Nonlinear Adaptive Control
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Week 9
Lecture No: 52
Tuning Function Adaptive Method

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Hello, everyone. Welcome to yet another session of our NPTEL on Nonlinear and Adaptive Control. I am Srikant Sukumar from Systems and Control, IIT Bombay. We have for a while now been motivated by our desire to drive uncertain autonomous systems such as the SpaceX satellite that you see in the background. And in our quest, we have already seen how to analyze and lately, how to design controller for uncertain dynamical systems. And these adaptive controllers, I hope that, will help you to design and develop algorithms for your own practical dynamical systems in your own research areas.

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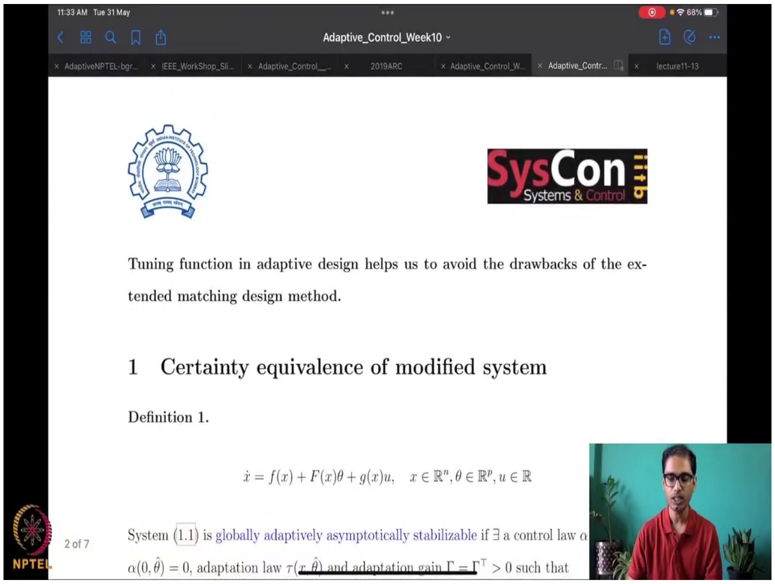
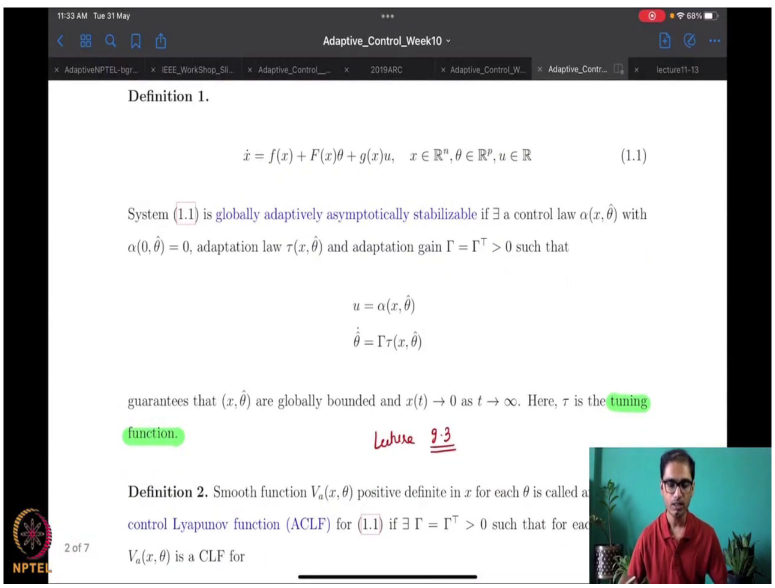
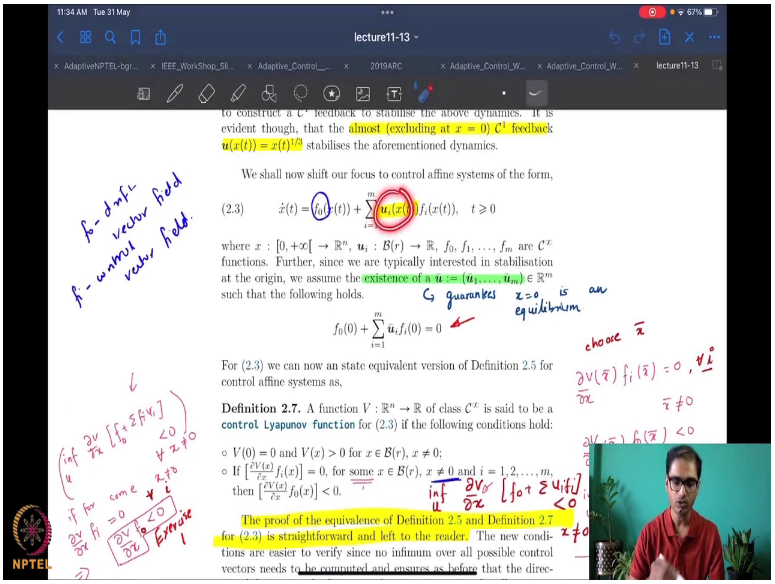






to construct a C^1 feedback to stabilise the above dynamics. It is evident though, that the almost (excluding at $x = 0$) C^1 feedback $u(x(t)) = x(t)^{1/3}$ stabilises the aforementioned dynamics.

We shall now shift our focus to control affine systems of the form,

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(x(t)) f_i(x(t)), \quad t \geqslant 0 \tag{2.3}$$

where $x : [0, +\infty[\to \mathbb{R}^n$, $u_i : \mathcal{B}(r) \to \mathbb{R}$, $f_0, f_1, \ldots, f_m$ are C^∞ functions. Further, since we are typically interested in stabilisation at the origin, we assume the existence of a $\bar{u} := (\bar{u}_1, \ldots, \bar{u}_m) \in \mathbb{R}^m$ such that the following holds.

