Nonlinear Adaptive Control
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Week 9
Lecture No: 51
Control Lyapunov Function

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Hello, everyone. Welcome to yet another session of our NPTEL on Nonlinear and Adaptive Control. I am Srikant Sukumar from Systems and Control, IIT, Bombay. We are into the ninth week of this course on Adaptive Control, and I think all of you would agree that we have seen a sufficient number of analysis and design methods for all of you to be able to actually pick up real applied practical problems from your field and directly apply the methods that we have discussed. I would strongly encourage all of you to do so and report your results to me or to the community or actually develop and design new technologies with these methods. I mean that is essentially the aim and scope of what we want to do here.

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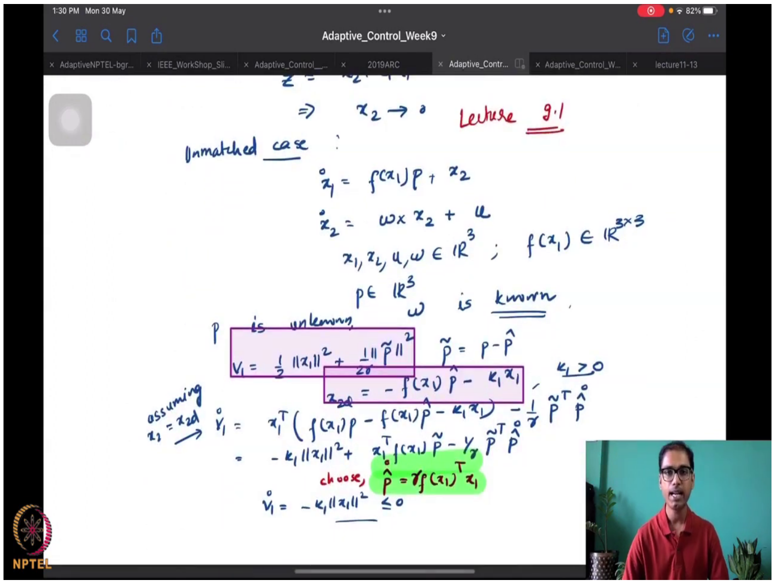

So, what we were doing until last time was of course, this sort of, we were essentially looking at examples at a particular example of the unmatched design, and we did the design for this vector system using both the standard adaptive integrator backstepping

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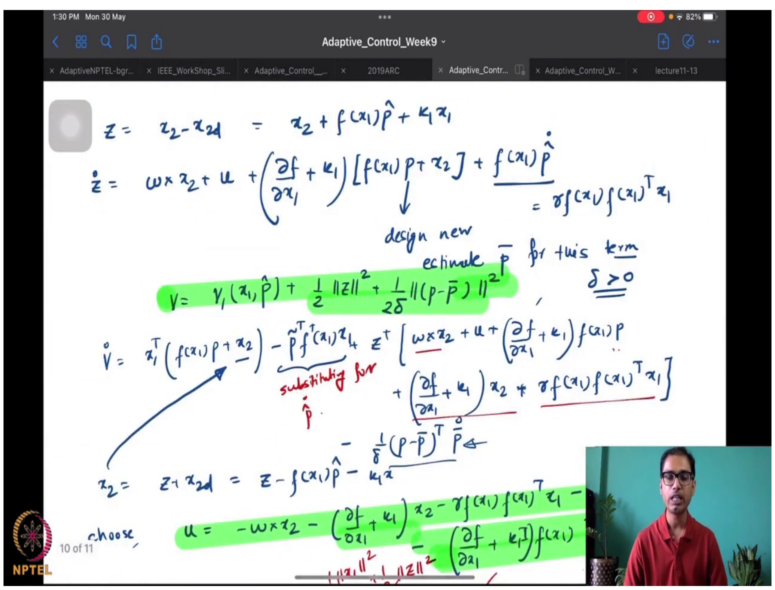

Which leads to over parameterization.

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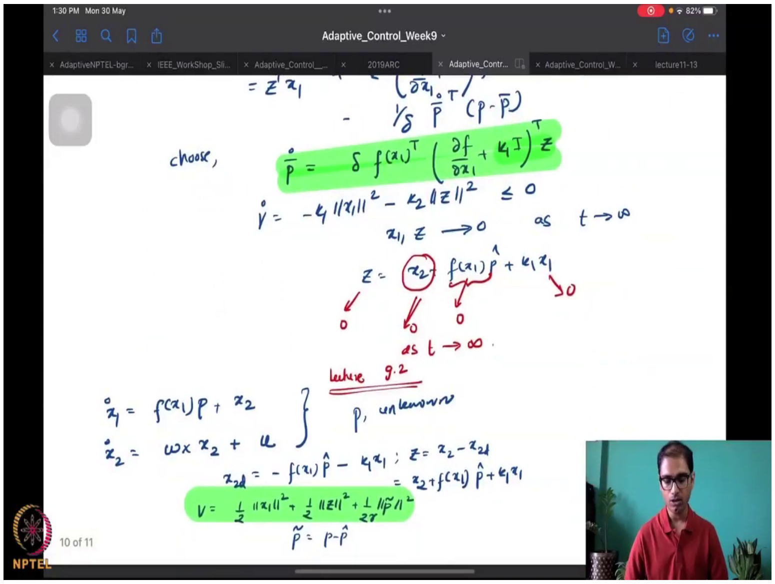

And also, the extended matching design method which does not lead to overparameterization.

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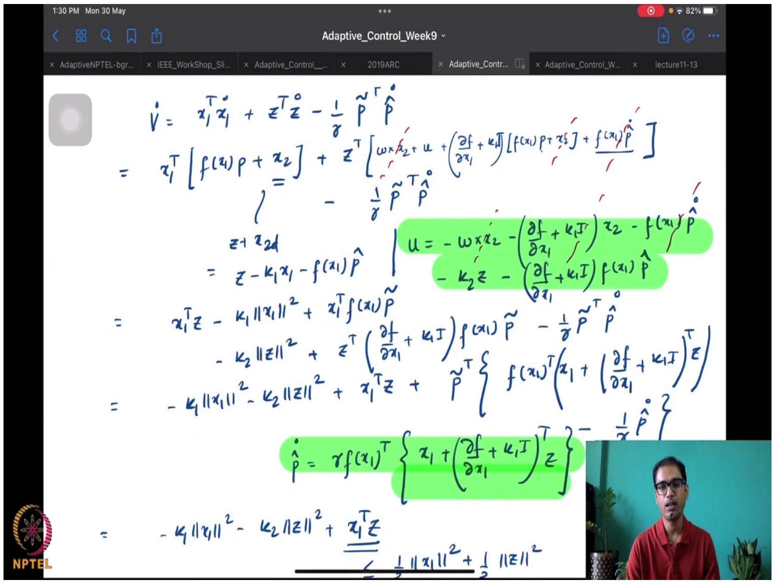

But yields a control law which contains the derivative of the parameter, parameter estimate.

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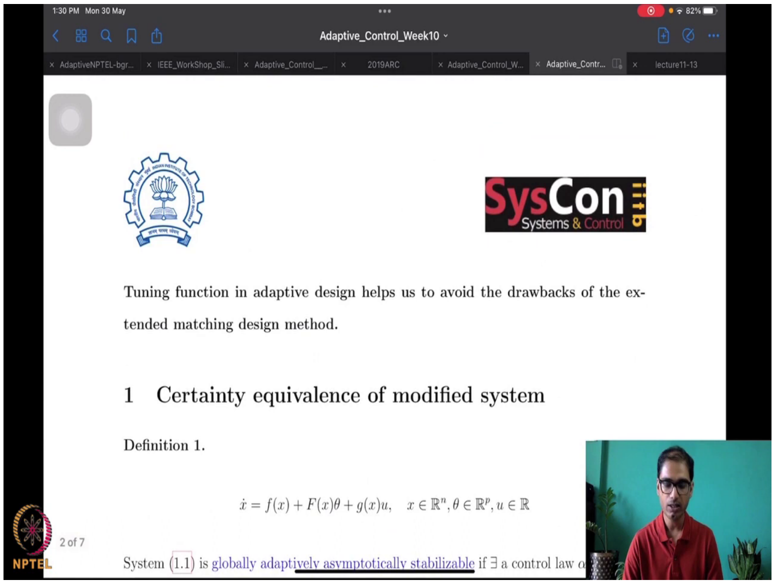

So, we then wanted to alleviate this issue and so we have started looking at the tuning function method for adaptive design.

