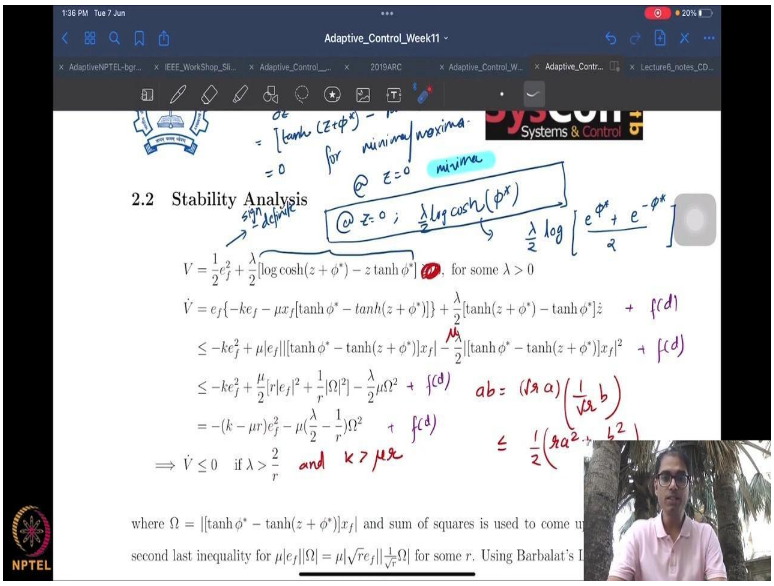

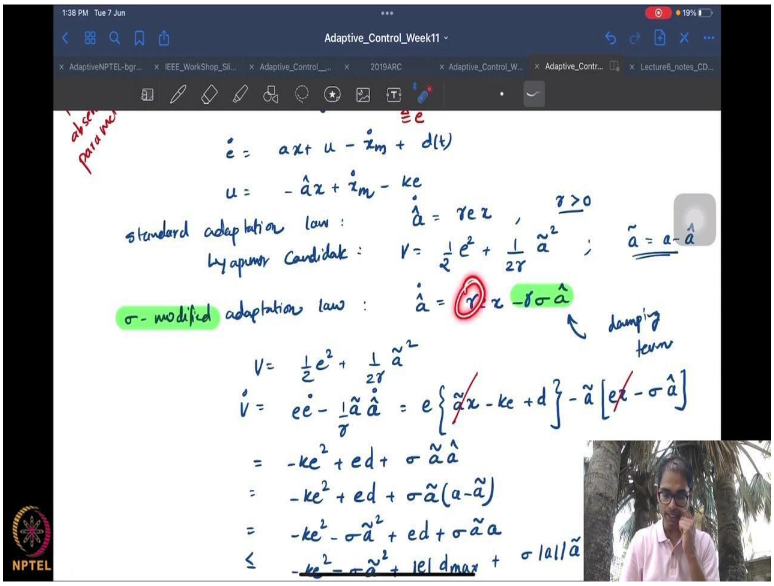

So, what we were looking at last time was essentially know projection based adaptive controllers is one way of implementing robustness. So, it was a rather nice way it was sort of a direct attack on the issue of boundedness of parameters, because when there was a disturbance what we noticed was the bad thing that happens is that the parameters get disturbed that is the parameters start to become unbounded.

And we directly attack this problem by using any prior information of parameter bounds that we might have. And with this knowledge, we use the projection operator using tan hyperbolic functions and show that we can actually design a control law and update law and filtered variables which are bounded. So, everything remains bounded in spite of disturbance, disturbance does not change anything because of how we implement things.

So, in our results are not impacted by disturbance. Of course, in the Lyapunov stability analysis, you still see that, I apologise, there seems to be a repeat of sheets here. So, now the stability analysis is not impacted, except for addition of some kind of disturbance terms in the V dots. And these will essentially result in some kind of residual set behavior, on the e f does not matter to us.

Because this is the best expected outcome in case of disturbance anyway, in standard Lyapunov based nonlinear control, or any control for that matter, because there is no way you can, counter disturbance using smooth feedback, of course, you have sliding mode, and other sorts of non-smooth feedback, which can actually do disturbance rejection also. But that is not within the scope of what we are doing in this course. So, it is a very nice method.

So, very useful. So, in fact, let me sort of get rid of this page, because this is repetition of this page if you do not mind before we go forward, just a second. So, then what we started doing pretty much at the end of the last session was talking about the sigma modification. So, we do know that there is a nice solution, which is essentially a projection solution in the presence of knowledge of the bounds of the parameter a.

But what happens if we do not have this kind of a knowledge, this kind of a prior information? In those cases, we have to devise some kind of modifications of the adaptive law. So, that is where we started, we started with the standard very, very simple model we had and we want to track some x m and because we defined e dot and a standard certainty equivalence control law. With that, we had a standard adaptation law, which is well known with the Lyapunov candidate.

And what we propose or what Ioannou and Kokotovic, proposed in the sigma modification adaptation law is that you add a damping term that is the right-hand side had no term in a hat. So, this is what we also mentioned in the earlier projection method, that non certainty equivalence-based methods do not have a term in the estimate, parameter estimate in the right-hand side of the parameter estimate update law.

And that is what we do here, you add a term with a hat on the right-hand side, which is a damping like term. So, it has a sigma term. So, that is why it is called a sigma modification, because they chose the gain sigma here. And gamma is of course retained as it is. So, this is the sigma modification.

