Nonlinear Adaptive Control
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Week 8
Lecture No: 43
Model Reference Adaptive Control: For Known and Unknown Parameters

Hello everyone, welcome to yet another session of our NPTEL on Nonlinear and Adaptive Control, I am Srikant Sukumar from Systems and Control, IIT, Bombay. So, we are entering the 8th week of this course on nonlinear adaptive control. So, we are officially in the final quarter, final innings or final few overs of this course.

And I hope that all of you that have been with me during the course have already learned or started to learn algorithm design, which is the critical component or most critical part of being able to drive autonomous systems such as the spacecraft orbiting the earth that you see in the background.

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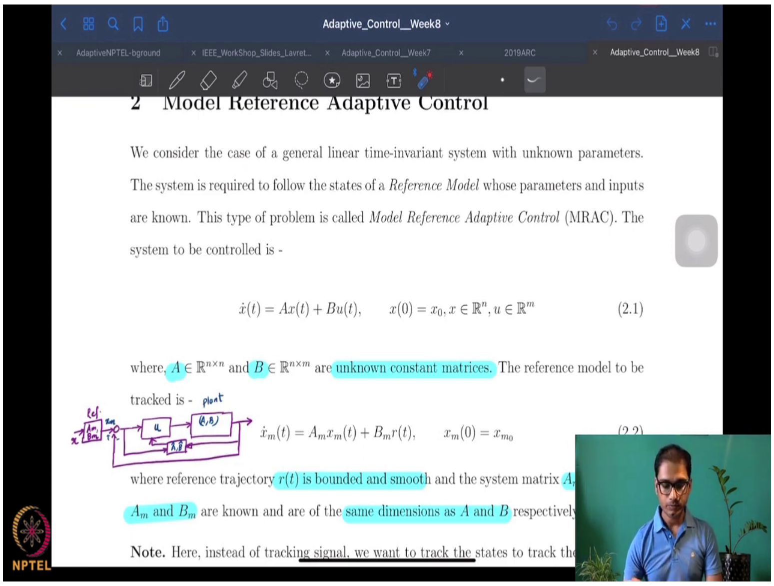

So, what we have been doing and we are, of course, going to continue what we have been doing in at the end of the last week is model reference adaptive control. As we had mentioned this is one of the most key paradigms in the field of adaptive control and this is the model reference adaptive control of linear systems.

So, unlike what we have done and what we had done until now, we use a sort of a stable system model and we generate a reference out of the stable system model which is what we try to track using our system states. So, this is why it is called a model reference adaptive control because there is a reference model instead of a reference signal, which is what we were using until now. Now, of course, you have the usual assumptions of nice boundedness Hurwitz of the model Am and so on and all the good properties.

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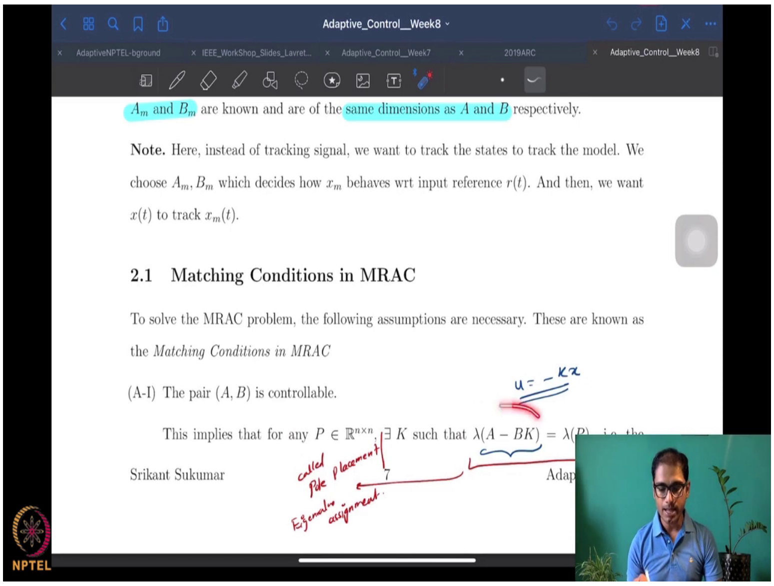

But we also have a lot of matching conditions, which we discussed in some detail last time. And we know we understand that some of these assumptions are rather restrictive but this is the, I mean, nature of the game. This is what we can do, those of you who can show that more can be done or better can be done of course you are free to sort of develop this and show this to the community and get a claim for it.

So, the first assumption is that this pair A B is controllable, which is pretty straightforward which essentially meant that you can match eigenvalues of A minus BK with any matrix B.