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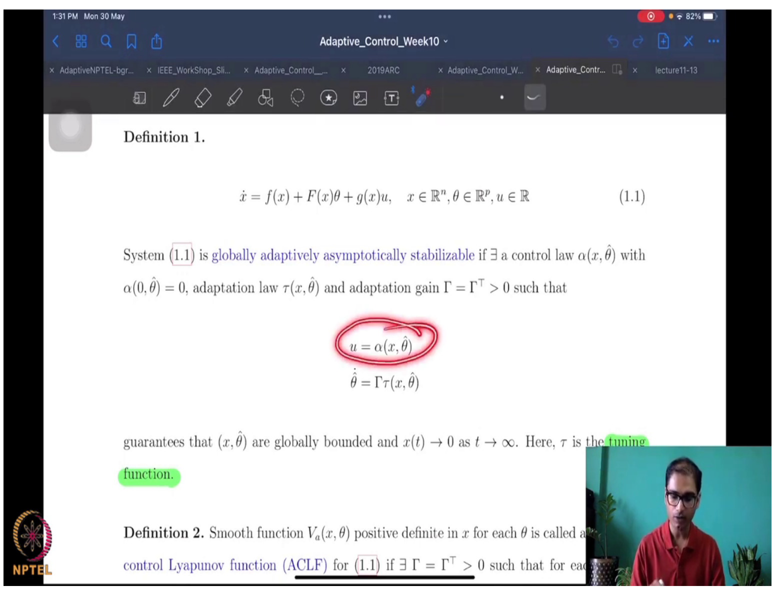

And the first basically piece that we saw last time was the definition for what is globally adaptively asymptotic stability. And essentially, this meant that for a system of this form 1.1, we are looking to have the existence of, and a feedback law which depends on the state and the parameter estimate.

And a parameter update law which again has a, depends on the tuning function tau and adaptation gain, tuning function tau and an adaptation again gamma, and if you have these two which guarantees that x and theta hat remain globally bounded and also go to 0 as t goes to infinity, then you can say that your system is globally adaptively asymptotically stabilizable.

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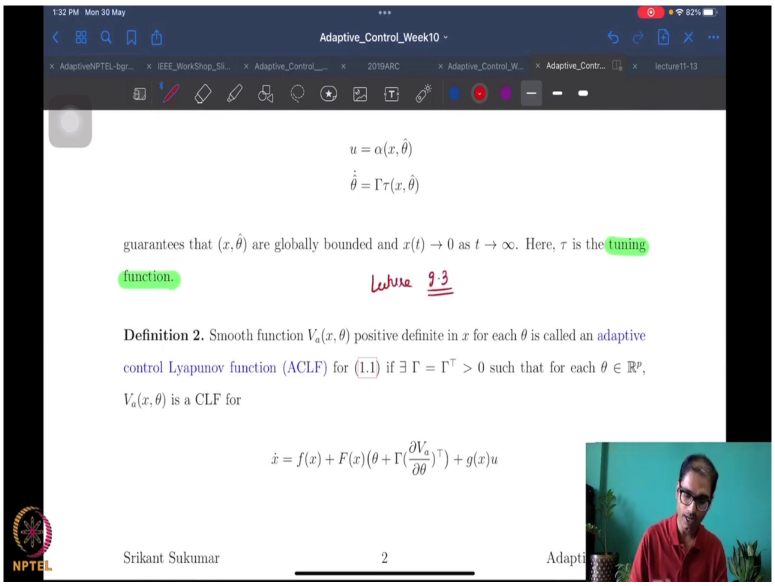

So, this is where we were and we want to continue from here. So, I am going to mark my lecture as lecture 9.3, I believe, right here, as lecture 9.3 right here because I believe I was doing. Let us see 9.2 here, excellent, excellent. So, we want to talk about adaptive control Lyapunov functions, but before we can actually do that, we first need to know what is a control Lyapunov function.

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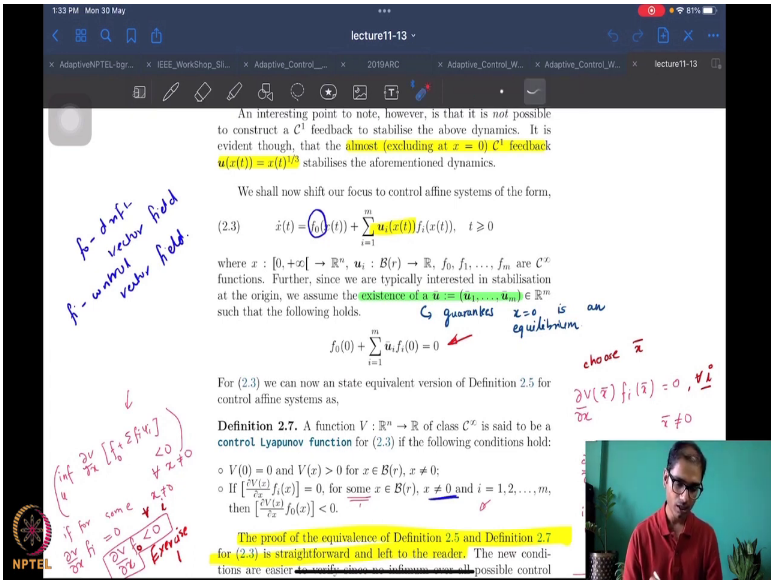

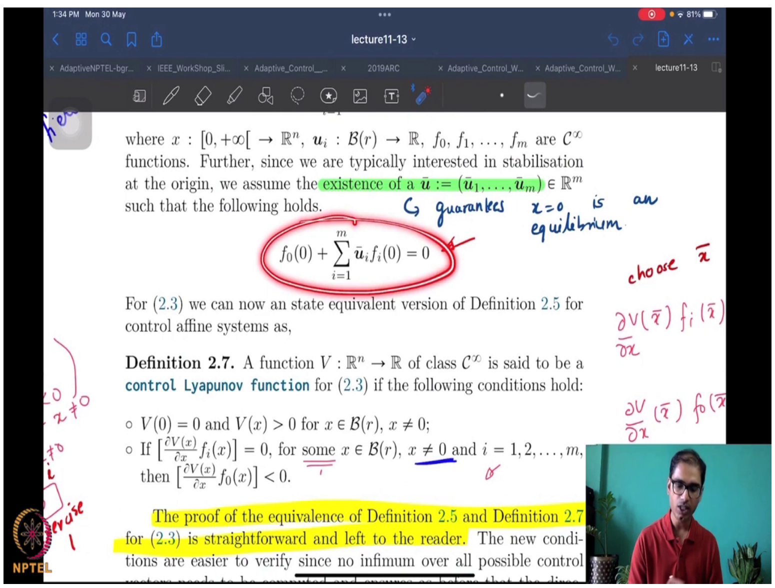

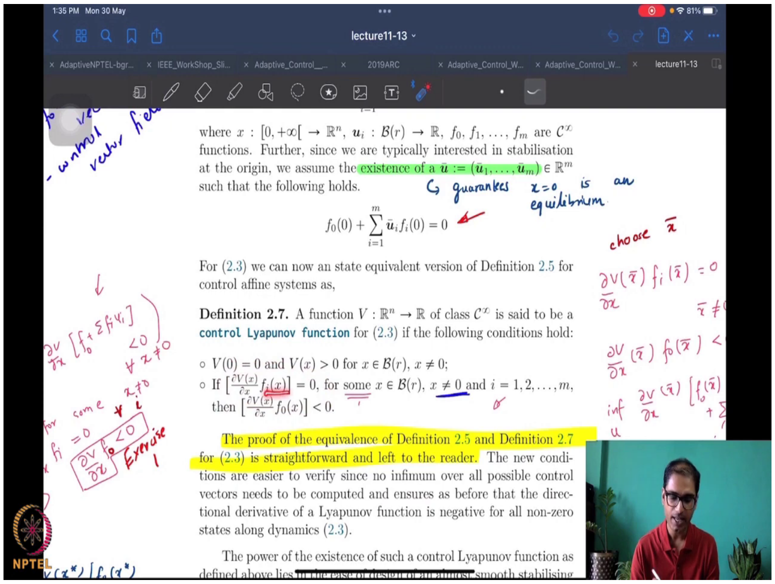

Those of you who have taken some nonlinear control course would know this notion but for those of you who have not, I want to talk a little bit about control Lyapunov functions. So, I am actually picking up from some notes that I have for my nonlinear systems course. I will of course post these. But I will cover a little bit of the material from here, not the proofs and so on and so forth. But this should be enough for you to follow what we are going to talk about.