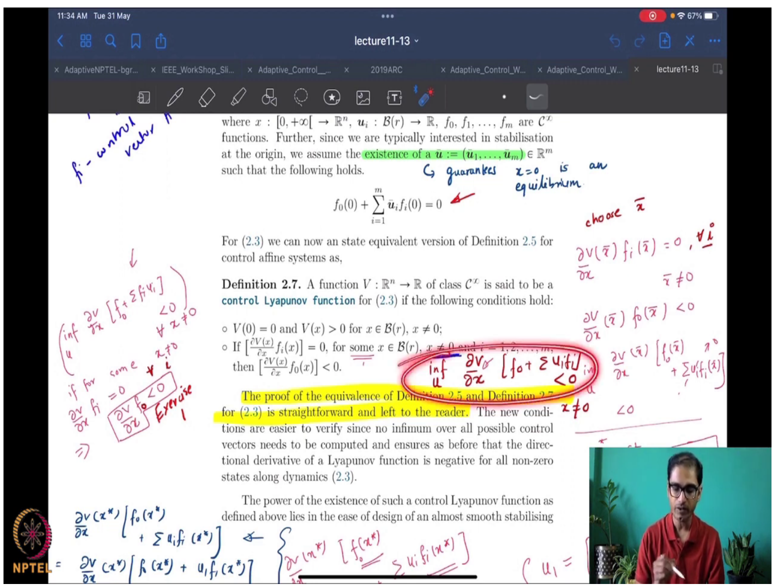
$$f_0(0) + \sum_{i=1}^{m} \bar{u}_i f_i(0) = 0$$

For (2.3) we can now an state equivalent version of Definition 2.5 for control affine systems as,

Definition 2.7. A function $V : \mathbb{R}^n \to \mathbb{R}$ of class C^∞ is said to be a `control Lyapunov function` for (2.3) if the following conditions hold:

- $V(0) = 0$ and $V(x) > 0$ for $x \in \mathcal{B}(r)$, $x \neq 0$;
- If $\left[\frac{\partial V(x)}{\partial x} f_i(x)\right] = 0$, for some $x \in \mathcal{B}(r)$, $x \neq 0$ and $i = 1, 2, \ldots, m$, then $\left[\frac{\partial V(x)}{\partial x} f_0(x)\right] < 0$.

The proof of the equivalence of Definition 2.5 and Definition 2.7 for (2.3) is straightforward and left to the reader. The new conditions are easier to verify since no infimum over all possible control vectors needs to be computed and ensures as before that the direc-



where $x : [0, +\infty[\to \mathbb{R}^n$, $u_i : \mathcal{B}(r) \to \mathbb{R}$, $f_0, f_1, \ldots, f_m$ are C^∞ functions. Further, since we are typically interested in stabilisation at the origin, we assume the existence of a $\bar{u} := (\bar{u}_1, \ldots, \bar{u}_m) \in \mathbb{R}^m$ such that the following holds.

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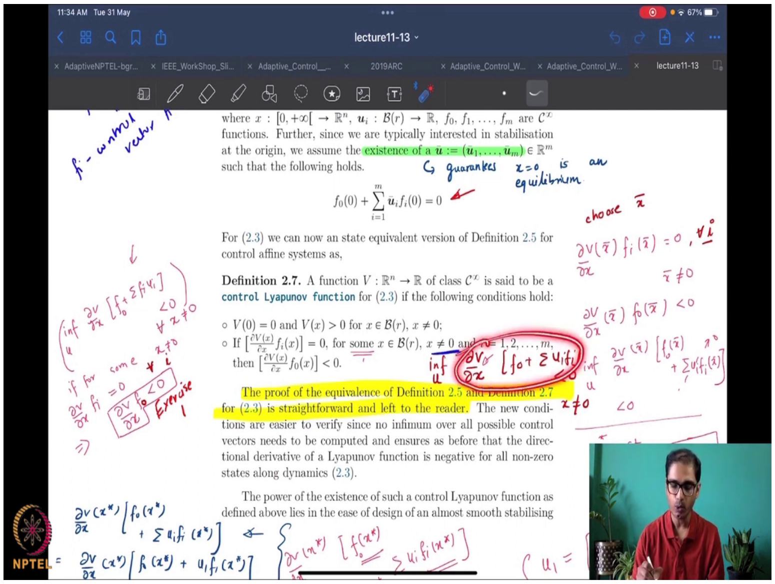
The proof of the equivalence of Definition 2.5 and Definition 2.7 for (2.3) is straightforward and left to the reader. The new conditions are easier to verify since no infimum over all possible control vectors needs to be computed and ensures as before that the directional derivative of a Lyapunov function is negative for all non-zero states along dynamics (2.3).

The power of the existence of such a control Lyapunov function as defined above lies in the ease of design of an almost smooth stabilising



So, what we were doing until last time was that we were starting to look at the tuning function method. And in the quest to understand the tuning function method, we had to talk about the notion of control Lyapunov functions. So that is what we did. We started last session with a control affine system, which is essentially a system which is linear in the control.

It is a multiple input system, but linear in the inputs. And so, there is a drift vector field f 0, and there is control vector fields f i. And we then defined what is a control Lyapunov function which is essentially similar to a Lyapunov function, but the derivative has the control term appearing in it. And therefore, when we talk about the negative definiteness, we take infimum over the all possible control signals.

We take the infimum over all possible control values, in fact, not signals, and then compute the derivative and we want that to be strictly negative for non-zero values of the state in a local domain B r. So, there were two alternative ways of defining this. One was using this infimum, the other was saying that if the control terms contribute nothing, then the drift term has to contribute negative values.

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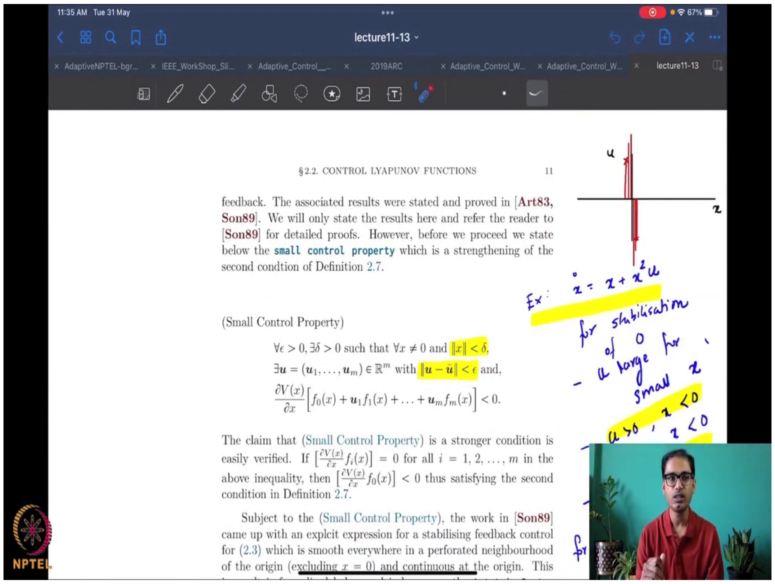