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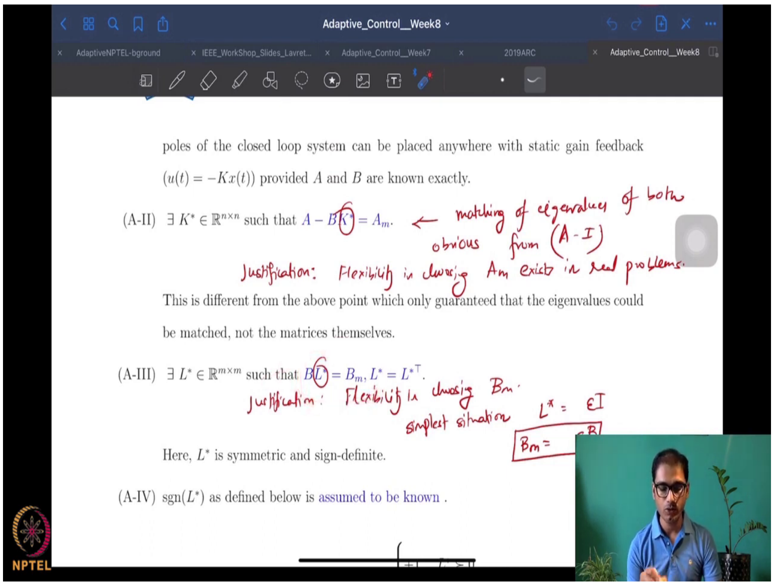

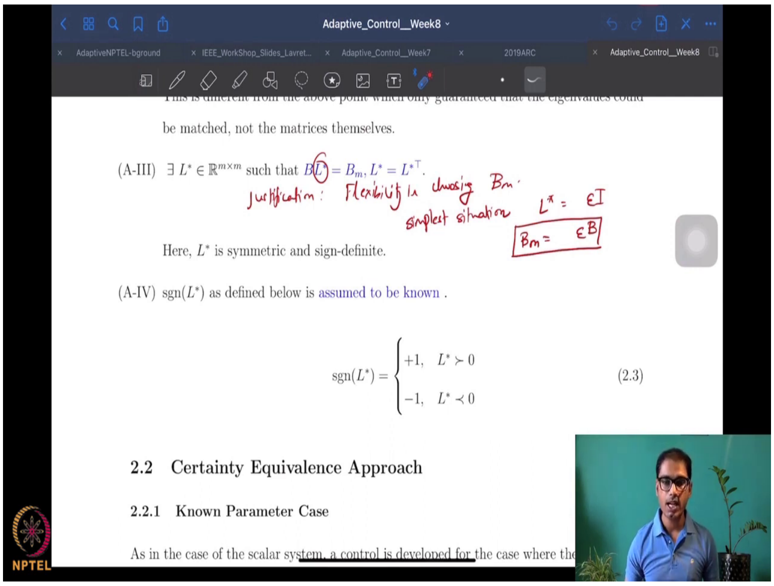

The second was beyond the matching, it actually said that A minus BK can match A m, there exists such a K star. The third was along the similar lines which said that there exist some L star which is in fact also sign definite such that B L star is equal to B m. So, we wanted a symmetric sign definite matrix which had this property.

And finally we needed to know the sign even in the scalar case, we needed information of sign of B which was the gain connected to the control this is actually a similar assumption that we need to know the sign of L star which is defined as such. So, this is sort of where we were in the model reference adaptive control to be honest we had essentially completed describing the problem and so what we are going to do is we are going to start with the design of the adaptive controller today.

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$$\operatorname{sgn}(L^*) = \begin{cases} +1, & L^* \succ 0 \\ -1, & L^* \prec 0 \end{cases} \tag{2.3}$$

Lecture 8.1

2.2 Certainty Equivalence Approach

2.2.1 Known Parameter Case

As in the case of the scalar system, a control is developed for the case where the parameters are assumed to be known exactly as follows:

$$u(t) = -K^*x(t) + L^*r(t) \tag{2.4}$$

poles of the closed loop system can be placed anywhere with static gain feedback $(u(t) = -Kx(t))$ provided A and B are known exactly.

(A-II) $\exists\, K^* \in \mathbb{R}^{n\times n}$ such that $A - BK^* = A_m$. ⟸ matching of eigenvalues of both obvious from $(A - I)$

Justification: Flexibility in choosing A_m exists in real problems.

This is different from the above point which only guaranteed that the eigenvalues could be matched, not the matrices themselves.

(A-III) $\exists\, L^* \in \mathbb{R}^{m\times m}$ such that $BL^* = B_m$, $L^* = L^{*\top}$.

Justification: Flexibility in choosing B_m: simplest situation $L^* =$

$B_m =$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\left(-K^*x(t) + L^*r(t)\right) \\ &= (A - BK^*)\,x(t) + BL^*r(t) \\ &= A_m x(t) + B_m r(t). \end{aligned}$$

$\dot{x}_m = A_m x_m + B_m r$

Now, defining -

$$e(t) = x(t) - x_m(t)$$

we have -

$$\begin{aligned} \dot{e}(t) &= \dot{x}(t) - \dot{x}_m(t) \\ &= A_m x(t) + B_m r(t) - A_m x_m(t) - B_m r(t) \end{aligned}$$

$$\boxed{\dot{e}(t) = A_m e(t)}$$

which is a Hurwitz system, and so, $e(t)$ goes to zero exponentially.