So, the first thing is we look at what is called control affine systems. What is a control affine system? It is essentially a non-linear dynamical system where the dynamics are linear in the control. So, control affine means linear in the control. And usually it is written as x dot is f 0 plus summation u i f I, u i times f i where i ranges from some 1 to m, so there are m controllers. You can see that there are m control laws.

And each of these f because, x is in Rn, therefore each of these f 0s and f i is also mapped to Rn. And they are sufficiently smooth, they are C infinity functions. This f 0 which does not get connected with the control is called a drift vector field and the f i's which are connected to the control are called control vector fields.

Notice that there is no unknown parameter and all that such here because this is just standard nonlinear control. We are not talking adaptive control and unknown parameters yet. So, now for systems of this form 2.3, we of course, first assume that there exists a u bar such that 0 is an equilibrium.

So, we of course want to talk about the 0-equilibrium stability and therefore we want the existence of u bar such that 0 is in fact an equilibrium. Then, we define a control Lyapunov function for this system as a function V which again takes the states and maps to real numbers. Again, and is assumed to be smooth. C infinity means smooth function such that it is 0 at 0, and it is positive definite for all x in some local domain around the origin.

B r is basically a ball of radius r around the origin. So, you want, so this is standard positive definiteness. As you can see, the first line is just asking for positive definiteness of this function V. This is standard requirement for also a candidate Lyapunov function, I hope all of you remember. Even for a candidate Lyapunov function, you need sufficient smoothness and positive definiteness. So, this is standard.

But the second piece is something slightly different. It says that if the partial of V with respect to x times f i x is 0 for some x and all i. So, basically what am I saying? I am saying that if I take the derivative of V with respect to this system, what do I have? I will have something like del V del x f 0 plus summation f i u i, what you see here on the left.

And now if I multiply only the control terms, I get del V del x f i. And if all these control terms are 0 for a particular value of the state then I want for sure that del V del x f 0 be negative, strictly negative definite. So, whenever x is such that the control terms become 0, I want the drift term to be strictly negative. This is what it means to be a control Lyapunov function.

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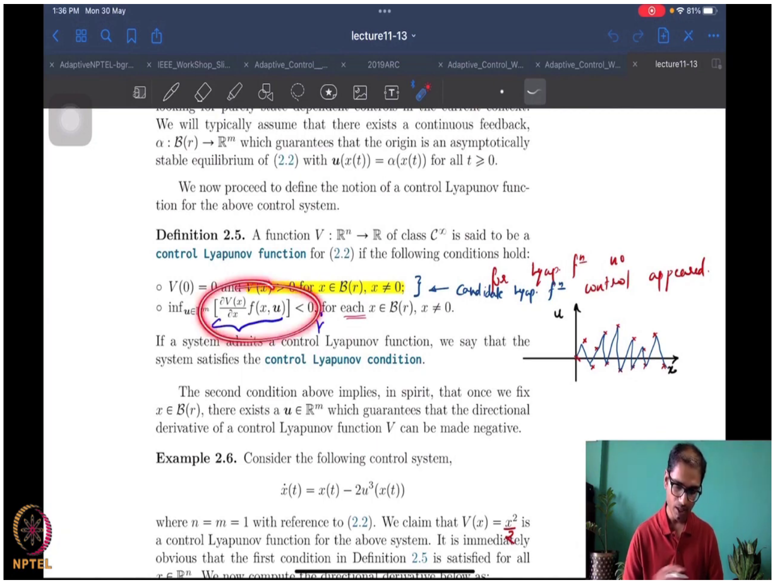