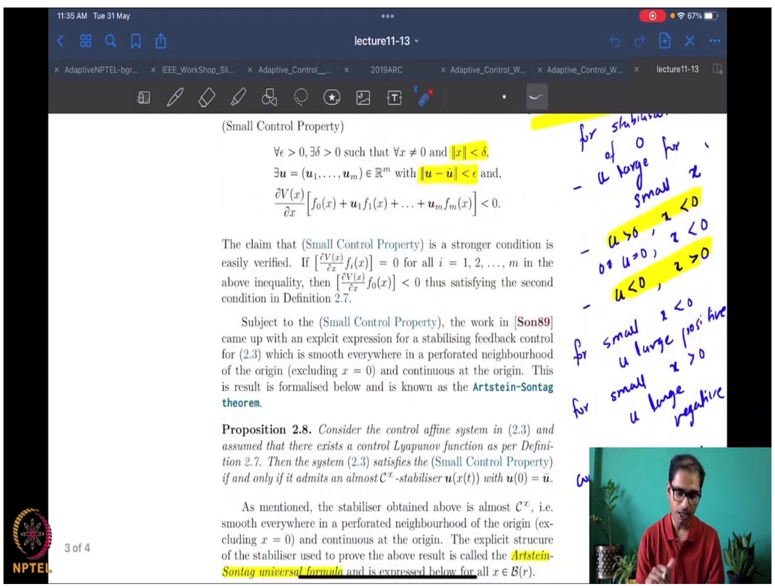

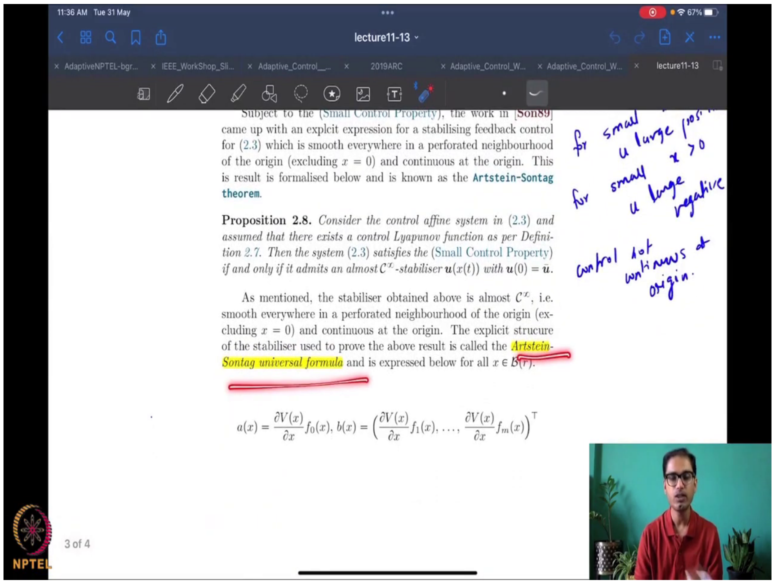

So along with this, and what is called the small control property, which is essentially, essential, which is essentially required for ensuring continuity of the control near the equilibrium, we are able to talk about a necessary and sufficient condition which is called the Artstein-Sontag theorem, which essentially says that having a control Lyapunov function is equivalent to having a C infinity stabilizer, almost C infinity stabilizer.

So, if you have a control Lyapunov function, then the system satisfies the small control property if and only if there is a C infinity stabilizer. So, the if and only if is what was rather important for us. The one side of it, that is if that a function V is a control Lyapunov function then the Artstein-Sontag universal formula actually gives us a way of constructing an almost C infinity stabilizer, which means that the controls so constructed is in fact smooth everywhere but at the origin, which is the equilibrium, where it is continuous at least.

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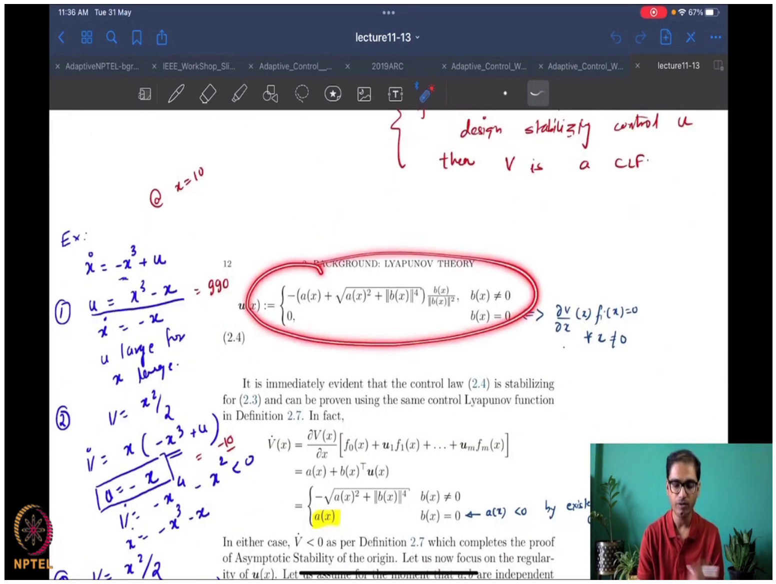