So, let me note it. So, week... lecture number 8.1. So, this is the first lecture of the 8th week of this course. So, as usual we first do the known parameter design. So, and this is the control, now one will ask why, the control structure is minus K star plus L star. So, since we are assuming that we have everything known. So, L star and K star are also known that satisfy these matching conditions. So, we assume the control to be of this form.

Once we have this we actually plug it back into our dynamics and see what happens. Because you will try to, the one of the questions you will have is that why the structure of the control and you will see very immediately why. So, once I substitute in Ax plus Bu I get A minus BK star x and B L star r.

And remember from matching I have this and I have this. So, the matching conditions that we have already assumed provides the existence of this K star and L star and obviously because we assume everything is known K star and L star is also known and A minus BK star is A m and B L star is B m this is exactly the matching conditions.

Now what is so cool you notice that now I have a Hurwitz matrix A m was assumed to be a stable a Hurwitz matrix. Therefore, now I have a Hurwitz matrix connected to x by virtue of this feedback and this is of course bringing in the r and if you remember what was x m dot I will actually write it out for you again. So, if you try to match these two you see that this B m r and this B m r are in fact the same.

So, just like in all the scalar cases here too we define our error as the difference of x and x m which is but natural because I want to drive my x that is the states of the system to the trajectory that I get from the model reference or from the reference model.

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$$= A_m x(t) + B_m r(t).$$

$$\dot{x}_m = A_m x_m + B_m r$$

$$= Ax + B\underbrace{[-K^* x + L^* r]}_{u}$$

$$= (A - BK^*)x + BL^* r \quad \text{matching}$$

$$\dot{x} = A_m x + B_m r$$

Now, defining -

$$e(t) = x(t) - x_m(t)$$

we have -

$$\dot{x} = A_m x - \cancel{A_m x_m} + A_m x_m + B_m r$$

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_m(t)$$

$$= A_m x(t) + B_m r(t) - A_m x_m(t) - B_m r(t)$$

$$\dot{x} - \dot{x}_m = A_m (x - x_m)$$

$$\boxed{\dot{e}(t) = A_m e(t)} \qquad (2.5)$$

which is a Hurwitz system, and so, $e(t)$ goes to zero exponentially.

Note. We have moved from unknowns A and B to K^* and L^*. This is similar to the scalar case, where we moved from unknowns a and b to θ_1^* and θ_2^*.

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_m(t)$$

$$= A_m x(t) + B_m r(t) - A_m x_m(t) - B_m r(t)$$

A_m Hurwitz $\exists\ \forall\ Q = Q^T \succ 0,\ \exists P = P^T \succ 0$ st

$PA_m + A_m^T P = -Q$ ← Lyapunov eqn

$$\boxed{\dot{e}(t) = A_m e(t)} \qquad (2.5)$$

$$\boxed{V = e^T P e > 0}$$

$$\dot{V} = \dot{e}^T P e + e^T P \dot{e} = e^T (PA_m + A_m^T P) e = -e^T Q e < 0$$

which is a Hurwitz system, and so, $e(t)$ goes to zero exponentially.

Note. We have moved from unknowns A and B to K^* and L^*. This is similar to the scalar case, where we moved from unknowns a and b to θ_1^* and θ_2^*.

2.2.2 Unknown Parameter Case

As in the scalar system case, the parameters in the control law for the known parameter case, K^*, L^* are replaced by their estimates using the CE principle,

$$u(t) = -\hat{K}x(t) + \hat{L}r(t).$$

$$\operatorname{sgn}(L^*) = \begin{cases} +1, & L^* \succ 0 \\ -1, & L^* \prec 0 \end{cases} \qquad (2.3)$$

Lecture 8.1

2.2 Certainty Equivalence Approach

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$$u(t) = -K^* x(t) + L^* r(t) \qquad (2.4)$$

So, once I do that. So, e dot is x dot minus x m dot here and if you just look at this it is very easy to see what happens these two cancel out. So, the first thing that happens is that the B m r cancels out in both the equations and I can take A m common and I get x minus x m and that is essentially A m times an e.

And you see again that we have achieved very nice performance, why? Because A m is a Hurwitz matrix. So, you could have also worked backwards in some sense you could have also worked backwards, how would it be, if you remember in the scalar case we chose a reference system to follow we said that we will take a for the double integrator we said e1 dot is e2 and e2 dot is minus k1 even minus k2 e2 and we will try to match this.

Now here if I think about it I am actually choosing this as my guide system I had to choose it smartly of course because it was asymptotically stable, of course and it somehow matches with the system. Now if I start here and I go backwards what happens. So, e dot is A m e. So, which means that x dot minus xm dot is Am x minus x m. So, therefore I have no control on xm dot. So, I plug it as it is. So, x dot is Am x minus Am xm plus Am xm plus Bm r.

So, these two are just xm dot. Now these two cancel out nicely. So, what is it I want x dot is Am x plus Bm r. So, this is the guide system the good guide system for the x dot system for the x dot equations. And here I will write it in terms of the variables K star and L star using the matching condition. And now it should be easy to see that this is going backwards this is equal to Ax and I can take a B common and I get a minus K star x plus L star r.