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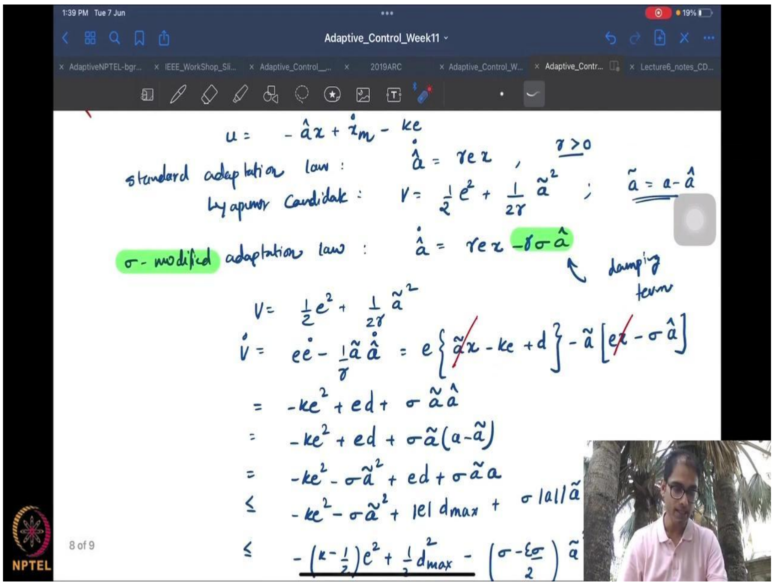
Adaptive_Control_Week11
$u = -\hat{a}x + \dot{x}_m - ke$
standard adaptation law : $\dot{\hat{a}} = \gamma e x$, $\gamma > 0$
Lyapunov Candidate : $V = \frac{1}{2}e^2 + \frac{1}{2\gamma}\tilde{a}^2$; $\tilde{a} = a - \hat{a}$
σ - modified adaptation law : $\dot{\hat{a}} = \gamma e x - \gamma\sigma\hat{a}$
damping term
$V = \frac{1}{2}e^2 + \frac{1}{2\gamma}\tilde{a}^2$
$\dot{V} = e\dot{e} - \frac{1}{\gamma}\tilde{a}\dot{\hat{a}} = e\{\cancel{\tilde{a}x} - ke + d\} - \tilde{a}[\cancel{ex} - \sigma\hat{a}]$
$= -ke^2 + ed + \sigma\tilde{a}\hat{a}$
$= -ke^2 + ed + \sigma\tilde{a}(a - \tilde{a})$
$= -ke^2 - \sigma\tilde{a}^2 + ed + \sigma\tilde{a}a$
$\leq -ke^2 - \sigma\tilde{a}^2 + |e| d_{max} + \sigma|a||\tilde{a}|$
$\leq -(k - \frac{1}{2})e^2 + \frac{1}{2}d_{max}^2 - (\sigma - \frac{\epsilon\sigma}{2})\tilde{a}$
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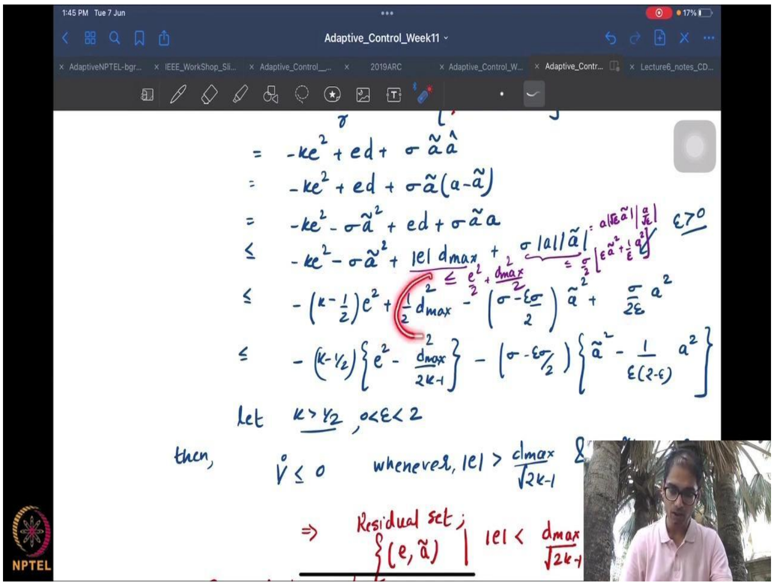
Adaptive_Control_Week11
$= -ke^2 + ed + \sigma\tilde{a}\hat{a}$
$= -ke^2 + ed + \sigma\tilde{a}(a - \tilde{a})$
$= -ke^2 - \sigma\tilde{a}^2 + ed + \sigma\tilde{a}a$
$\leq -ke^2 - \sigma\tilde{a}^2 + |e| d_{max} + \sigma|a||\tilde{a}|$
$\epsilon > 0$
$\leq \frac{e^2}{2} + \frac{d_{max}^2}{2}$
$\leq -(k - \frac{1}{2})e^2 + \frac{1}{2}d_{max}^2 - (\sigma - \frac{\epsilon\sigma}{2})\tilde{a}^2 + \frac{\sigma}{2\epsilon}a^2$
$\leq -(k - 1/2)\{e^2 - \frac{d_{max}^2}{2k-1}\} - (\sigma - \epsilon\sigma/2)\{\tilde{a}^2 - \frac{1}{\epsilon(2-\epsilon)}a^2\}$
let $k > 1/2$, $0 < \epsilon < 2$
then, $\dot{V} \leq 0$ whenever, $|e| > \frac{d_{max}}{\sqrt{2k-1}}$ &
$\Rightarrow$ Residual set ; $\{(e, \tilde{a}) \mid |e| < \frac{d_{max}}{\sqrt{2k-1}}$
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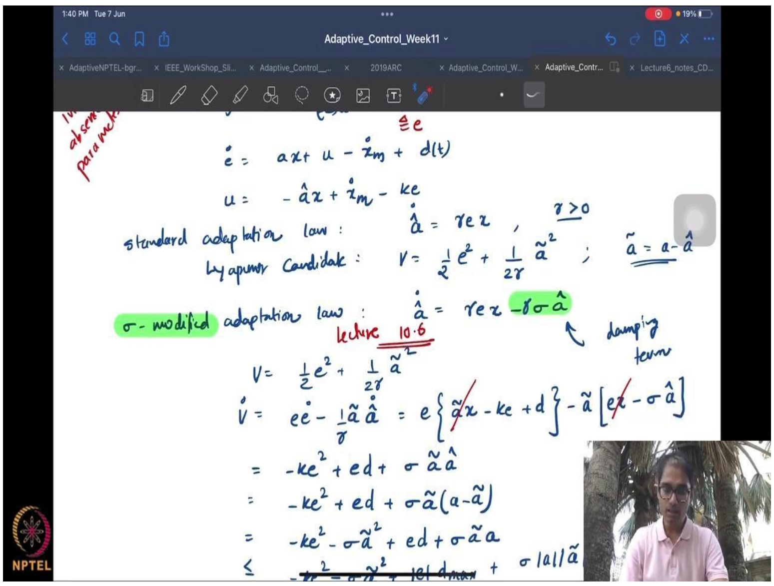
Adaptive_Control_Week11
$\dot{e} = ax + u - \dot{x}_m + d(t)$
$u = -\hat{a}x + \dot{x}_m - ke$
standard adaptation law : $\dot{\hat{a}} = \gamma e x$, $\gamma > 0$
Lyapunov Candidate : $V = \frac{1}{2}e^2 + \frac{1}{2\gamma}\tilde{a}^2$; $\tilde{a} = a - \hat{a}$
σ - modified adaptation law : $\dot{\hat{a}} = \gamma e x - \gamma\sigma\hat{a}$
lecture 10.6
damping term
$V = \frac{1}{2}e^2 + \frac{1}{2\gamma}\tilde{a}^2$
$\dot{V} = e\dot{e} - \frac{1}{\gamma}\tilde{a}\dot{\hat{a}} = e\{\cancel{\tilde{a}x} - ke + d\} - \tilde{a}[\cancel{ex} - \sigma\hat{a}]$
$= -ke^2 + ed + \sigma\tilde{a}\hat{a}$
$= -ke^2 + ed + \sigma\tilde{a}(a - \tilde{a})$
$= -ke^2 - \sigma\tilde{a}^2 + ed + \sigma\tilde{a}a$
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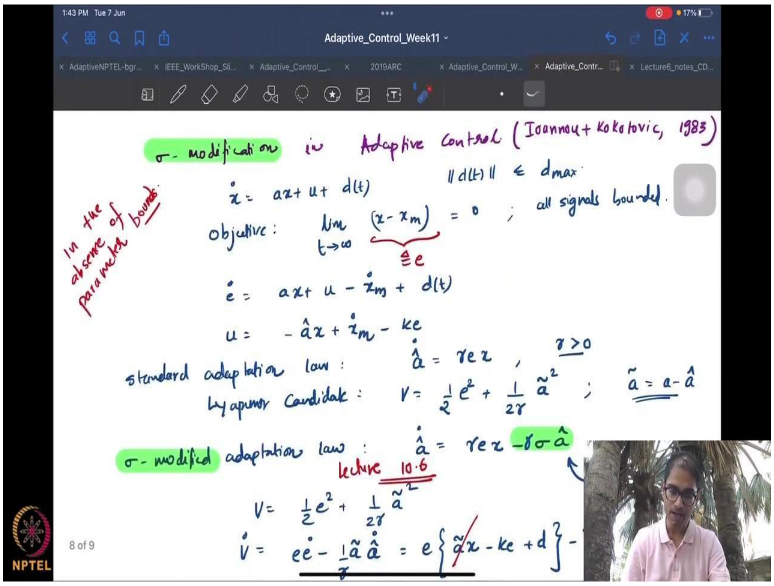