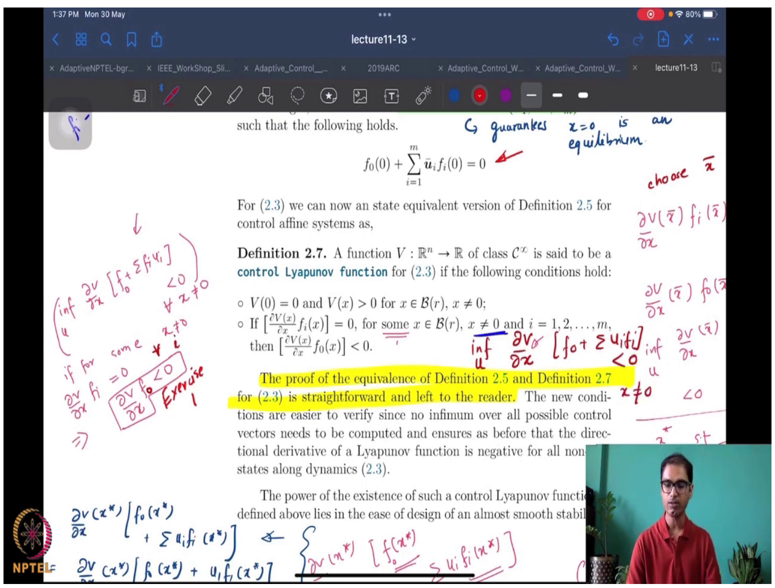

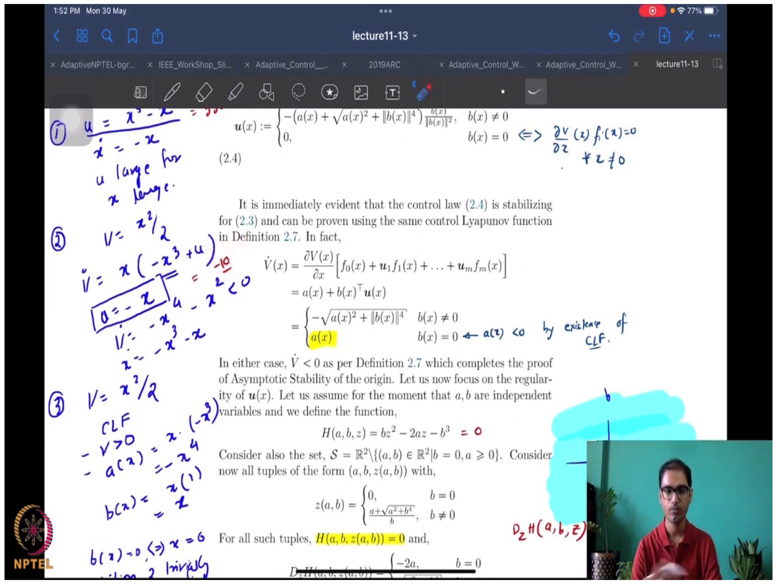

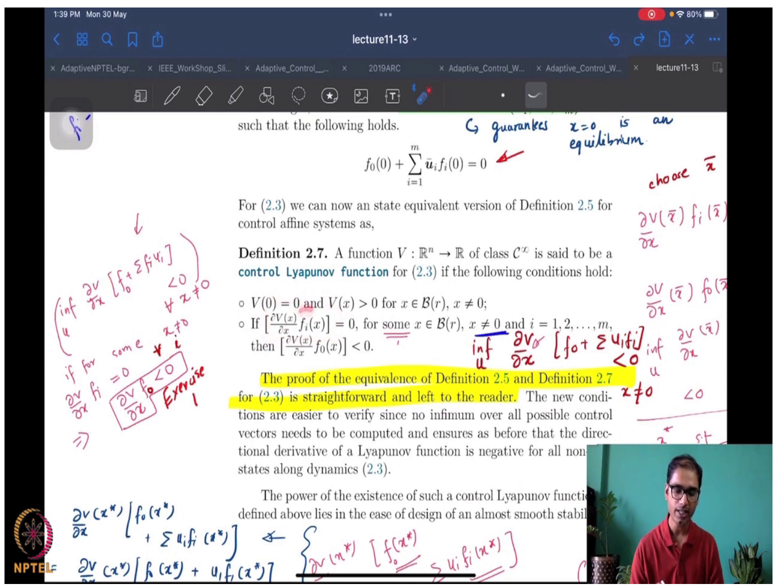

And we also say, I mean although I did not really talk about the earlier definition, we also say that this is equivalent to definition 2.5 which is the original definition of a control Lyapunov function which is in this form, but let us not worry about this form. What we want, what is the equivalent version is that there exists u such that inf over u, well actually I should not say that.

We want infimum over u del V del x f 0 plus summation u i f i should be less than 0 whenever x is non-zero. So, this is essentially what you need from a control Lyapunov function. By the way, I, I am not proving it here. I mean the proof is again in these notes and I will post these. If you are interested, you can look at the proof, but that is not required for our scope of our discussion here.

So, these two are equivalent. This point and this are equivalent. Either you can say that if I take an infimum over the control, and this is basically V dot, because it is del V del x times x dot, and this has to be strictly negative for all non-zero x. Or I can say it equivalently as if the control terms do not contribute anything, they are exactly 0 for a particular value of x. Then at that particular value of x, this has to be negative.

So, this is what it means to be a control Lyapunov function. What is the difference here? When we are talking about standard Lyapunov functions, there was no mention of the control. Control never appeared in V dot. We always talked about Lyapunov analysis for systems of the form x dot is f t x. There was no control. Here, there is a control. So, therefore, we have to account for what the control does to the system.

Therefore, there is the concept of having a infimum. So, there is the concept of having an infimum. And similarly, there is, in this case also, here, we do not have an infimum in this particular version, but we sort of have partial with respect to V of x, del V del x multiplied by the f I, and which, which if it is equal to 0, then you want del V del x times the f 0 to be strictly negative.

So, we have to account for the contribution of the control when talking about the definiteness. Essentially, these are, the second condition is sort of a definiteness condition on V dot, which is the same as what you had for Lyapunov functions also. But here we have to account for the contribution of the control. And that is what makes it a control Lyapunov function.

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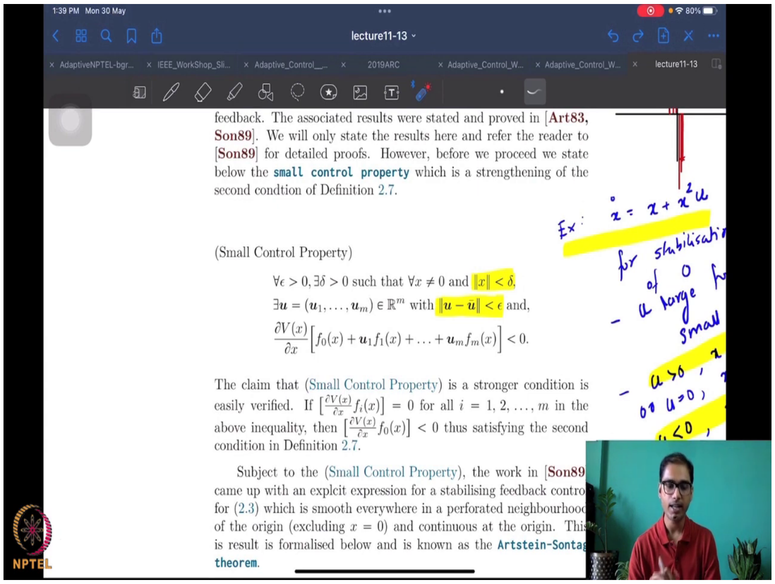

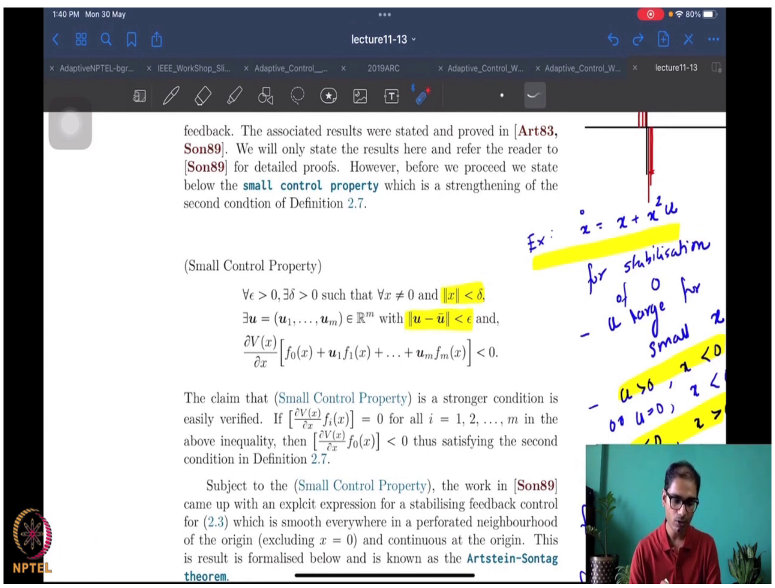

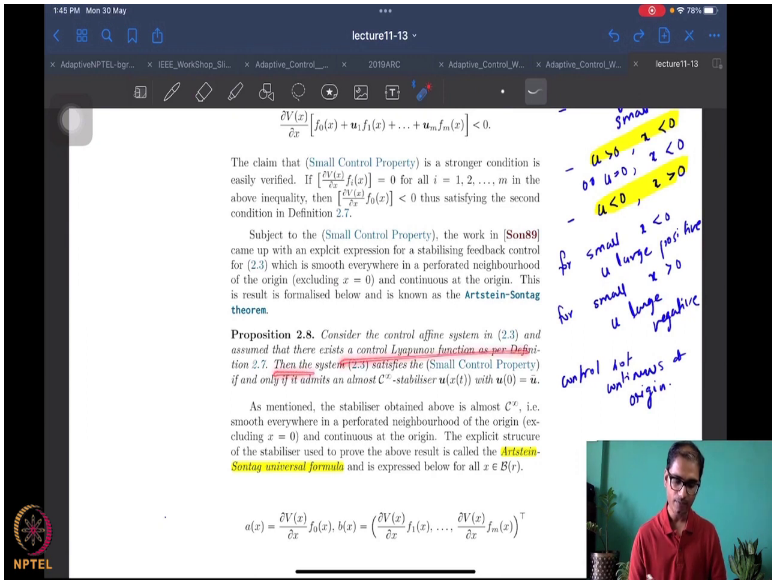

Now, why do we care about control Lyapunov function? There is very nice results by Sontag and Sussmann, which and of course also, also Artstein which say that it is possible to construct smooth, almost smooth feedback if there exists a control Lyapunov function. So, existence of a control Lyapunov function is equivalent to being able to construct a feedback. And this is where very strong, very powerful result.

But before we can say that we have to also talk about what is a small control property. So, what is the small control property? It says that, essentially, it is in this epsilon delta form. It says that for all epsilon positive, there exists a delta such that for all non-zero x and x smaller than delta there exists the control which is close to the equilibrium control in terms of epsilon. And you have the V dot to be strictly negative.