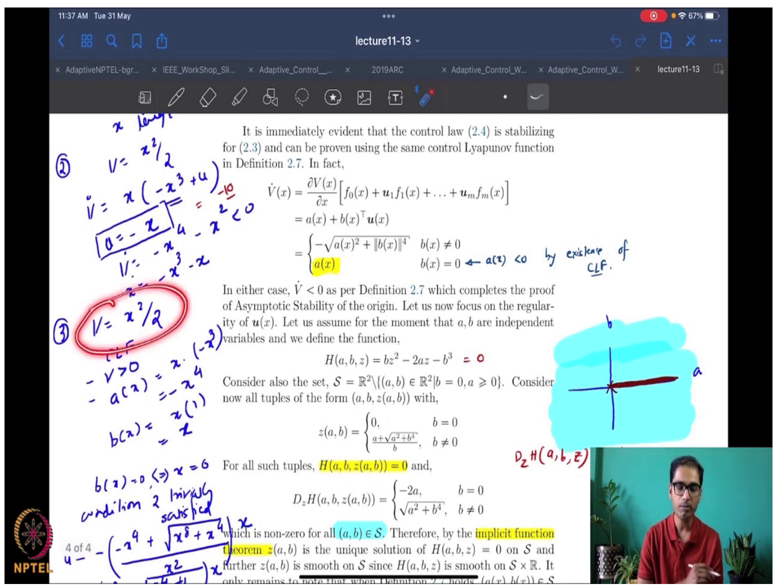

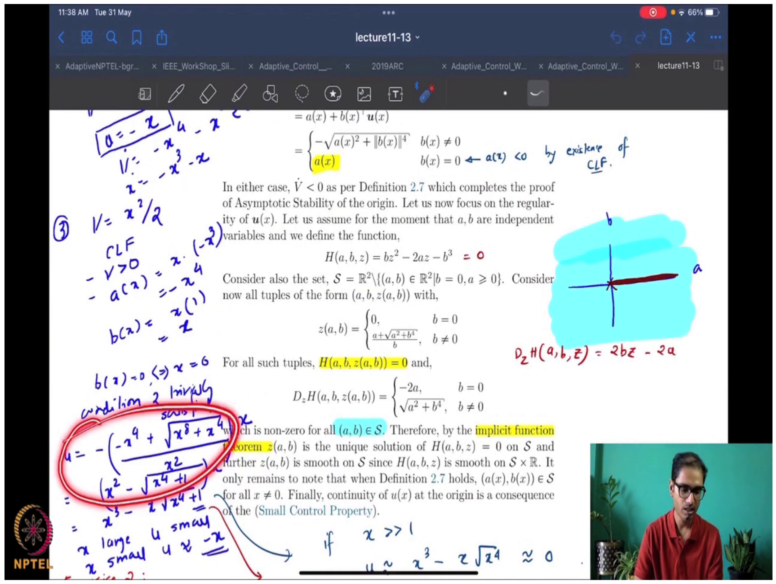

And so, there was a nice expression for such a control law. And the other way around which is also very important for us to understand because that is what we have been doing in the past is that if you design, if you choose a V which is positive definite and smooth and all that, and then you take the derivative and you are able to come up with a control law such that this V becomes negative definite, then this automatically also means that the V function that we chose to begin with was a control Lyapunov function.

And this is what we have been doing, we have been designing our control laws we have and the update laws for the estimates using a suitable Lyapunov function or a candidate Lyapunov function. And so, these were in fact control Lyapunov functions. So instead of choosing whichever control came out of this particular method of just intuitively choosing a control, just like we see in this example, where we took a system x dot is minus x cube plus u and we took a V which was x squared by 2, and after taking the derivative, sorry, we took the V as x squared by 2, and after taking the derivative we somehow have an intuitive idea of what the control should be so we just choose it as minus x.

And this is, this is in fact a smooth controller, everywhere, including the origin so therefore it means that this V is a control Lyapunov functions. So instead of choosing this particular project, this particular choice of controller, that is u equal to minus x, we could have in fact use the information that V is a CLF to construct a controller using the Artstein-Sontag formula.

So, so anyway, so the idea is that we can use either the Artstein-Sontag formula to construct a controller or we can also use our own intuition using a particularly or suitable choice of control Lyapunov function. So, this is essentially, the, this is essentially equivalent methods, essentially equivalent methods. So that is really the idea, so, of what we did last time.

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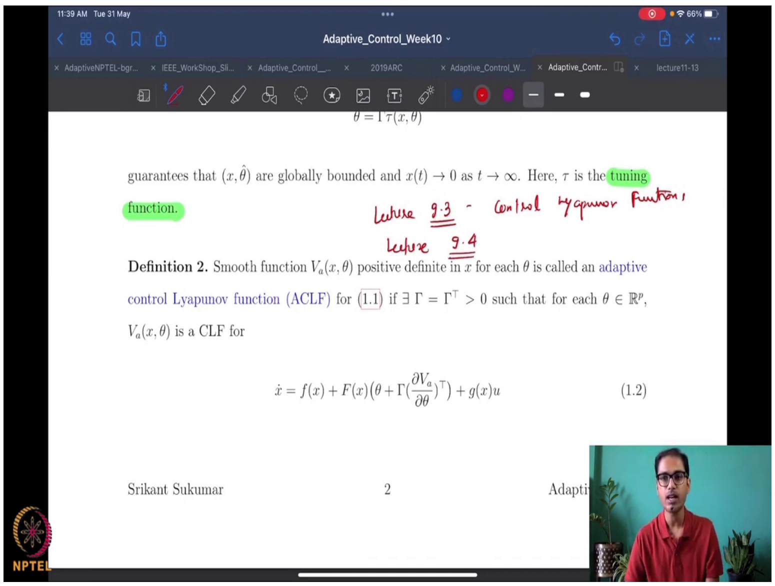

So, what we want to do now, what we want to do now is to continue our discussion on adaptive control Lyapunov functions because if you notice until now, we did not, I am going to, we did not actually talk about any uncertainties in the system at all. We only talked about a standard nonlinear control system. There was no uncertainties any.

So now we are in a, of course in this course, we are in a situation where there is uncertainties also. So, we want to talk about what is called adaptive control Lyapunov functions. So, let me first mark what we did in the last lecture. We did lecture 9.3, which was control Lyapunov functions. And now, we are on lecture 9.4. And this is where we start with the definition of an adaptive control Lyapunov function.

So, we already saw what is, what it means to be globally adaptively asymptotically stabilizable system. I want to remind you of this because this is going to show up in a discussions soon. So a system of the form 1.1 is said to be globally adaptively asymptotically stabilizable if you can find a control law that is a feedback law which depends on the state and the parameter estimate, and also a tuning function tau, again depending on state and parameter estimate, and an adaptation gain such that you have an adaptation law and a control law, and such that both x and theta hat states are globally bounded, and x goes to 0 as t goes to infinity.

So, this is the, what it means for a system to be, for the system 1.1 to be globally adaptively asymptotically stabilizable. So, let us begin with this rather longish introduction, let us begin with our definition 2, which is that of an adaptive CLF. So, what is an adaptive CLF or an ACLF?

It is essentially a smooth function now, of both x and theta, because there is a parameter involved. Remember that theta is a constant. Is it, because we are talking about theta and not theta hat. So, theta is a constant. This is just a constant parameter. So, so essentially an adaptive CLF is, a function V a is called an adaptive CLF for this system if what happens, there exists some positive definite gain gamma such that for every theta, remember for every theta, which is a vector in Rp V a is a CLF for this modified system.

What is the modification? This term. This term is the modification. So, V a, we will see the reason why this modification and not any other modification, V a which is a function of the state and the parameter, the constant parameter is an adaptive CLF if there exists positive definite gain gamma such that for every value of parameter theta, V a is in fact a CLF for this modified system 1.2. That is, it. So, CLF and ACLF are just related by this modified system. That is all. And we will soon see, like I said, why this particular modification and nothing, and nothing else.