So, we want to start to look at it in more detail today, in this session, so I am going to mark our lecture 10.6. And now we of course, add the disturbance, I mean, we already had the disturbance in the dynamics, and of course, the design cannot account for any of this disturbance, obviously, the design remains exactly the same as before, the only difference being the sigma modification. So, we again, take the same Lyapunov candidate so, no reason to choose another one.

And you get the derivative as e dot minus 1 over gamma a tilda a hat dot as always, then a substitute for the closed loop dynamics. So, I substitute this guy a here, and I get an e dot, which is a tilda x. So, this cancels minus k e from here, and a disturbance from here. So, this is what we get an e dot. And then we have an a tilda a hat dot, and a hat dot is just this. So, gamma e x times gamma sigma a hat and the gamma cancel out, so, you are left with the e x minus sigma times a hat.

Now, we continue. So, you notice that this e x term cancels this, that is essentially the purpose of the update law, that the update law is chosen so that these two cancel out. So, it should not come to you as a big surprise. So, once these two cancel out, you have the nice negative term in e, again, not a surprise, then you have term e and d, which is the disturbance term that we saw anyway in the beginning of this week’s lectures.

And finally, you have a term, which is the new term, which is sigma, a hat, a tilda, this guy. Now, what we do is we write our a hat in terms of a tilda as a minus a tilda from here. So, a minus a tilda just from this expression right here. And so, what happens, you see that I start to get a minus sigma a tilda squared term. So, now, I have two quadratic terms, negative quadratics minus k e squared minus sigma a tilda squared.

Remember, when we started this discussion on this robustness, I had mentioned that this issue of disturbance causing some states to grow unbounded is primarily an outcome of the fact that we do not have strictly Lyapunov functions in adaptive control. Even if we start with a strictly Lyapunov function for the known case, when we start to do the adaptive control becomes a non-strictly Lyapunov function, because the V dots always come out to be the same as the non-adaptive case, but the V had an additional a tilda term.

So, therefore, it became a non-strict Lyapunov, and that was the problem. So, this, notice what this sigma modification has done. It has introduced a negative term in the a tilda. And this is what is somehow giving the strictification if you may. This is what is giving a strictification. Of course, this is not for free, because you already had the disturbance term, which was okay. Let us not worry about it. But you also end up with a sigma a tilda a term. So, you have an additional disturbance like term here.

Notice that both these terms are linear in the state, so this is linear in e, this is linear in a tilda, a is a constant, unknown, but a constant, sigma is a constant, and d is not a constant, but it is bounded by a constant. So, now what do we do we use our standard. Well, I mean, first of all, we do a norm bounding, which is like this is less than so I put norms everywhere on absolute values everywhere, because these are scalars. If these are vectors, I would put norms.

So, this is just less than equal to absolute value of e times absolute value of d, which is less than equal to d max. So, this becomes absolute value of e times d max, and then I just put absolute value everywhere sigma times absolute value of a times absolute value of a tilda. And now, I simply do a standard, a b less than equal to a squared plus b squared by 2 on both of these terms, well, I do not do a standard, I actually do something slightly more.

So, this guy is actually, this is written as less than equal to e squared by 2 plus d max squared by 2. So, this term is the standard one. But this term I am writing slightly differently, as this is written as sigma is multiplying here and by 2 then I have epsilon a tilda squared plus 1 by epsilon a square. We already saw this kind of a decomposition also, it is possible because I am simply writing this is equal to a square root epsilon a tilda times a over square root epsilon.

So, this, I can do this, this is just a multiplication and division by square root of epsilon. So, nothing is changing. So, we have already done this kind of decomposition. So, that is what we do for this term. And now, you just combine these with the square term, because I have a squared term here. So, I have k minus 1 half e squared, and then sigma minus sigma epsilon by 2, a tilda squared here, and then I am left with the d squared terms in these unknowns.