Again this should remind you of the scalar case because even in the scalar case there was a B common outside and then we redefined the parameters by dividing by this B in some sense this is exactly what is happening I have taken the B outside and my new parameters are K star and L star. So, this becomes my control.

This is my control and now you see that this is the right choice minus K star x plus L star r minus K star x plus L star r. So, if I start with this target system for e I will end up with this by just working backwards and using the matching condition of from here to here I had to use the matching conditions. So, I can even mark this. So, I had to use the matching condition. So, you see how valuable this matching condition was.

So, I have this nice construction for u using the same idea of target systems and so on and so forth. Just that I had to be very smart about how I chose the target system this it is simply motivated by the fact that there is already a Hurwitz matrix in the reference and so I use the same Hurwitz matrix same stable matrix and I do not try to pull out a different matrix which would make no sense in this case.

So, as we said we have moved from unknowns A and B to K star L star this is very similar to the scalar case where we move from a b to theta 1 star theta 2 star because we also took B common there very similar it is just a one might say a multi-dimensional vector extension of what we did in the scalar case.

So, now see I mean we did not really talk about the stability analysis but it is pretty easy I will choose my V as a e transpose P e, where P P e there (ex) from because Am is Hurwitz implies for all Q Q transpose positive definite there exists P P transpose positive definite such that P Am plus Am transpose P equals minus Q. This is the Lyapunov equation all of you are supposed to know this from linear systems theory.

So, this is the well-known Lyapunov equation. This is the well-known Lyapunov equation. You are all supposed to know this from linear systems theory if you do not I again urge you to revise this. So, once I have this kind of a condition I choose my V as e transpose P e that I get from here and if you take a derivative you will see you get e transpose P e dot plus e dot transpose P e which is actually equal to e transpose P Am plus Am transpose P times e which is equal to minus e transpose Q e which is negative definite, which is negative definite.

So, this is how the Lyapunov analysis will be. So, I will mark this as a Lyapunov function for the known case because we do need it. We need a known case Lyapunov function because in the unknown case we simply extend this Lyapunov function by adding terms corresponding to the unknown parameters. So, now we are ready to move to the unknown parameter case. So, we are ready to move to the unknown parameters, what do we do we apply the certainty equivalence principle.

The control was minus K star x plus L star r. So, I replace it with the estimates for K star and L star which is minus K hat x plus L hat r.