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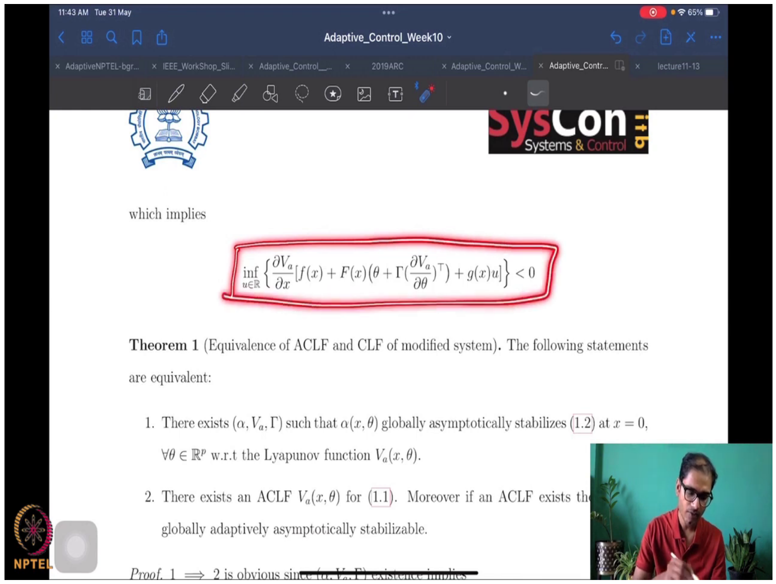
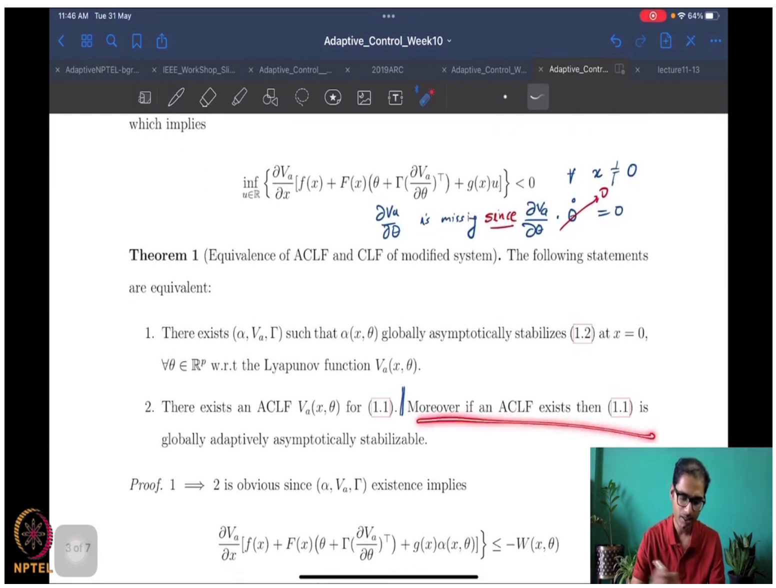
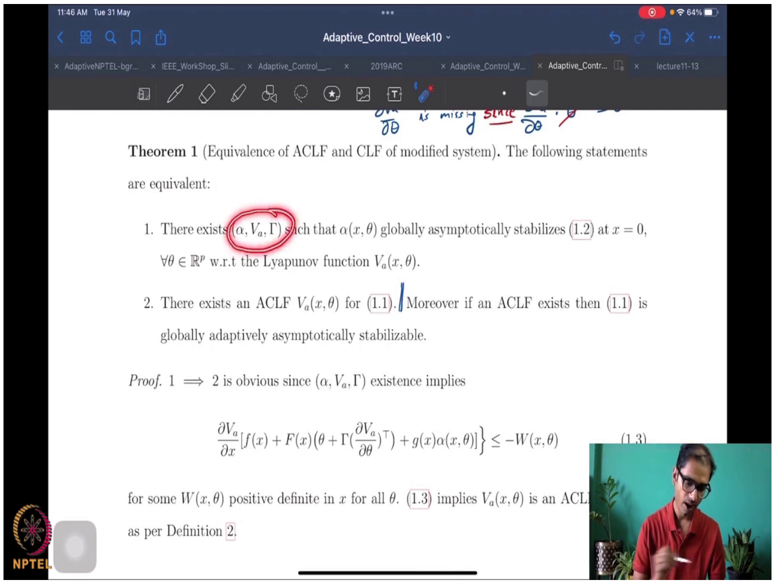






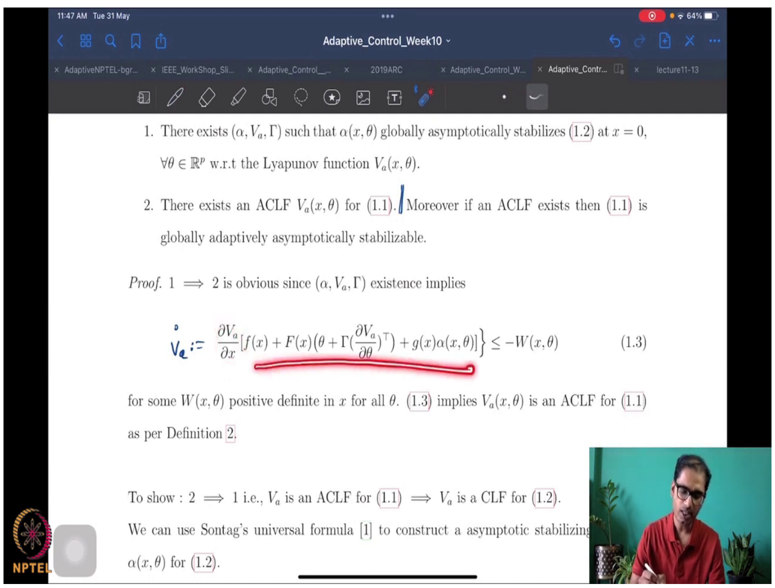
Theorem 1 (Equivalence of ACLF and CLF of modified system)**.** The following statements are equivalent:

1. There exists (α, V_a, Γ) such that $\alpha(x,\theta)$ globally asymptotically stabilizes (1.2) at $x = 0$, $\forall \theta \in \mathbb{R}^p$ w.r.t the Lyapunov function $V_a(x,\theta)$.
2. There exists an ACLF $V_a(x,\theta)$ for (1.1). Moreover if an ACLF exists then (1.1) is globally adaptively asymptotically stabilizable.

Proof. 1 $\implies$ 2 is obvious since (α, V_a, Γ) existence implies

$$\frac{\partial V_a}{\partial x}\Big[f(x) + F(x)\big(\theta + \Gamma(\frac{\partial V_a}{\partial \theta})^\top\big) + g(x)\alpha(x,\theta)\Big]\Big\} \le -W(x,\theta) \quad (1.3)$$

for some $W(x,\theta)$ positive definite in x for all θ. (1.3) implies $V_a(x,\theta)$ is an ACLF as per Definition 2.