So, this is sort of the interesting, interesting thing, I mean. What is the interesting thing? You know that at the equilibrium, the control is u bar. We know that at the equilibrium, the control is in fact u i. Now, what we are saying is that close to the equilibrium, that is, close to the equilibrium value of the control, there exists such a control u, which is close to the equilibrium value which makes my V dot negative definite. So, there exists such a control.

So, this is what it means to have small control property. It means with a small control, if I am starting close to my equilibrium, with a small control I can sort of make my V dot negative. And if my V dot is negative, it means that I am going close to the equilibrium. That is the philosophical idea of the Lyapunov analysis, that V dot negative definite means asymptotically converging and also asymptotically stable.

Therefore, we are saying that with a small control, which is, we are saying which is a control which is close to the equilibrium value of the control, I can push my states starting from near the equilibrium to the equilibrium value. So, near the origin in this case to the origin. So, just to give an example, it is not always possible to have small control for stabilization.

If you look at a system of like this, x dot is x plus x square u, what if you think about this kind of a system, if you, if you want to stabilize the origin, if you want to stabilize the origin what would you have to do? You would sort of have to get a minus x sort of, to cancel this guy out. What I want to have a minus x.

And so, what it means is that if say x is, if x is small, then this is really tiny, really tiny. So, u will have to be very large and negative. u will have to be very large. So, let us look at different situations. u has to be large for small x. That is important. If u is positive. So, u is positive for x negative. If x is negative, this is anyway positive, does not matter.

If x is negative, u has to be positive because it has to counter this guy. Similarly, if x is positive, u has to be negative. So, the sign of the u is opposite to that of x. So, near the origin, that is, the equilibrium, u has to be opposite sign. So, x u has to be positive for negative x, u has to be negative for positive x.

But as you go closer and closer to the origin, u needs to be larger and larger. So, very close to the origin, u is minus infinity on one side almost, and positive infinity on the other side. So, it is not at all a small control point. In fact, this creates a discontinuous control. Even though the states are very close to the origin, starting very close to the origin, the control that is needed to actually push them towards the origin is very large in the, very large and almost plus minus infinity.

So, this is sort of the problem that you want to avoid. And therefore, we declare this as a property called the small control property. So, I really would strongly suggest that you think carefully about this problem. I may have spoken too quickly, but I would urge you to think about this very carefully, this sort of an example very carefully, and why the control becomes discontinuous if you do not have the small control property.

So, the proposition which is, leads to what is called the Artstein-Sontag universal formula which we will look at soon, essentially says that if there exists a control Lyapunov function, then a system satisfies the small and control, small control property if and only if it admits almost C infinity stabilizer. And almost C infinity stabilizer means there exists a control u which is smooth everywhere except at the origin where it is only continuous, which is still very nice.

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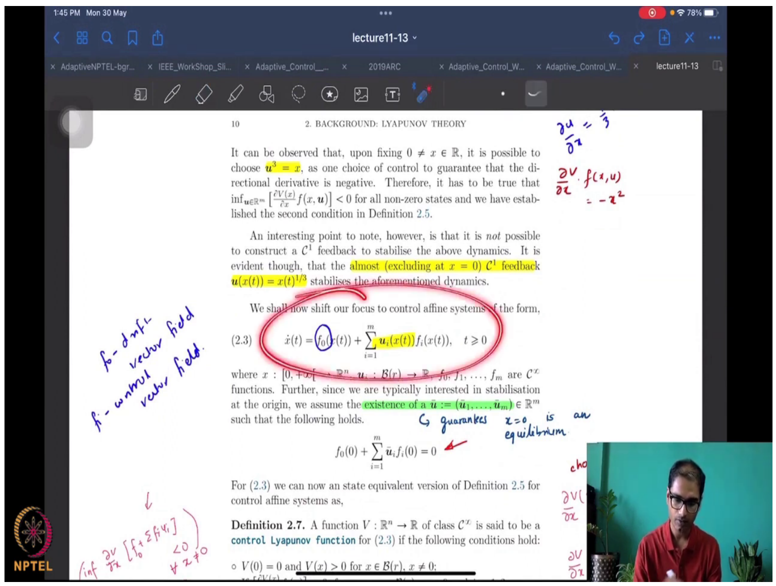

So, the proof of this kind of a proposition which is called the Artstein-Sontag theorem, very, very famous result, very famous result, was done using an actual control construction u for this system, for this system, for this system 2.3, controller affine system. And that construction is what we showed. So, we declare a as, and this construction relies, of course, on the control Lyapunov function V. And we define a as del V del x f 0 and b as the vector containing the del V del x f i.

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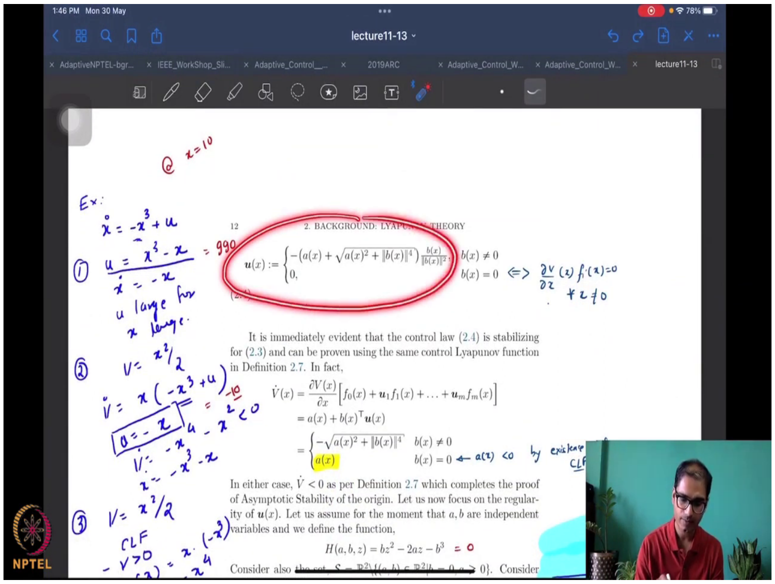