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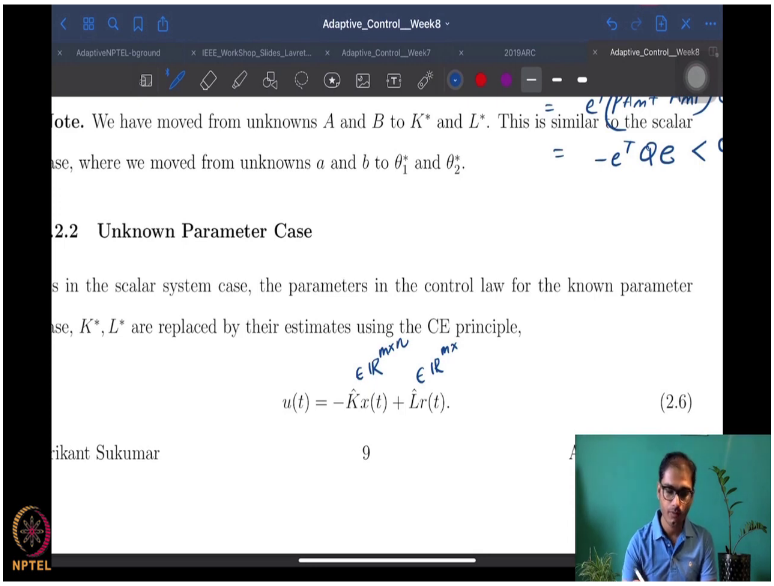
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Note. We have moved from unknowns A and B to K^* and L^*. This is similar to the scalar
case, where we moved from unknowns a and b to θ_1^* and θ_2^*.
$= -e^T Q e <$
2.2 Unknown Parameter Case
As in the scalar system case, the parameters in the control law for the known parameter
case, K^*, L^* are replaced by their estimates using the CE principle,
$\in \mathbb{R}^{m \times n}$ $\in \mathbb{R}^{m \times}$
$u(t) = -\hat{K}x(t) + \hat{L}r(t).$ (2.6)
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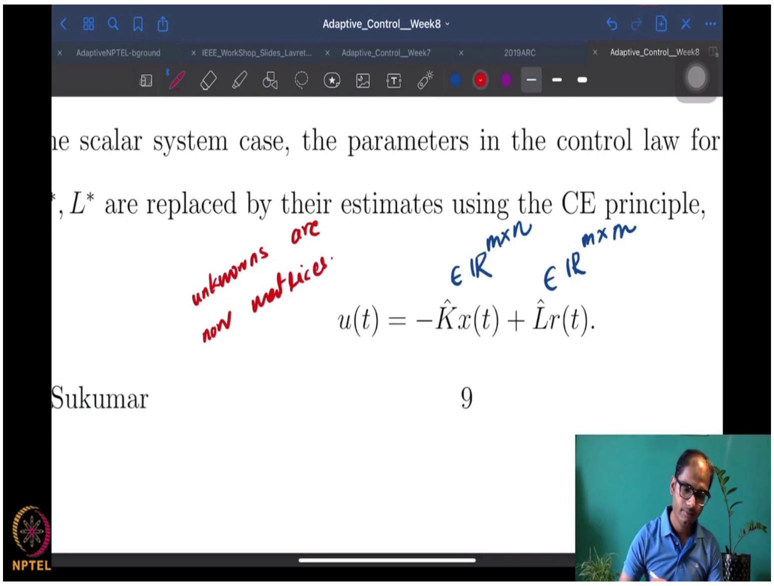
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the scalar system case, the parameters in the control law for
L^* are replaced by their estimates using the CE principle,
unknowns are now matrices.
$\in \mathbb{R}^{m \times n}$ $\in \mathbb{R}^{m \times m}$
$u(t) = -\hat{K}x(t) + \hat{L}r(t).$
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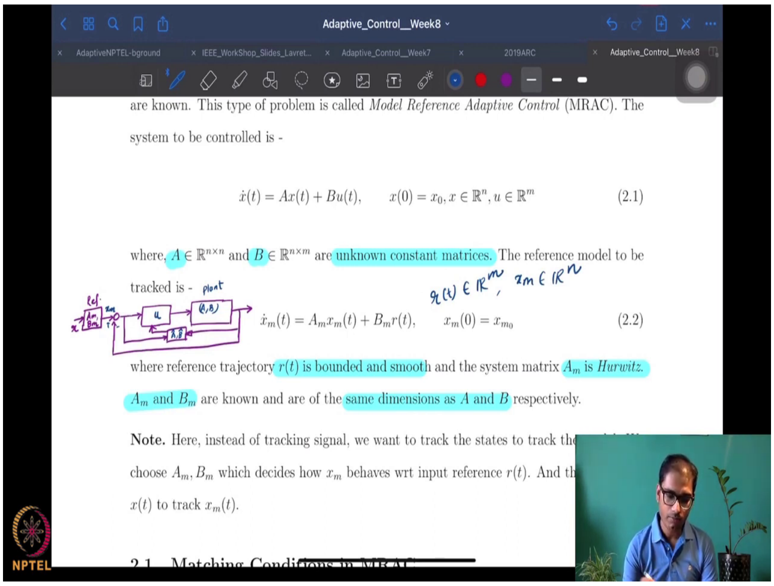
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are known. This type of problem is called Model Reference Adaptive Control (MRAC). The
system to be controlled is -
$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, x \in \mathbb{R}^n, u \in \mathbb{R}^m$ (2.1)
where, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are unknown constant matrices. The reference model to be
tracked is - plant
$r(t) \in \mathbb{R}^m$, $x_m \in \mathbb{R}^n$
$\dot{x}_m(t) = A_m x_m(t) + B_m r(t), \quad x_m(0) = x_{m_0}$ (2.2)
where reference trajectory $r(t)$ is bounded and smooth and the system matrix A_m is Hurwitz.
A_m and B_m are known and are of the same dimensions as A and B respectively.
Note. Here, instead of tracking signal, we want to track the states to track th
choose A_m, B_m which decides how x_m behaves wrt input reference $r(t)$. And th
$x(t)$ to track $x_m(t)$.
2.1 Matching Conditions in MRAC
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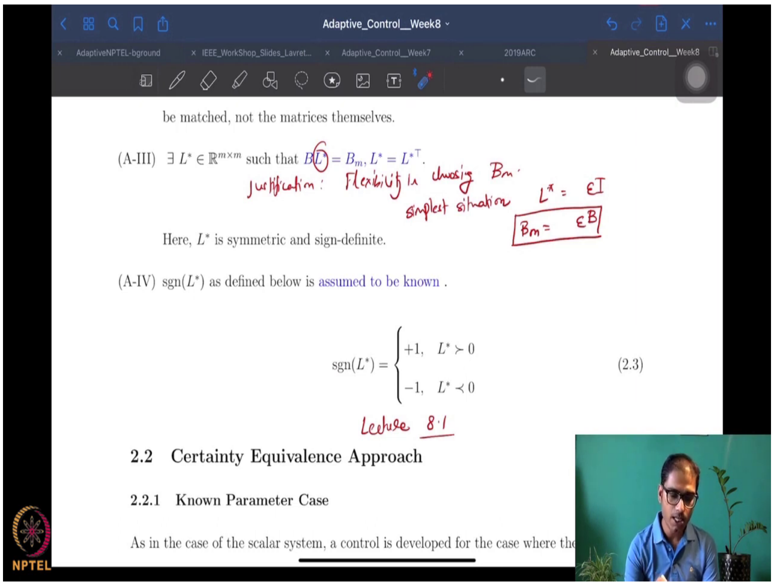

Now remember these are this is where things get maybe messy but also very important that this is now what is the dimension this is no longer a scalar that is the important thing to remember. It is n well it is a dimension of the control. So, it is m cross n and L hat is m cross I would say m because I think r is assumed to be the same dimension as let us see r is assumed to be the same dimension as the control.