So, these are unknown constant, positive constant. And then I, of course, combine this guy and this guy. And I get something like this and something like this. It does not matter I could have combined it with one or the other and all that mess.

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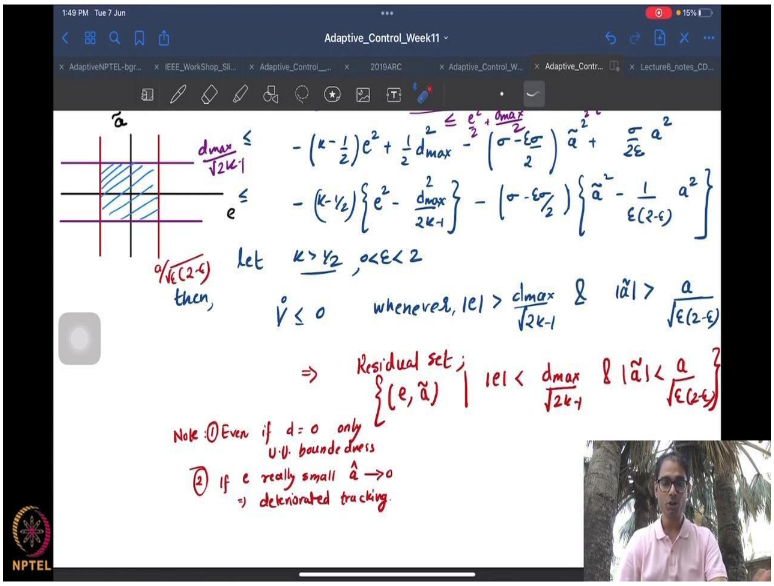

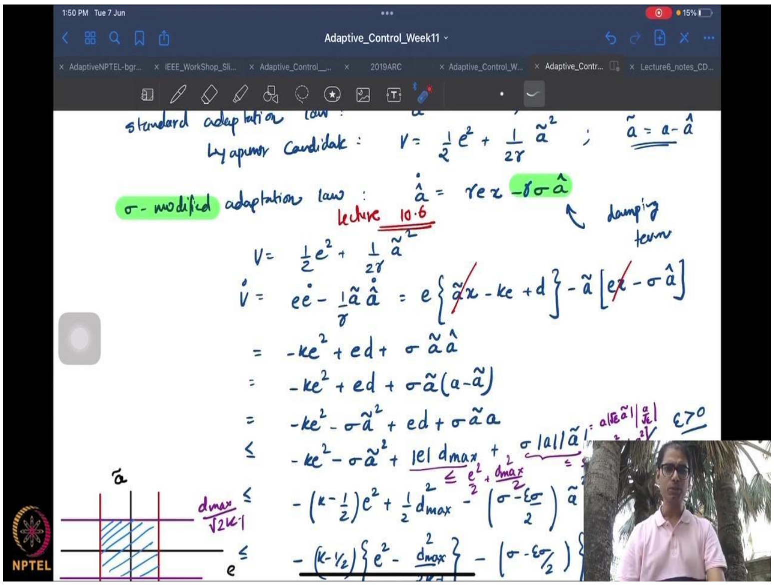

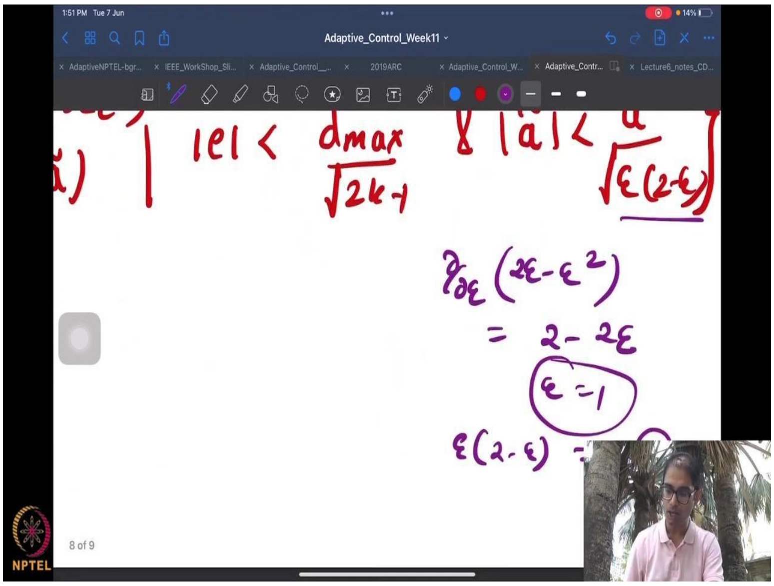

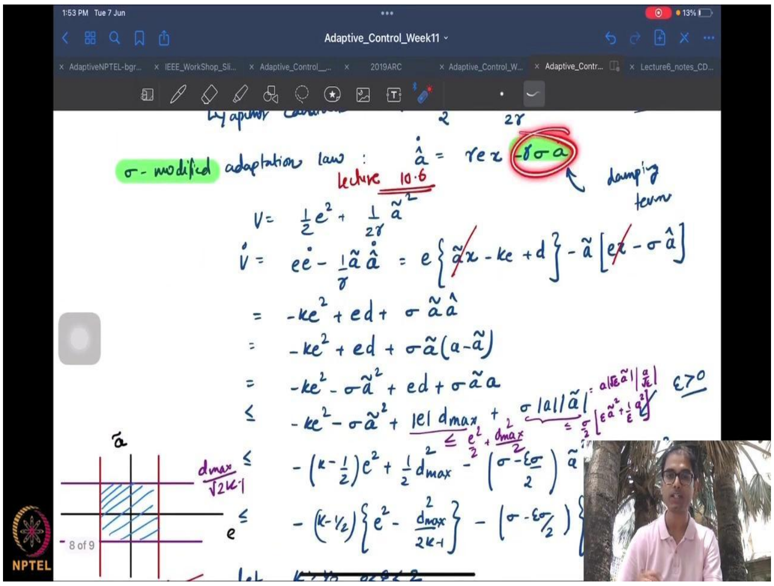

But it is sort of irrelevant, you would essentially get similar outcomes, similar sort of residual sets. Now, what is happening here, if you notice is that if you assume k is greater than half and epsilon is less than 2, of course, that is important. Otherwise, these quantities are not positive.