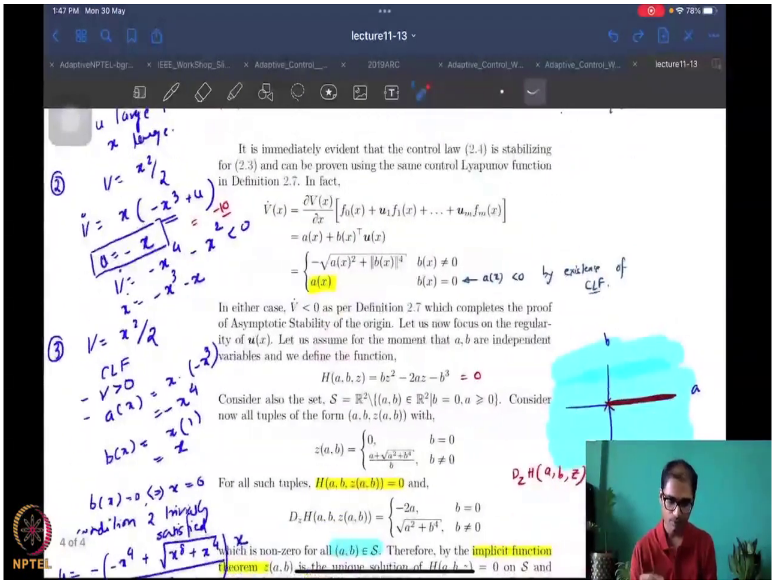

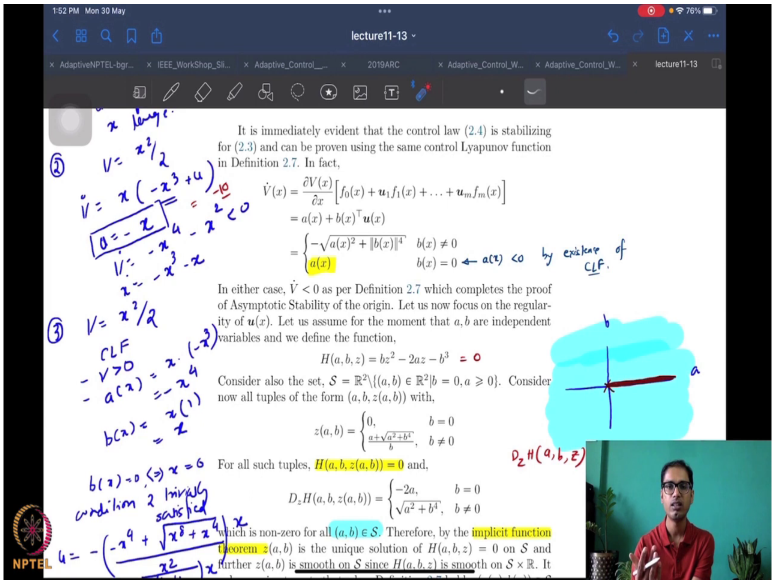

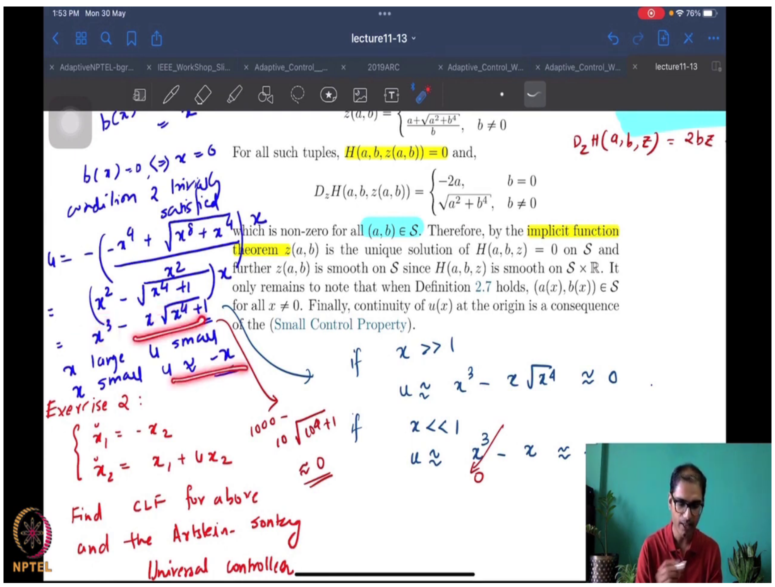

And the control is actually defined in this way. Whenever b is non-zero, b non-zero means at least one of them is non-zero. And V 0 means every entry of b is 0. This is essentially the control Lyapunov condition. If b is 0 means all of these are 0, for all n. These are 0 for all n. And that is what it means for b to be 0. And b non-zero means at least one of them is non-zero.

So, if one, at least one of them is non-zero, then you have this nice construction which, with a division by b, norm of b x. And this is essentially what you have. And this is non-zero. So, because b is non-zero, so norm is non-zero, so this is well defined. But when b is 0, this norm is not well defined. So, control is defined as 0.

And this Artstein-Sontag actually show that this is a smooth control law. They show that this is a smooth control law. V, there is also a proof here which I am not going to go through, but you can actually compute V dot here using our standard way of computing V dot, which is del V del x. So, V dot is just this guy for this control affine system.

And if I plug in for the control u 1 to u m because u is the vector of u 1 to u m, if you plug in this guy, this is a x plus b x transpose u, and you, you essentially get this expression. And you know very well that a x is negative by existence of CLF, and this quantity is also negative.

So, in both cases, whether b is 0 or b x is non-zero, in either case, you have a negative quantity here. Therefore, V dot is negative definite, therefore, you have asymptotic stability by standard Lyapunov theorems. So, what happened is that the existence of a control Lyapunov function actually gives you a control design.

In fact, if you remember, our control designs are also based on first designing the Lyapunov function. So, indirectly we are, in fact, designing control Lyapunov functions. Although we do not call them control Lyapunov functions, and we do not use the Artstein-Sontag formula, we are in fact using the V to design a control law. Therefore, we are designing control Lyapunov functions. So, I hope you understand this.

I know that this formula is a complicated one, and we will see this. Suppose I look at this kind of an example here, something, very simple system. x dot is minus x cube plus u. So, we can do, design in multiple different ways. First, I can directly prescribe a controller. I cancel this term and introduce a minus x. So, x dot is minus x.

What happens? Because of the structure of the control u is large when x is large because if x is even, if x is 10, then this is, this takes value 990. Now, the second kind of design is when I use a V, which is possibly a control Lyapunov function. So, so notice, what, what does the theorem say? Before I proceed to this example more carefully, what does this theorem say?

The theorem says that the control Lyapunov function, with the control Lyapunov function, the small control property is equivalent to existence of a stabilizer. So, there are two ways to talk about a control Lyapunov function. One is, I have a control Lyapunov function and from that, I can use an Artstein-Sontag formula to get a controller.

The second is, I have a function V, and using that function V, I come up with a stabilizing controller u. And if I can do that, then it means that the V I chose was a control Lyapunov function. So, this is the important message to remember. If with a V, I can design stabilizing control u, then V is a CLF. This is by virtue of the if and only if condition.

And this is what we have been doing. We have been picking up a V and then with that we have been coming up with a u which was stabilizing. Therefore, all the V that we chose until now were control Lyapunov functions without explicitly saying so. The other side is of course I know that the V is a control Lyapunov function and I use an Artstein-Sontag formula.

So, let us look at this example in more detail and try to connect it to what we have been doing. The first is I just chose blindly a controller because I know this is giving me a target system x dot is minus x. Let us do the second thing. Choose V is x squared by 2, compute a V dot which is x, x dot which is x minus x cube plus u.

I know that this is minus x 4. So, it is a good term. I do not need to do anything about it. I choose u as minus x, I choose u as minus x and I am done. I, I get a negative definite V dot. So, I know that this V is a CLF because I can choose a stabilizing control u. So, this, what is the value of this control, u equal to minus 6? It is minus 10, it is minus 10.

So, no problem, very good. Let us look at, again the V being x squared by 2, I know that it is already a CLF because of this previous case and because I know it is positive definite and it is, there exists a stabilizing controller, if I choose a V and I can compute a stabilizing controller using that V, then it is a control Lyapunov function because of the if and only relationship.

So, therefore, V is a CLF. Now, what do I do? I apply the Artstein-Sontag formula. And I get this complicated expression for control. It is, it is something like this, something like this. Now, what is the cool thing about this controller? The cool thing is that when x is large, then notice that this is almost nothing. So, this is almost 0, almost 0.

When, for large values of state, that is, again, when x is 10 this control came out to be 990, this came out to be minus 10, and this quantity will come out to be almost 0. And when x is small, this is of the order of minus x because this is almost 0, this is almost 0. So, I have x of the order of minus x, which is the same as the second control.

So, somehow, the Artstein-Sontag formula, though complicated looking, I mean the controller is rather complicated looking as compared to what we intuitively chose, but it has better properties. It gave me a small value of control when the state is really large. And when the states are really small, the starting states are really small, then it is almost acting like a minus x. So, this is excellent control, it is a very good controller.

Typically, when you do a control design, your. your control value is very large for large states. So, but this is actually very small for large states. The control value is very small for large values of states, and when the state becomes very close to the equilibrium here, then the control is of the order of minus x, which is also relatively small, quite okay. So, so this is the idea of a control Lyapunov function.