We are not saying it, has to be has to be. So, r of t also belongs to R n and of course xm belongs to R n is also evident xm belongs to R n because otherwise I cannot create an error x minus xm and r has to belong to R m because otherwise Bm one of the matching conditions is that Bm and B have to be connected by a positive definite sign definite matrix which is a non-singular matrix.

So, if B and B m are different dimensions you cannot connect them by a non-singular matrix L because it is a non-square matrix which is naturally single. So, B and B m are also the same dimension therefore r is the same dimension as the control xm is the same dimension as the state x. So, the important point for us is that now the unknown parameters are matrices.

So, we started to get into complicated more complicated domain I mean we knew this was going to be the case because the original unknowns were also A and B. So, what was the dimension of A and B same, so it was the same dimension. So, A and B well I mean A was an n by n matrix and then B is of course m by m matrix sorry B was an n by m matrix.

So, now it is slightly different the number of terms are different but it does not matter to us. The point is the unknowns were matrices and the redesigned, redesigned unknowns are also matrices, the number of parameters that we are identifying may be less or more. We are not so worried about it, the point is the unknowns are now matrices no longer vectors are no longer scalars. So, remember. So, I will actually write it down unknowns are now matrices.

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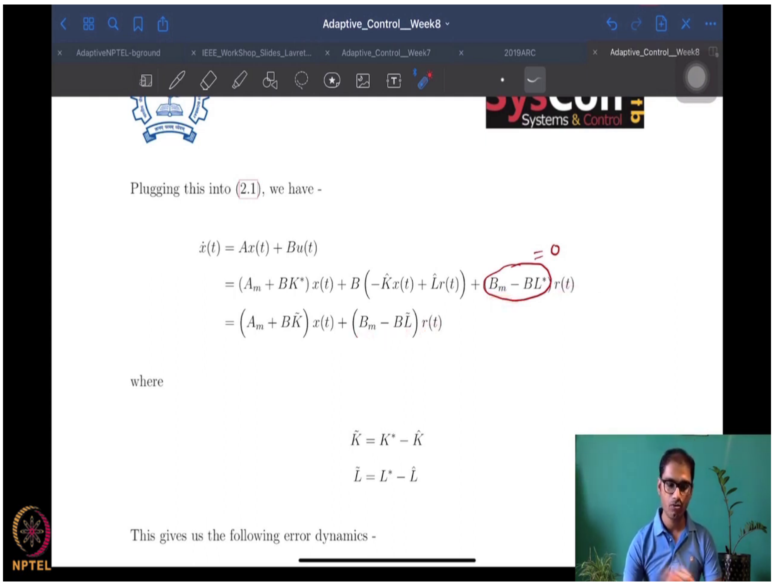