So, this is just to sort of ensure that you have this quantity positive and this quantity positive, that is all then we have V dot to be negative semi definite whenever this is positive, and this is positive. And that is given by these conditions, when e is greater than this, and also a tilda is greater than this. And notice what happens? What happens here is we get two residual sets, two residual sets, meaning we get intersection of two residual sets.

So, if I was actually to draw a picture, just like we did for the standard case, If I was to make a picture like this, let us try. And this was say my e axis. And this was a tilda axis this is rather not perfectly done. Do not worry about it. Let us see, I am sort of drawing only a projection of this I am only drawing a projection of this set. So, if I draw this guy. So, there will be residual set in e just like this, and then there will be a residual set in a tilda which is like this. So, your trajectories, so it can be very easily shown that your trajectories will in fact, a lie here. Your system trajectories will in fact lie here.

So, this red line, of course, represents your a tilda, which is a over square root epsilon 2 minus epsilon, and your this magenta line, which is the e bound is basically d max over square root 2 k minus 1. So, you can see that by choosing epsilon and k of course, you can modulate these a little bit, but the basic idea is there is a residual set in both. So, you are so you can see that. You can never go unbounded here. The ideas will never go unbounded. Why? Earlier, you had some V dot negative semi definite only in terms of the e dynamics.

Now, what happened in terms of the e variable. Now, I have a residual set, I have sort of this V dot negative semi definite condition. For both, it is an and condition, it is and it is not an or, it is not one or the other. So, if even one of them if a tilda of the parameters tried to escape this bound, if the parameters try to escape this bound, the V dot will, again become negative, and pull them back.

And similarly, if only e becomes e starts to escape the bound, then also you will be pulled back by the negative V dot. So, you move out in any direction, it does not matter, you will be pulled back out. So, therefore, both e and a tilda will remain bounded. When you see and there is an and condition in the residual set. So, this is the rather nice feature of the sigma modification that, first of all, it looks very simple compared to projection all of you have to agree, we all like simple things.

So, all we did was really just change the added damping term, which would have sounded very, very logically obvious to all of you. I mean, and, hence Ioannou and Kokotovic, came up within with it in 1983, really far back, that was the logical conclusion, the absence of the damping term was doing some bad thing was essentially killing the robustness. So, obviously, this is a very, very intuitive answer, add a damping term.

So, in the presence of disturbance, you have very nice performance, in the sense you have bounded performance. And if I increase my k, I can of course, reduce my residual set. Epsilon is of course, the point is, epsilon is of course, bounded.

So, if you, it does not matter what you, you cannot do too much with this bound, is what I am trying to say, you cannot do really too much with this bound. And we will see why? And why we cannot do too much? I mean, one obvious reason why nothing much can be done is look at how the expression is. It is epsilon times 2 minus epsilon and epsilon is a choice of epsilon is just between 0 and 2.

So, whatever you do here, or the maximum value is either bounded. I mean, it is lying epsilon is between 0 and 2, if you choose epsilon to be very close to 2, then you have some really small quantity here. Sorry, I mean, the way to make this small is to make this large.

So, the largest this value can take is not much, honestly speaking, I mean, you in fact, I do not, we have to see, it is essentially not difficult to compute what will be the largest value this guy can take. So, this is 2 epsilon minus epsilon squared. So, if I take partial with respect to epsilon, this will be 2 minus 2 epsilon. So, at epsilon equal to 1 you have an optimum and what is that? And this is essentially the denominator is equal to 1 at epsilon equal to 1. So, that is what the denominator is exactly equal to 1.

Now, the question is this the maximum value? Yes, because if I take second derivative of this guy and other derivative, it is minus 2 is negative 2. Therefore, this is in fact a maximum. So, the largest value this can take is 1.

So, the idea is that the residual set. So, the minimum so it looks like from this the residual set the maximum or whatever the least you can make this is a, which means your error will be less than a and not smaller than that. So, if your true value is a, there will be at 2 a or minus 2, 2 a or 0. Because the minimum error is a, a tilda is the error in the parameter estimate, remember. Your parameter estimation error is a which means that the true value is a then you are either a 2 a or 0, which is not saying much. So, the residual set is large.

On the other hand, the parameter residual can be made really small by choosing large control gain, that is still there. But in any case, we are getting bounded performance, all the cool things. So, we should not be complaining too much, I guess. However, one of the two things, first thing is even if the disturbance is 0, you still get only bounded performance. Why is that? Suppose I put d equal to 0 in everything, then what happens? This d max term is what goes away is half goes away and the d max goes away. All of this goes away. So, this is not there. This is not there.

It does not matter if this is there or not just some scaling, but this term goes away. Now, if this term goes away, what changes? I still have some term here some residual term here. Some disturbance like term here because of my introduction of the damping. So, my introduction of the damping creates a disturbance like term, which is not going to go away, even if you kill the external disturbance.

So, you can also think of a some internal disturbance, this damping has some internal disturbance, that is what it behaves like. So, this term is not going anywhere. So, all that will happen is that this will be not there, this will be 0.

Now, this term, can go to 0, but you will still have a bound here. And if you have a bound, then you will still get boundedness e will still not necessarily go to 0, because you have some kind of residual, no additional term here, because even if this term goes missing, you still have this guy. And you can see, this is like a disturbance term. And this is not going to allow V to go to 0, even in the absence of disturbance. So, what happened?

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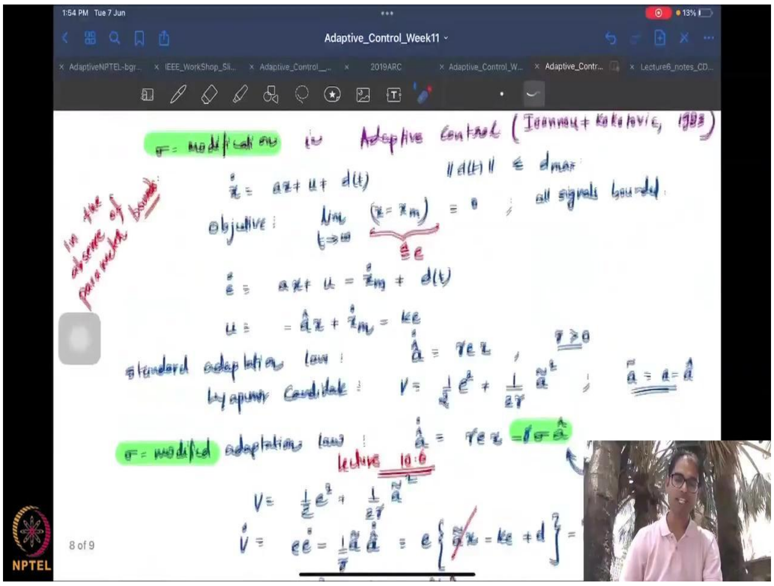