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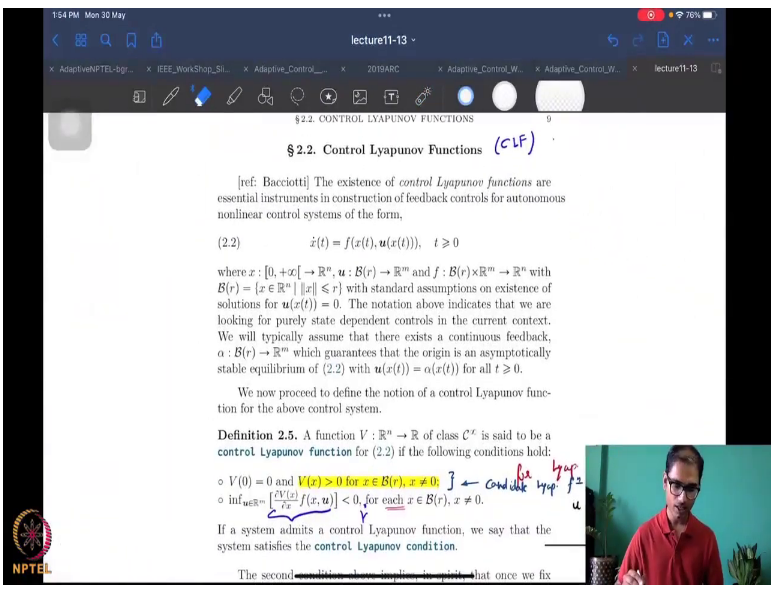
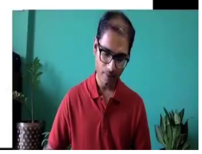


It can be observed that, upon fixing $0 \neq x \in \mathbb{R}$, it is possible to choose $\boldsymbol{u}^3 = x$, as one choice of control to guarantee that the directional derivative is negative. Therefore, it has to be true that $\inf_{\boldsymbol{u} \in \mathbb{R}^m} \left[\frac{\partial V(x)}{\partial x} f(x, \boldsymbol{u})\right] < 0$ for all non-zero states and we have established the second condition in Definition 2.5.

An interesting point to note, however, is that it is *not* possible to construct a $\mathcal{C}^1$ feedback to stabilise the above dynamics. It is evident though, that the almost (excluding at $x = 0$) $\mathcal{C}^1$ feedback $\boldsymbol{u}(x(t)) = x(t)^{1/3}$ stabilises the aforementioned dynamics.

We shall now shift our focus to control affine systems of the form,

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} \boldsymbol{u}_i(x(t)) f_i(x(t)), \quad t \geqslant 0 \tag{2.3}$$



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$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} \boldsymbol{u}_i(x(t)) f_i(x(t)), \quad t \geqslant 0 \tag{2.3}$$

where $x : [0, +\infty[\rightarrow \mathbb{R}^n$, $\boldsymbol{u}_i : \mathcal{B}(r) \rightarrow \mathbb{R}$, $f_0, f_1, \ldots, f_m$ are $\mathcal{C}^\infty$ functions. Further, since we are typically interested in stabilisation at the origin, we assume the existence of a $\bar{\boldsymbol{u}} := (\bar{\boldsymbol{u}}_1, \ldots, \bar{\boldsymbol{u}}_m) \in \mathbb{R}^m$ such that the following holds.

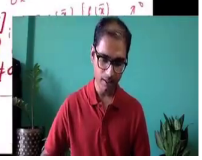
$$f_0(0) + \sum_{i=1}^{m} \bar{\boldsymbol{u}}_i f_i(0) = 0$$

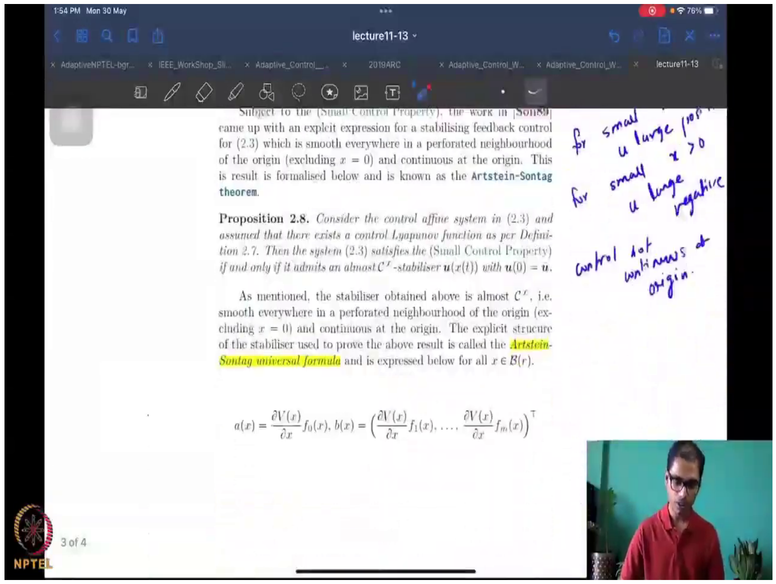
For (2.3) we can now an state equivalent version of Definition 2.5 for control affine systems as,

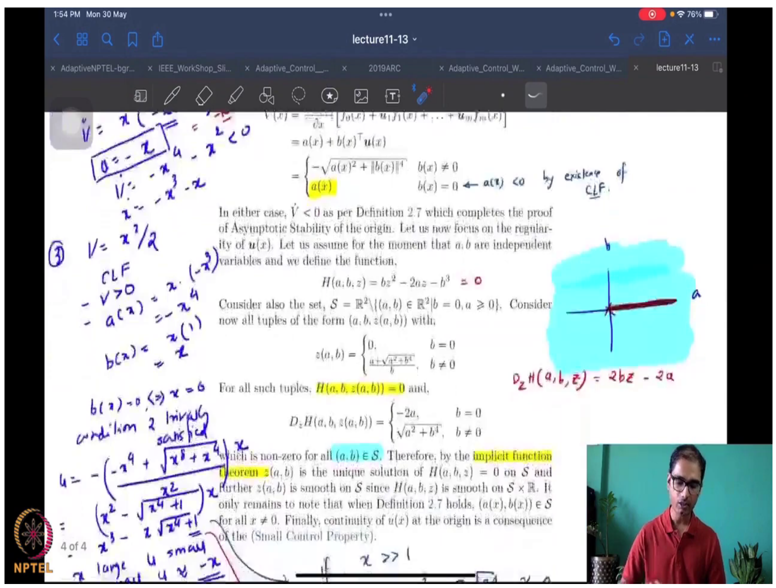
Definition 2.7. A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ of class $\mathcal{C}^\infty$ is said to be a `control Lyapunov function` for (2.3) if the following conditions hold:

- $V(0) = 0$ and $V(x) > 0$ for $x \in \mathcal{B}(r)$, $x \neq 0$;
- If $\left[\frac{\partial V(x)}{\partial x} f_i(x)\right] = 0$, for some $x \in \mathcal{B}(r)$, $x \neq 0$ and $i = 1, 2, \ldots, m$, then $\left[\frac{\partial V(x)}{\partial x} f_0(x)\right] < 0$.

The proof of the equivalence of Definition 2.5 and Definition 2.7 for (2.3) is straightforward and left to the reader. The new conditions are easier to verify since no infimum over all possible control vectors needs to be computed and ensures as before that the directional derivative of a Lyapunov function is negative for all non-zero states along dynamics (2.3).







Now, I know this is a very, very, unfortunately a very, very minimal lecture on control Lyapunov functions. As you can see, this is a rather relatively longer lecture. I start here, started here. I only talked to you about this guy, that is definition 2.7. Then we looked at the small control property and then Artstein-Sontag, I did not talk to you about the smoothness, how it is smooth, how to prove it smooth, but this is sufficient for our understanding.

What do we need to remember? We need to remember that there is two aspects to a control Lyapunov function. One is that it is the, it is a function such that if I take infimum over all possible control of V dot, then it has to be negative definite. This is what is a control Lyapunov function.