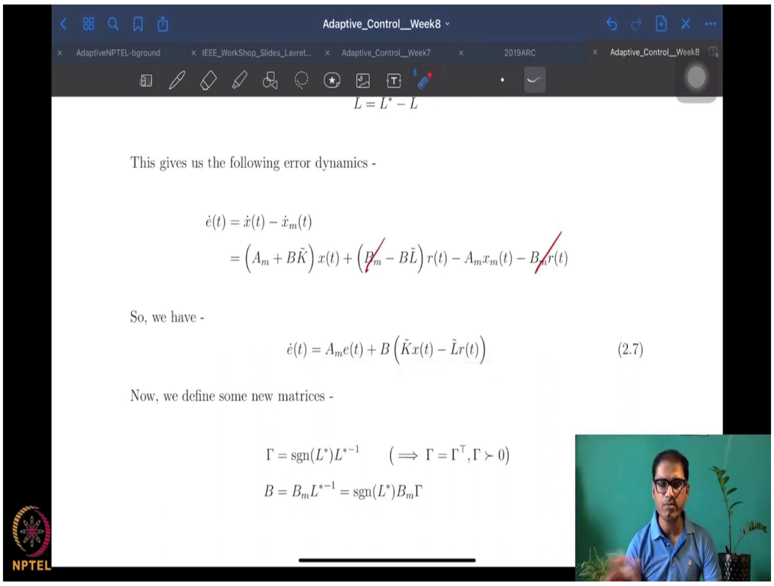

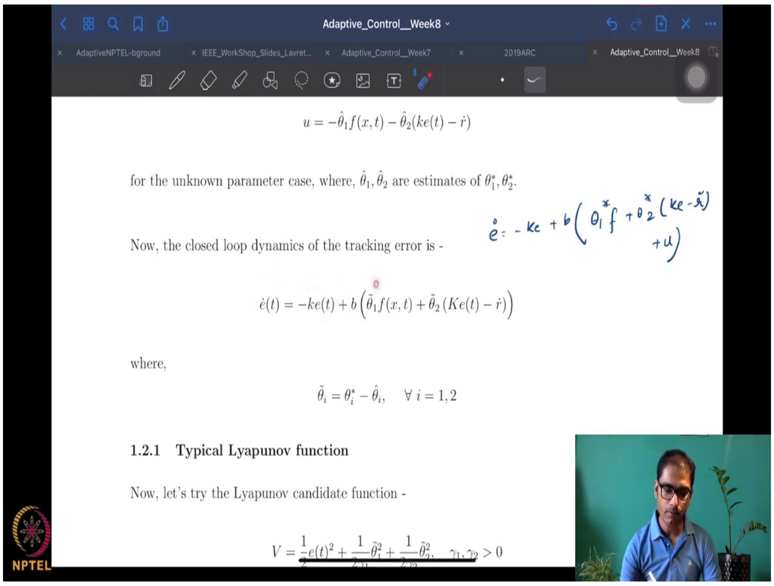

So, now what do we do we plug this into the system dynamics now, this is again something similar to what we have been doing nothing new when we plug this into the system dynamics I also write the matching conditions smartly because so now I can write A as Am plus BK star using the matching condition and B as B L star sorry B as wait a second.

So, as of now I am only writing A as Am plus B K star let us continue and then in the B I substitute the control here and then I know that Bm is equal to B L star. So, I just introduce a 0 term. So, this is you can see is actually equal to 0 yes I have not introduced anything I am just like adding and subtracting terms.

This is just adding and subtracting we are not introducing anything new here. So, once I have this why we do this is because you get these identical terms. So, you want to sort of compare them that is about it. So, this B K star x and B K hat x can be combined to get B K tilde x. So, where case K tilde is K star minus K hat and similarly this B L hat r and B L star r.

So, this B L hat r and B L star r can be combined to give you B L tilde r. So, let us see I want to see that I want to make sure the sign is correct I have a B L hat r minus B L star r. So, that is a negative B L tilde correct this is a negative B L tilde r and then there is already a B m r. So, this is exactly what we get. Now what do we do we define we want to compute the error dynamics. So, the error dynamics is e dot x dot minus x m dot.

So, this is the x dot and what is the x m dot which is just this guy. So, what do we have we have cancellation of this, actually we do not have any cancellation wait a second, well we do have a cancellation of this term with this term. So, these two terms get cancelled that is it.

These two are combined together these are combined together to give me Am e and I am left with B K tilde x minus B L tilde r. So, this is exactly what we had in the known case.

And now we have two terms in the unknown again very natural very much like the scalar case. I mean let us see if I look at the scalar case I will actually pull it out. Wait a second here you go, whatever you had in the known case plus a B multiplying two terms in the unknowns. Exactly what you had in the known case plus the B that is the control gain multiplying the unknown parameter terms.

Exactly similar is what we get here also exactly similar this is this is what you would have in the known case and you have B the control gain multiplying terms in the unknowns. The terms look slightly different but in structure identical. Now remember we are dealing with unknowns which are matrices. So, the method of analysis and design of the parameter update law is also slightly different that is the idea in keeping that in mind we define a few other new matrices one is this matrix called gamma which is signum L star times L star inverse.

So, remember that signum L star depends on the sign definiteness. So, signum L star is positive if L star is positive definite, signum L star is negative if L star is negative definite. So, this implies that first of all this is symmetric why because L star is symmetric, it is assumed to be symmetric. And further it is positive definite because I multiplied the signum with the L star inverse. So, the sign of L star and L star inverse are the same, if L star is positive definite, L star inverse is also positive definite.

Similarly, if L star is negative definite, L star is in inverse is also negative definite, it is just standard this is pretty simple and standard idea.

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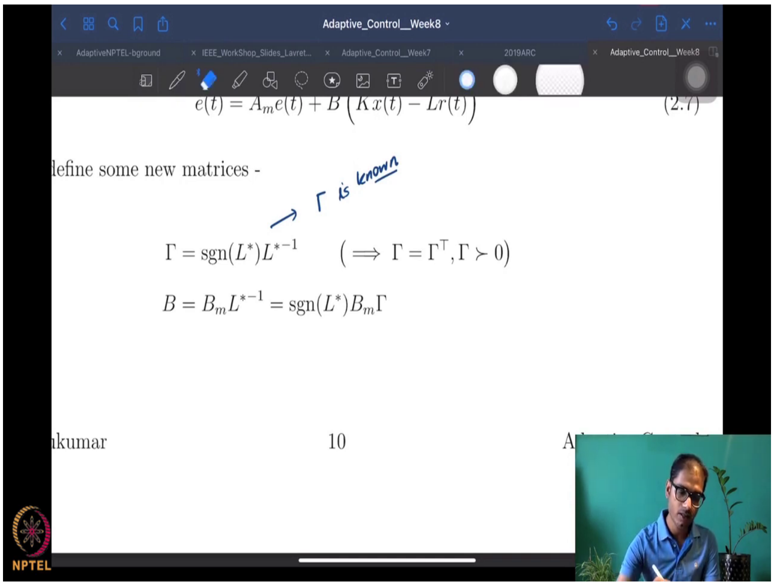