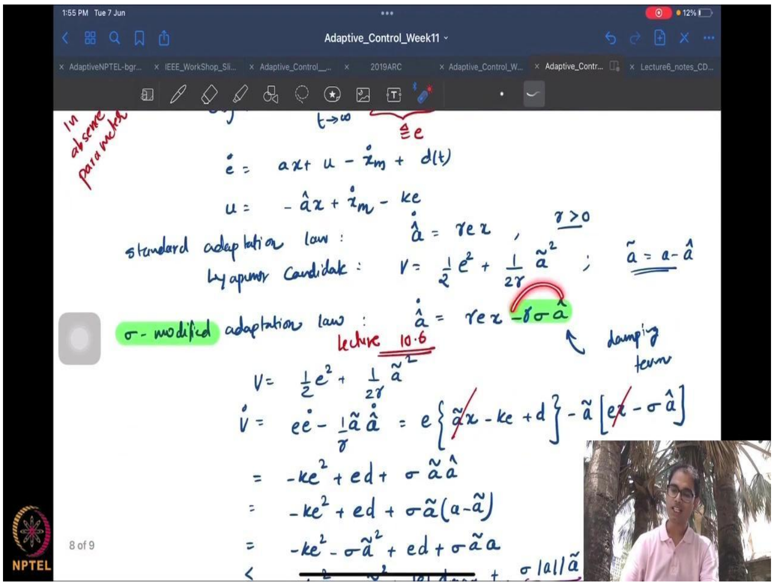

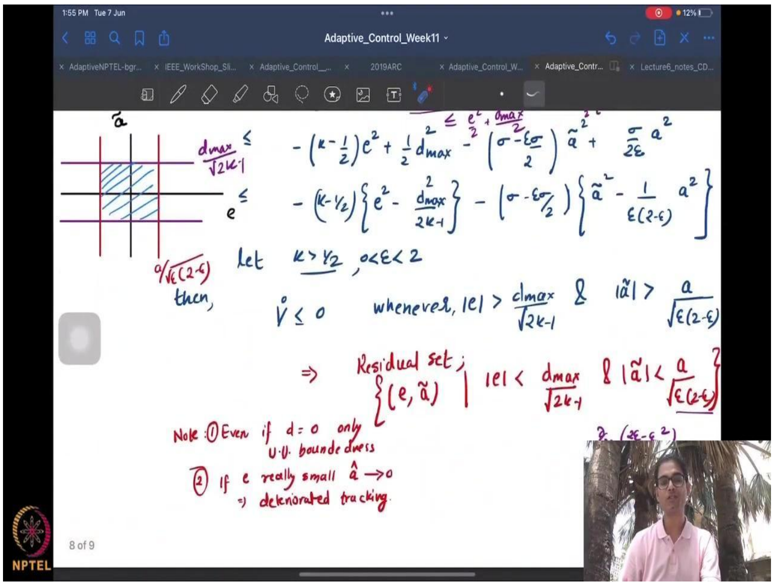

The Sigma modification did not need knowledge of the bounds on the parameter. But of course, it cannot do everything it is always a give and take in life. This is the important lesson, whenever you see a very technical presentation where there are no assumptions and everything can be done. Be very, very skeptical, because there is no free lunch.

So, we did not require knowledge of parameter bounds, the construction was rather simple. But even in the absence of disturbance, it is only going to give bounded performance. If you are happy with that. Great. If you are not, then you want to use projection. The final thing is, when e is really small. When e becomes really small, you have poor tracking performance. Why? When e becomes really small, this guy goes to 0 almost. This guy goes to 0.

So, what happens? You just have this damping term, which is going to push the parameter value to 0, because it is just exponential decay. So, this is going to push the parameter value to 0. And I mean, if you have learned the value of the parameter, suppose you are getting very close to the true value of the parameter. And then suddenly your errors became 0 in the system, then you are essentially going to unlearn what you just learned. You learned the value of the parameter after significant effort, I hope, but you are going to unlearn it, because of this kind of problem.

So, in order to resolve this second issue, the first issue cannot be resolved. Does not matter if you will always get only bounded performance with these kinds of modifications where you do not have any projection.

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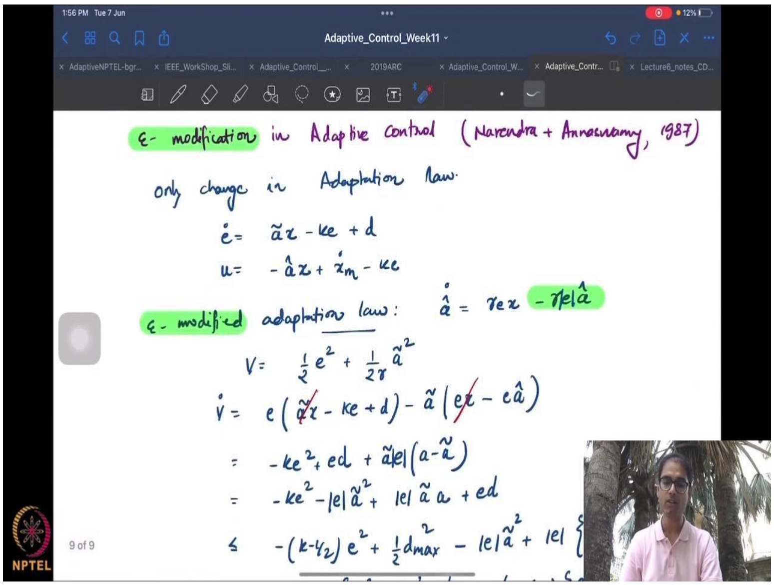