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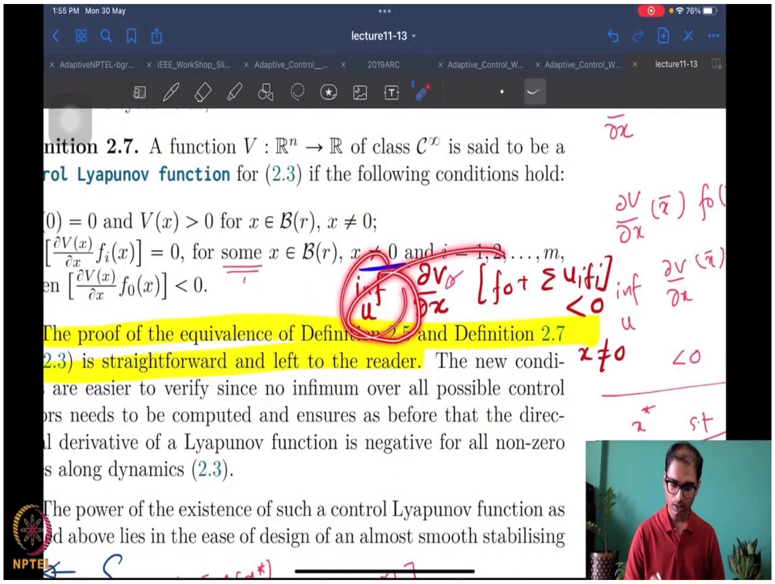

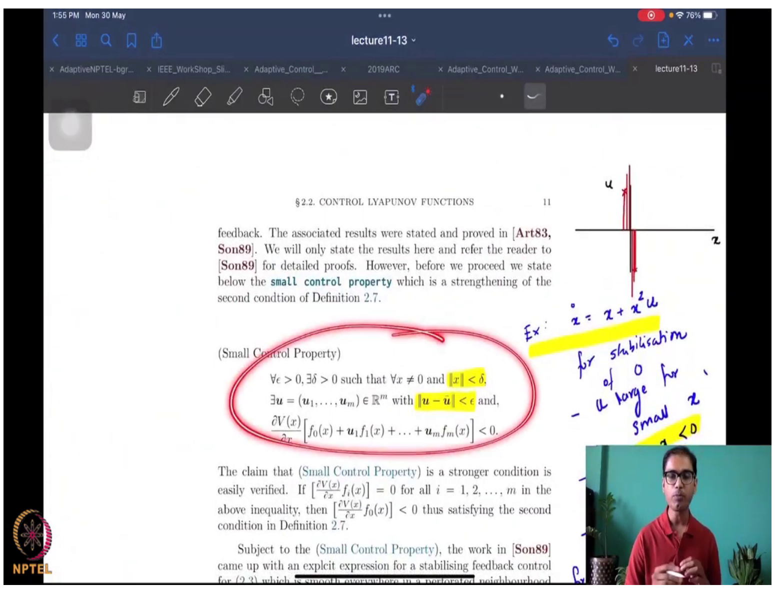

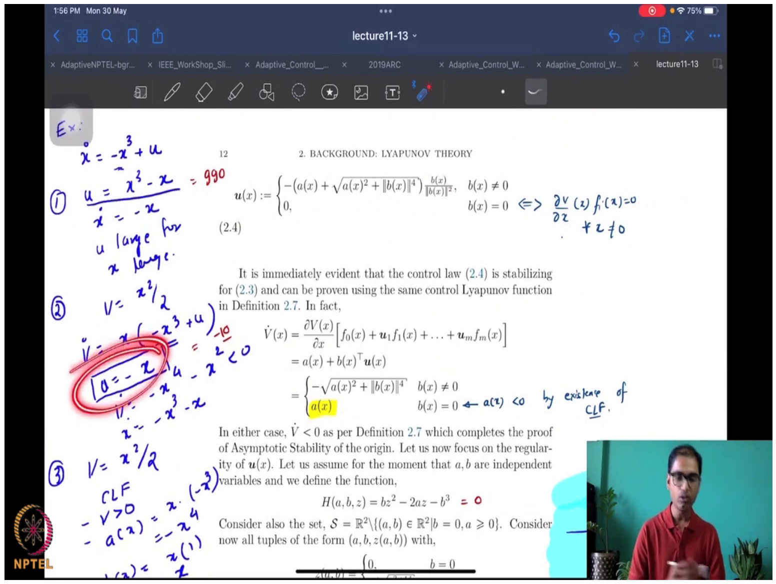

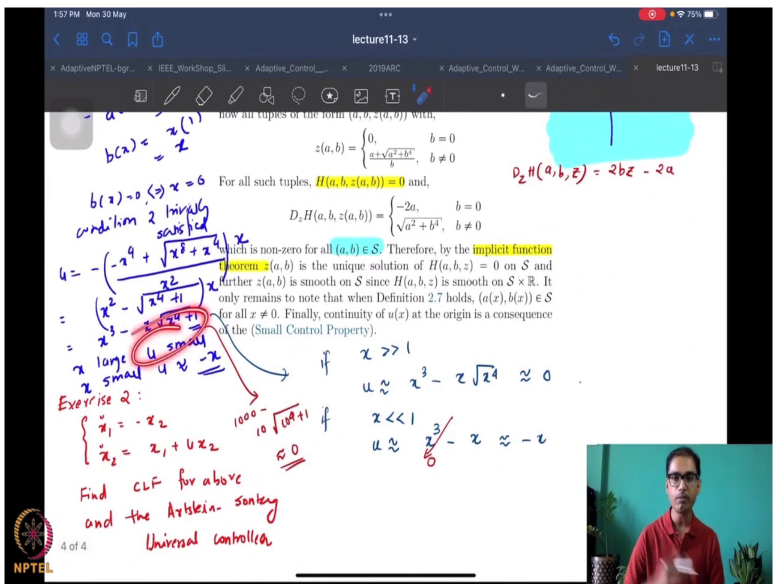

So, if I take an infimum over all possible control of V dot, then it has to be negative definite. This is what I need for function to be CLF. Then, there is also the small control property which is a critical property for continuity of the control near the equilibrium or the origin, in this case. And being able to choose a V and compute a control, stabilizing control from here with this V, means that this V was a control Lyapunov function. So, that is the back.

And there is the converse implication, the forward implication is if V is a controllable function, and you know that, then you can use the Artstein-Sontag formula which is this formula, to devise a smooth stabilizing control. The converse is, of course, what we have been doing until now, before ins.

Although, we did not call it a control Lyapunov function, we have been doing this, we have been taking a V, and define, designing stabilizing controllers out of that. And that is essentially implying that the V we chose to begin with, was a control Lyapunov function. So, instead of choosing that particular stabilizing controller we chose, for example, with the same x squared by 2, I, I might have chosen intuitively this u equal to minus x as my stabilizing controller, but I need not have.

Once I knew using this u, once I can com compute a u, and I know that V is a CLF, I can actually go back and use the Sontag, Artstein-Sontag universal controller. That is also another choice. So, it is like for the same V, I can get multiple choices of control designs with, with different properties, of course. I mean in this case, we saw that u equal to minus x is larger in magnitude, that is, depending on the state. But whatever I got from the Artstein-Sontag formula is actually very small in magnitude for large states. So, that is rather nice.

So, what did we look at today? We sort of started talking about control Lyapunov functions for nonlinear systems that are not, that are not uncertain, did not have any uncertainties. We tried to compare it with what is a Lyapunov function itself, and essentially it includes the role of the control in the Lyapunov function and it is derivative.

The idea of positive definiteness and negative definiteness, positive definiteness of V and negative definiteness of V dot still appear. It is just that the role of the control gets explicitly mentioned here, in the case of a control Lyapunov function. And the cool thing about control Lyapunov functions and why they are powerful, is that using the Artstein-Sontag universal formula, you can actually design a control corresponding to a control Lyapunov function.

So, control Lyapunov function comes with the controller. And this is a smooth controller, almost smooth controller, that is, it is smooth everywhere except at the origin, where it is at least continuous. So, you have a rather nice result. I mean, it is a little bit involved, the theory of control Lyapunov functions, and of course you will read the Artstein-Sontag papers. They are also a little bit more involved and complicated, but the crux is this.

And as we mentioned, we have already been using this idea of CLF. We have been using CLFs to design controllers, we have just not been calling them CLFs. So, in the upcoming session, we will be able to start discussing adaptive control Lyapunov functions for uncertain nonlinear systems. So, I hope you will join me there. Thanks.