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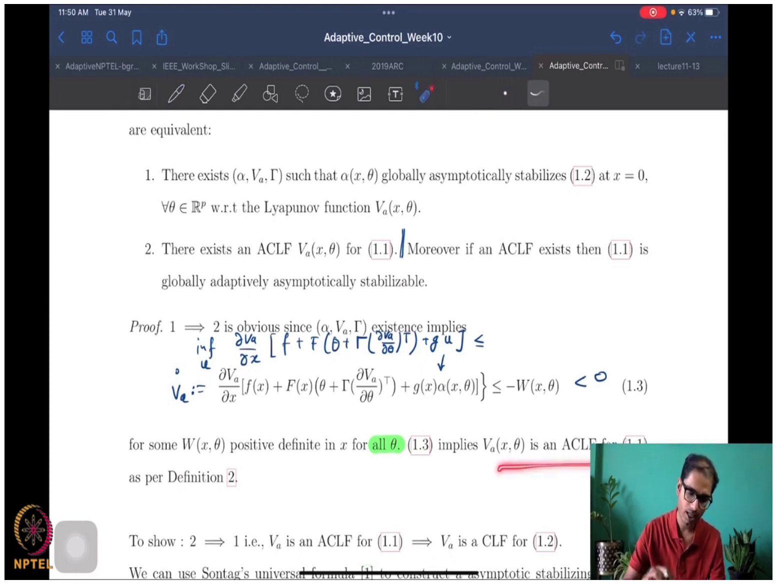
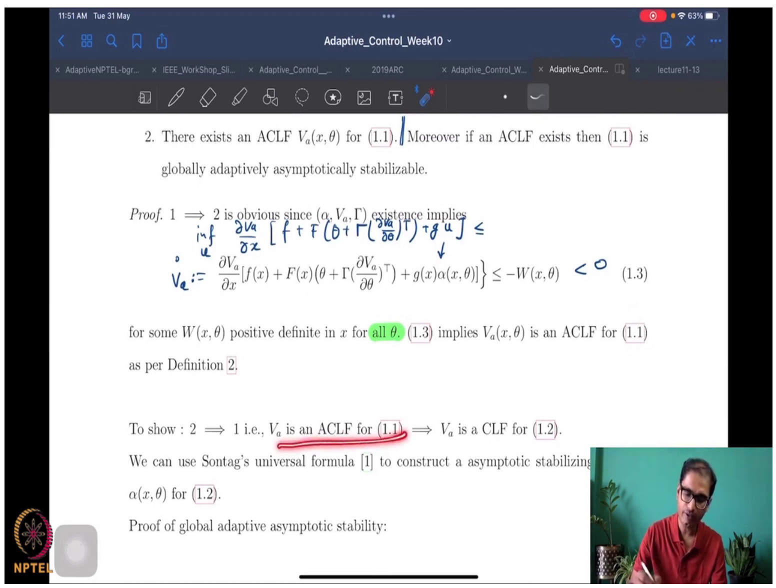
$$\dot{V}_a := \frac{\partial V_a}{\partial x}\Big[f(x) + F(x)\big(\theta + \Gamma(\frac{\partial V_a}{\partial \theta})^\top\big) + g(x)\alpha(x,\theta)\Big]\Big\} \le -W(x,\theta) \quad (1.3)$$

for some $W(x,\theta)$ positive definite in x for all θ. (1.3) implies $V_a(x,\theta)$ is an ACLF for (1.1) as per Definition 2.

To show : 2 $\implies$ 1 i.e., V_a is an ACLF for (1.1) $\implies$ V_a is a CLF for (1.2).
We can use Sontag's universal formula [1] to construct a asymptotic stabilizing $\alpha(x,\theta)$ for (1.2).





This, what does it imply for a V a to be a CLF of this modified system? This is essentially our definition. We have assumed u is in a real number for now. So essentially, we need that the infimum of, over the control values V dot, V a dot, that is del V a del x, this is negative. So, this has to be more precise for all x not equal to 0, for all non-equilibrium values. This has to hold only if x not equal to 0. At x equal to 0, we do not need this to hold because V dot may be 0 at x equal to 0. So, this has to hold for non-zero x only. Now, let us see. Is there anything else I want to say?

You might ask why del V a del theta is missing? But that is because del V a del theta times theta dot is 0 because theta is a constant, which is why del V a del theta term is missing in this expression. Just remember that. So then, we have a, a nice little equivalence result. What does the equivalence result say?

It says that, it is essentially the equivalence of the adaptive CLF and CLF of the modified system. What does it say? It says that there exists a feedback, an adaptive CLF and a gain gamma, such that alpha globally asymptotically stabilizes 1.2 at the origin for all theta in Rp with respect to the Lyapunov function V a. We are calling this a Lyapunov function because now the feedback alpha is already specified. So no longer, there is no longer a control. The control is already specified to be alpha. Therefore, we can treat this as a Lyapunov function directly, not a control Lyapunov function.

So, we are saying that statements 1 and 2 are equilibrium, equivalent. The statement 2 is just this. There exists an ACLF V a for 1.1. So, this rest of the statement is separate. So, let us not worry about the rest of the statement. The equivalence is between existence of ACLF of V a for 1.1 which you already defined, and existence of feedback V a and gamma such that alpha globally asymptotically stabilizes equilibrium 0 for all theta with respect to the Lyapunov function V a.

It has the same function V a, but here it is an ACLF and here it is a Lyapunov function because here the feedback is already been implemented. So, so what, why the equivalence between 1 and 2? Again, like I said, the equivalence is only until here. This is a separate statement, that is not required, I mean that is not part of the equivalence, but anyway I mean this is important too. We will read it soon.

So, 1 implies 2 is obvious, we say, because existence of a alpha V a gamma combination implies what? Implies this is true, implies this sort of an inequality holds. What is this sort of an inequality? This is, the left-hand side is just the definition of V a dot. This is just, I am going to rewrite it, sorry. V a dot along the system trajectories. Notice again, that theta is a constant still. We just do it for all values of theta.

So, theta dot is constant, it is just a constant parameter. So therefore, there is no del V a del theta term. So, this is just del V a del x times x dot. The only difference is that now in x dot, the control is substituted with an alpha. And anyway, the gamma appears here. What do we know?

That if you have an alpha V a gamma combination available or a tuple, three tuple available, then you will always, which globally asymptotically stabilizes the origin with respect to the Lyapunov function V a.

It essentially means that the V a dot along the system trajectories substituting for alpha is going to be less than equal to some negative definite function which means W is a some positive definite function in x for all theta. So, this is important. Always for all theta, because theta is not, cannot play a role. Just by changing theta, I cannot lose my asymptotic stability.