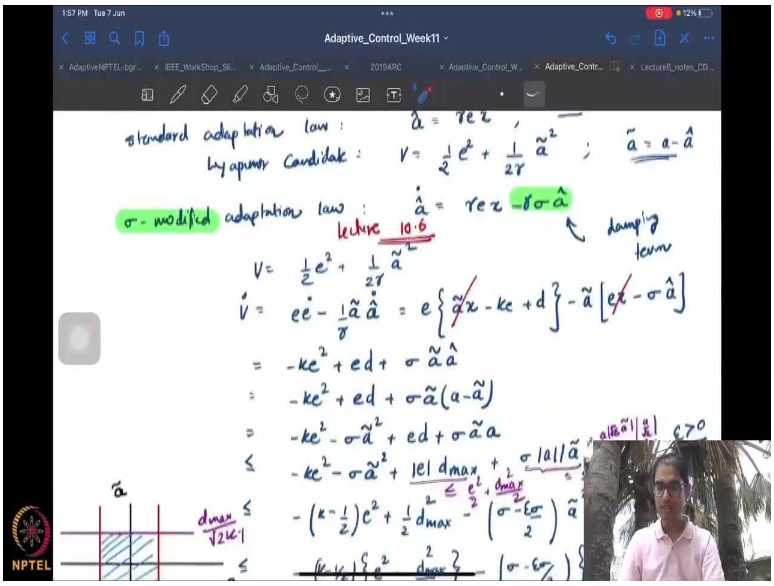

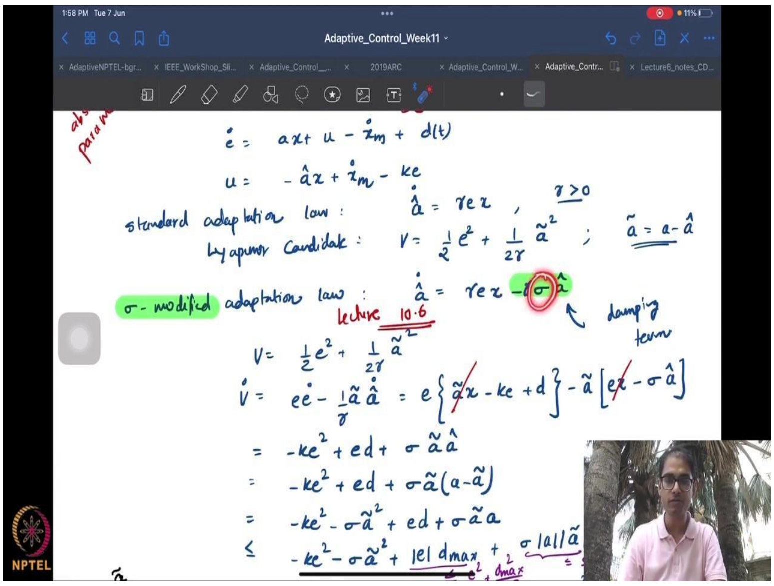

But for the second issue, you can do some nice things. So, if we a and that is called epsilon modification. It came four years after the sigma modification by two other big names in adaptive control. Narendra and Annaswamy. And what they do is? Very simple again, I mean, very, very intuitive and nice simple results that is the best part of this, instead of having just a sigma here, they put a term which is dependent on the parameter error and an absolute value.

So, this is not exactly smooth, not exactly smooth. So, they put an absolute value here of error, instead of the sigma, that this is called a epsilon modification, how does it change things? As usual, you have the same Lyapunov candidate have the same cancellations.

And you are left with the same sort of terms here. The only difference is, the only difference is instead of the sigma appearing in these terms, the absolute value of e appears here. Does not change anything, because this is non negative. And this is important absolute value is important. Because otherwise, you will have, you could have negative quantities. And so just to ensure that these are non-negative quantities, this is important.

So, you have an absolute value of e here, because sigma if you notice was a positive gain. It is obviously if you want a damping term, which is a negative term, this has to be positive, cannot have arbitrarily signed quantities here. To ensure that there is at least not negative, you have to put an absolute value here. Otherwise, this is not a damping term. It can be a blowing up term also. So, therefore, the absolute value e is very essential here very critical. So, all that this does is it changes here, instead of sigma, you have an absolute value of e here.

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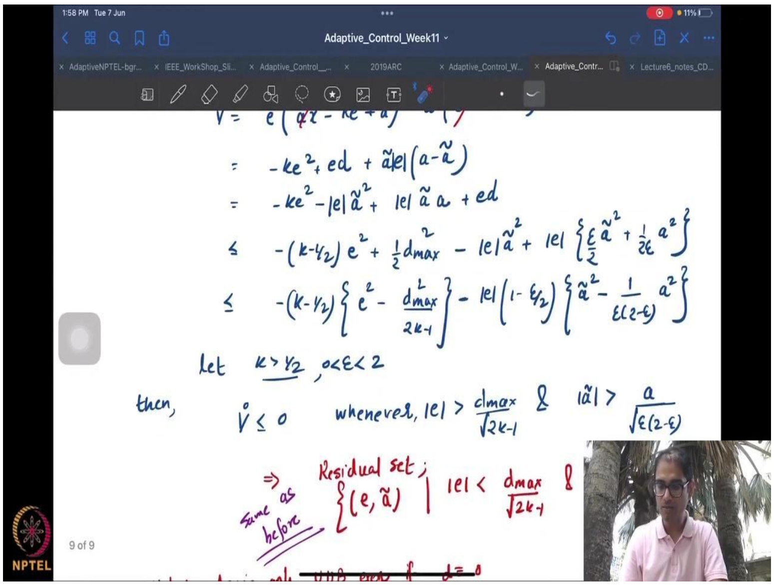

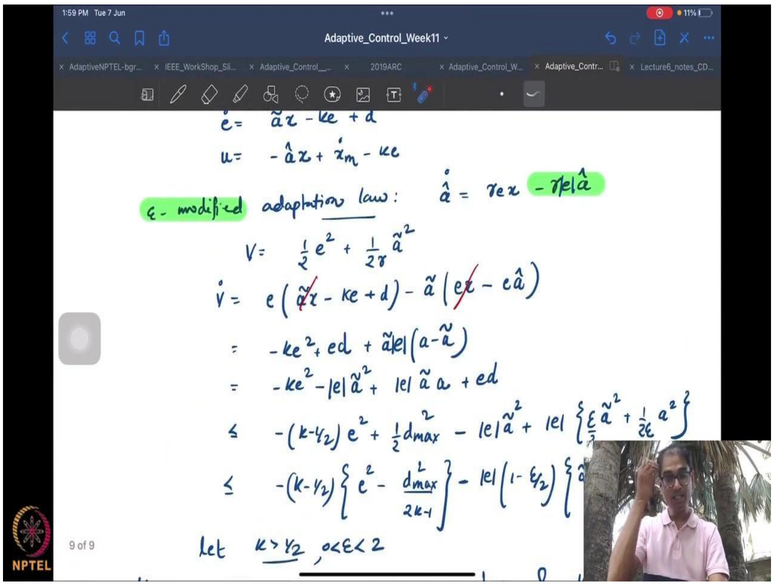

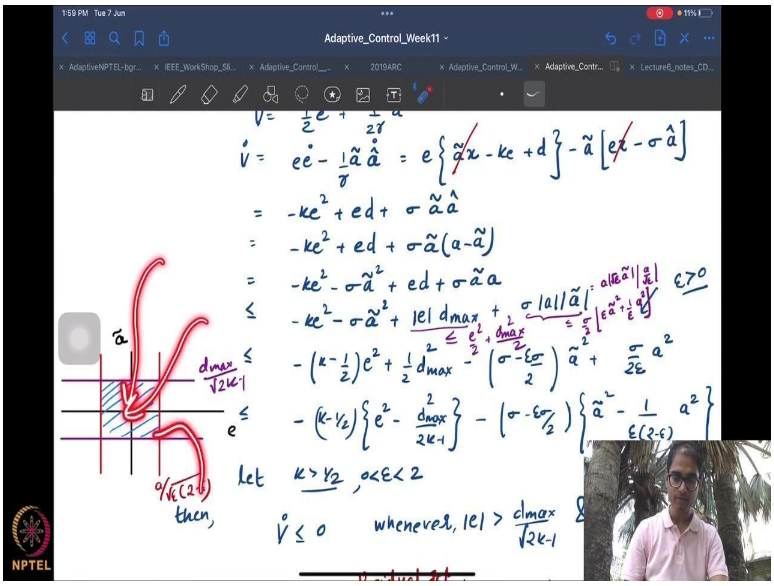

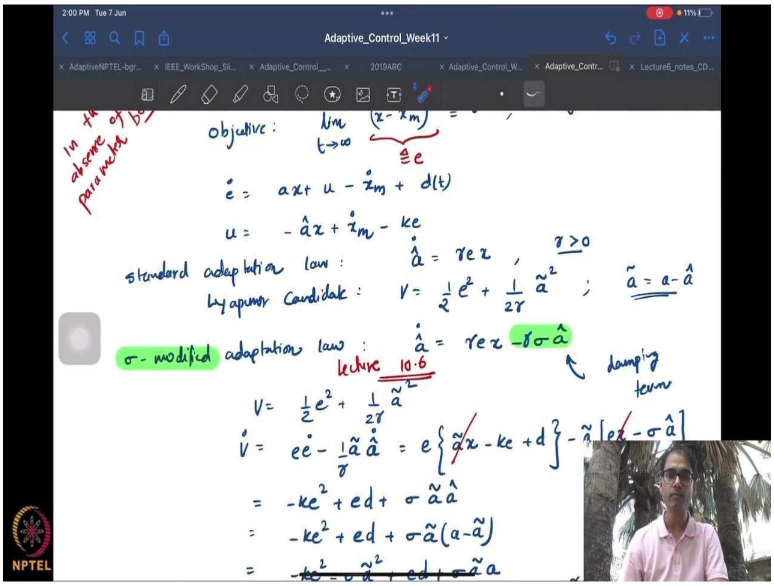

And then rest of it is pretty much exactly the same. And in fact, the residual set expressions also turn out to be exactly the same, because nothing changes except for this sigma being replaced by this absolute value of e here. This has the same issue, of course, it is still only uniformly ultimately bounded if d is equal to 0, why we call it uniform, ultimate boundedness.

It is uniformly bounded because initial conditions do not impact the bound. You will still reach the same residual set. You see there is no initial conditions appearing here, that is why it is uniform. Why is ultimate bound? Because it ultimately goes to the, that is it may start outside, but it eventually goes into the set. That is asymptotically. It goes into this set. Not necessarily in one step or anything like that. Therefore, it is uniform, ultimate boundedness if you look at large time it is uniformly bounded. So, this bound is not impacted by the initial condition, but is valid after large time.

So, now what is the difference that happens here is in the adaptation law, notice what happens? Earlier, the issue was that when e became small, this term has no impact. And this term pushes a hat to 0 and leads to unlearning. Now, what did Annaswamy and Narendra do? They did the smart thing, the absolute value of e here and could not have put e here. So, they had to put an absolute value.

And what this does is? When e gets close to 0, both terms are equally small. So, there is no unlearning if a had reached close to the true value of a, then you continue to remain close to it, because the right-hand side of the derivative is small, because e small so, derivative is small. So, obviously, you are not moving too far away from the well learned value. Therefore, your tracking performance is going to be better. That is the whole idea of epsilon modification, which is sort of one can say an improvement over the sigma modification, remember, although that you did put in a non-smooth term in the update law.

Many, again, many theoreticians and maybe even applied engineers may not like this, this is a high frequency term in some sense. Because if your e is oscillating between negative and positive values around the origin, then this is sort of giving you some trouble can potentially give you trouble and because of the sign. And this could become plus minus plus minus and all that.

So, remember that this may not always be the best option, but whatever. I mean, this is what is the epsilon modification, one can also smoothen this I guess, if you want but that's on you, that is also a possibility. A smoother version of this is also possible. And you can again, use tan hyperbolic or sigmoidal type functions.

So, what did we see today, we looked at an alternate way of doing robustness or imparting robustness to adaptive controllers via the sigma and epsilon modification. The idea is that the lapse or the flaw in the earlier design is the absence of a damping term. So, both sigma and epsilon modifications add a damping term to the update law adaptive update law. While in sigma modification, it is a constant sigma gain, in the damping term, in the epsilon modification, it is the gain is actually absolute value of the error.

The advantage of epsilon modification is that when the parameter has been learned and the error goes close to 0, then the sigma modification will lead to unlearning of the parameter and a hat will become also 0 because of the damping term.

But in the epsilon modification, because of the absolute value e scaling, your parameter value will not be unlearned, in spite of the fact that your error grows close to 0 that is the advantage, in both these cases, there is a give and take because the bounds are not used. Like in the projection case, you get only bounded performance even if your disturbance is 0.

So, there is of course, like I said, a give and take input here. So, I hope you found this week rather interesting and learn a little bit about robustness in adaptive control. So, I will see you in the upcoming week's sessions. Thank you.