So, what does 1.3 imply? 1.3 immediately implies that V a is an ACLF. Why? Why, you might ask. What do you need for an ACLF? Let me make this smaller. You need that for sum, you take the infimum over all possible values of control. And you want it negative. Yes? For non-zero x, of course, for non-zero x, of course.

Here, I am saying there exists one value of control for which this is negative definite. So, if there exists one value of control for which it is negative definite, that it is negative, then it is done. I am satisfying this already. This implies this. Because I can find one choice alpha for which this holds, because I am taking infimum, the smallest over all possible choices of control.

And here I am already saying there exists this choice alpha such that this holds because this is the same as this, substituting for alpha against u. Therefore, because I can find a u equal to alpha such that this holds, it means that infimum also has to be smaller than that at least. Infimum cannot be larger than that. Infimum of this for all values of u has to be smaller than what you get for alpha. So, it is as simple as that.

So let me, so essentially, I mean if you want me to write it, I can write it as inf over u del V a del x f plus cap f theta plus gamma del V a del theta transpose plus g u has to be less than equal to this guy, is always less than equal to this guy, which means this is, which means that you have what you want. So therefore, V a is an ACLF for 1.1. And that is what is, I apologize, and that is what is this statement, that there exists an ACLF, V a is exactly this ACLF, V a is exactly this adaptive CLF.

Now, if you want to show that 2 implies 1, that existence of ACLF implies being able to find stabilizing feedback alpha and gamma, is pretty obvious because ACLF definition in fact contains the gamma itself because this is what you have. ACLF definition, in fact, contains the gamma itself. And now, because V a is, V a is an ACLF for 1.1 implies V a is a CLF for 1.1, is a control Lyapunov function for 1.1 because that is the definition.

And if you have a control Lyapunov function for a system, what do I know? I know that I can use the Artstein-Sontag formula to construct a feedback alpha because V a is an ACLF of 1.1. By definition, it is an ACLF for 1.2. And if you have a control Lyapunov function for any system, this is equivalent to being able to construct a feedback, almost smooth stabilizing feedback, alpha. So, you have constructed an alpha.

And the gamma is coming from the definition of ACLF itself, the gamma is right here. Gamma is right here. So, that is it. So, these two are equivalent. So, this is the very straightforward idea. I mean you are saying, it is not a big deal. You are saying equivalence of ACLF is essentially the same as CLF of a modified system. Anyway, this should be sort of obvious from the definition, and that is what we have used, except for the Artstein-Sontag universal formula.

Now, there is an additional statement here which is an important thing and we do want to try and prove that. If an ACLF for, exists for 1.1, if you have an adaptive CLF of 1.1, then it is globally adaptively asymptotically stabilizable. So, existence of an ACLF gives you an adaptive control law for the system. That is the idea. And that is what we want to see now.

And that is an adaptive controller for the original system, not for 1.2 which is the modified system. You get an adaptive controller for this system. So, having an ACLF for 1.1, which is this guy means that I can construct an adaptive controller for the system. And that is what we want to see, how to do that, or why that is the case.

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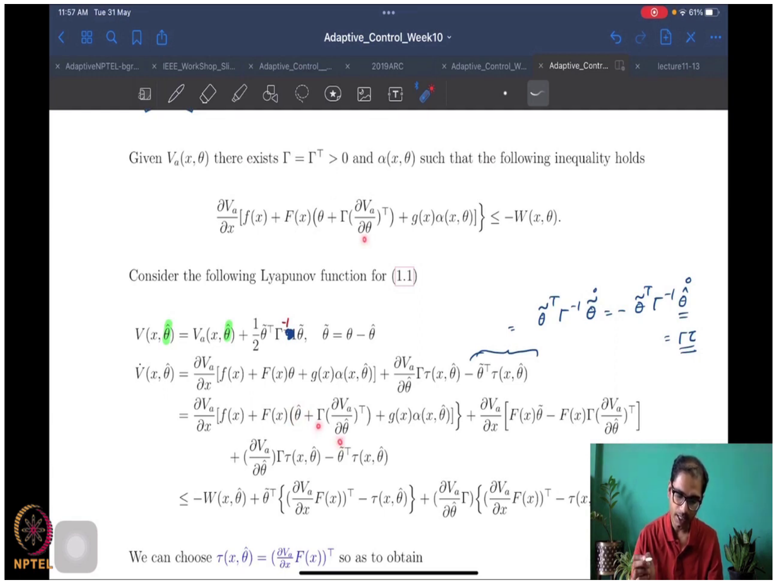

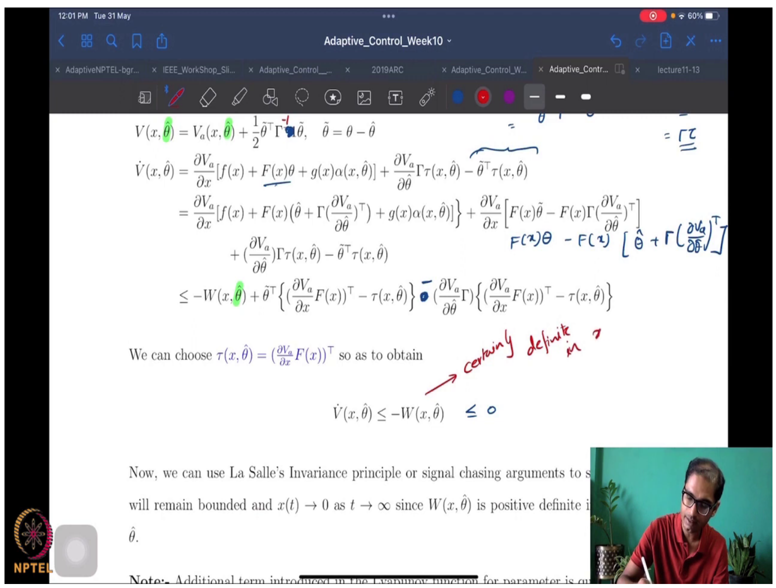

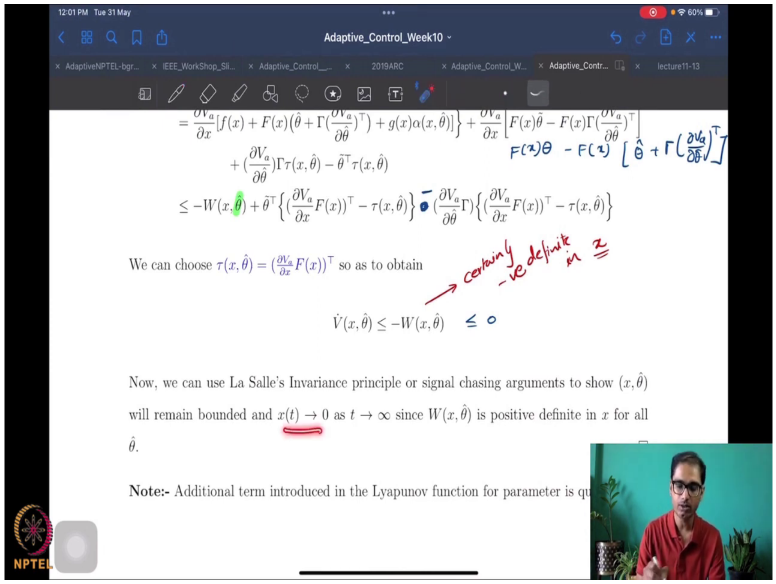

So, let us sort of look at this proof, a little bit. If I am given this V a, it implies that there exists a positive definite gamma and an alpha because by the equality, by the equality of the, by this theorem 1 essentially, like existence of an ACLF implies all of this happens. So, which means that you have this sort of a relationship already with this gamma and alpha.

So now, what we do is we consider a Lyapunov function for the original system. And what is this Lyapunov function? We construct in a specific way. We take a V a, but now with a theta hat. Notice, it is not because, because you are talking about adaptive control now. So, theta is unknown. So, we replace all the thetas by the theta hats, all the thetas get replaced by the theta hats.

And then we add a term, again, a quadratic term corresponding to the parameter error, theta tilde. And of course, scale it with a gamma inverse. So, this is, sorry, there is a typo here. This should be gamma to the power minus 1, this is gamma inverse, the same gamma. And now we take the derivative carefully. All we have done is we have taken the same V, but in place of the theta which is an unknown, we have used the estimate theta hat. And of course, we are going to prescribe an estimation law. We do that. So, we take a derivative here.

Now, theta hat is no longer a constant. So, we take derivative with respect to both of them. So, I have del V a del x times x dot, essentially the original system not the modified system, the original system, and I have a del V a del theta hat times this tuning function, gamma times tuning function. We do not know what the tuning function is yet. We will actually compute it.

And then we have theta tilde transpose times gamma inverse times gamma, the tuning function. So, this term comes from, this is actually equal to, I mean with the negative sign, is equal to theta tilde transpose gamma inverse theta tilde dot, which is minus theta tilde transpose gamma inverse theta hat dot, which is equal to of course gamma times tau.

And that is what you have here, that is what I get from here. So, once you make this substitution, what I want to do is I want to replace this with the modified system because, because this inequality I have on the modified system. So, what do I do? I take f x and replace it in terms of the modified system. So, all the theta's become theta hats.

So, this is theta hat plus gamma del V a del theta cap transpose. And everything else remains exactly the same, if you notice. But now because I have added this term, I also have to subtract this term. So, I take del V a del x common. Then I have, what do I have here? I have this guy coming in. So, that is f x theta. And then I have subtraction of f x theta cap plus gamma del V a del theta cap transpose.

So, these two combines to give me f x theta tilde, and this term is written as it is. So, this term gets written as it is and these two combines to give me f x theta tilde. And then, of course I have these two terms. So, del V a del theta cap gamma tau minus theta tilde transpose tau. So, these two terms remain as it is. Now, I know that this guy is now minus W x theta cap.

Again, remember because all the thetas were replaced by theta cap. So, this is not theta, W x theta, but minus W x theta cap, that is the only difference. And then I have these terms now what do I do? I take the theta tilde terms common. So, this theta tilde term, this is a theta tilde term. So, I take a transpose because everything is a scalar. So, I take transpose of scalar as much as I want.

I take theta tilde transpose so I get del V a del x f x transpose. And then I have a theta tilde transpose here. So, I get a minus tau. Then I take del V a del theta cap times gamma common, this guy. So, I get one term, I get is, let us see plus tau. So, in fact this is, this should be a plus tau here, and this term, if you see, del V a del theta cap gamma, gives me a del V a del x f x whole transpose with a negative sign. So actually, this should be a negative sign. So, there is a sign error. This should be a negative sign.

But notice one interesting thing. This term in this curly bracket and this term in this curly bracket is exactly the same. These are exactly the same terms. So, if I choose my tuning function as this quantity, both of these non-definite terms go to 0. So, if I choose tau as this guy, then this goes to 0 and this is 0. And both of these become 0 and I am left with V dot x theta cap is less than equal to minus x W x theta cap which is now only semi-definite, notice.

Why? Earlier it was negative definite, because theta was not a state, it was just a constant parameter. But now, theta cap is in fact a state because it appears in the Lyapunov function also. Therefore, this is no longer definite but it is negative semi-definite. And, but, but we know that we can use, it was definite, if, this, this W x theta cap was certainly definite in x.

So, otherwise certainly definite in x correct because otherwise I cannot claim this to be negative definite. If it is not definite in x, then this is not negative definite. And that would be ridiculous because that is what it means to have an ACLF. So, this is definite, this W is certainly definite in x, positive definite in x.

So, so therefore, this whole thing is certainly negative definite in x, which means that I will be able to show that x goes to 0 by La-Salle invariance or Barbalat's lemma signal chasing, and that is it, because it is, W x theta hat is positive in x for all theta hat. And you will have bounded x and theta hat.

So, x theta hat will be bounded because this is at least negative semi definite, therefore V is not increasing over time, therefore the states here have to remain bounded. And because V a is positive definite because it is a CLF, so therefore it is positive definite in x, therefore it is, both x and theta tilde have to remain bounded because V is not increasing. And further, x will certainly go to 0 as t goes to infinity, this is by signal chasing type analysis.

So, the important thing to remember is that the additional term introduced is a quadratic term in the parameter. And we have essentially proved that the original system 1.1 by virtue of having an ACLF is in fact adaptively asymptotically stabilizable. And that is why we had this funny looking additional term so that this sort of a nice cancellation happens. That was the purpose of this additional term.

So, what have we discussed today? We started with our discussion on, of ACLF, an adaptive control Lyapunov function. And how it is equivalent to having a control Lyapunov function for a modified system. And we also proved that having this control Lyapunov function for a modified system means that for the original system, we have an adaptive asymptotic stability property.

That is, we can design an adaptive controller which is a feedback and an adaptation law such that the original system has bounded states and the x states, that is, the system states and not the parameter error states, they go to 0 as t goes to infinity. So, we will continue in this vein, talking about more about the tuning function method in the subsequent session also. This is where we stop. Thank you.