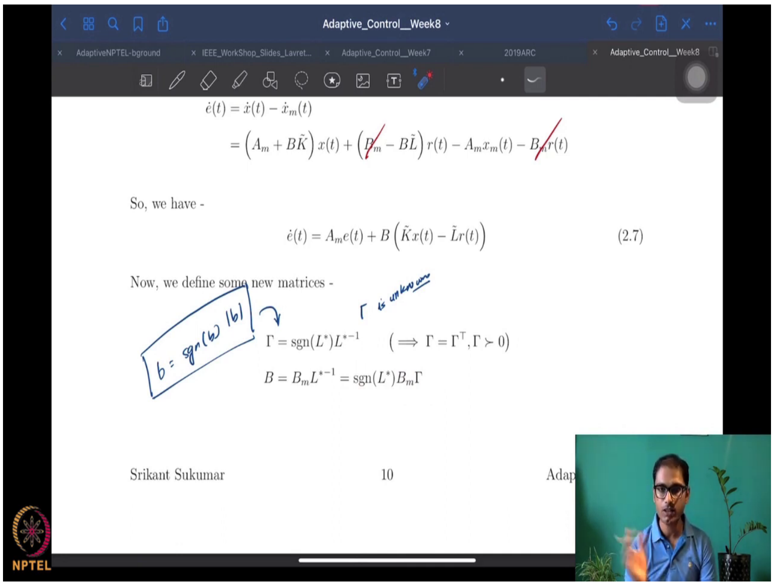

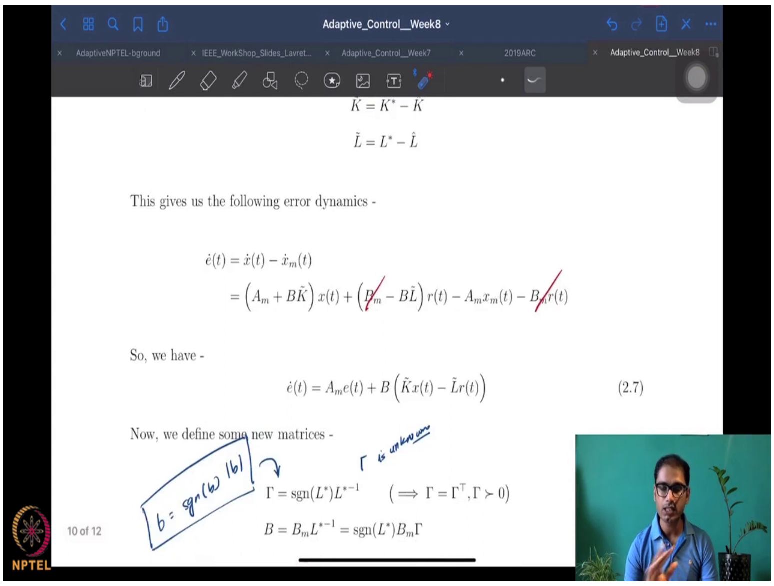

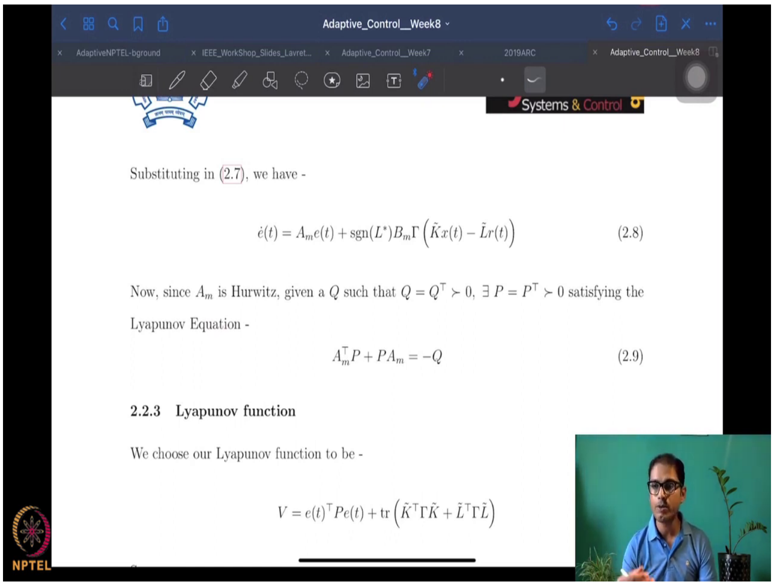

So, what we have essentially done is we have constructed out of L star using the inverse a positive definite matrix and remember that this matrix gamma is known we have constructed a known I am sorry wait a second, this is not known sorry sorry I apologize not known. So, I will I will be careful I will say gamma is unknown. Why? We do not know L star this is what we are trying to find gamma is unknown.

Now how we use this is we know that B m is, B is equal to B m L star inverse because of the matching condition. Now this L star inverse can now be written as gamma times sigma L star. So, basically this is simply using this idea something similar to this B is equal to sigma B times absolute value of B, simple as that. It is basically using this similar notion for matrices it just using a similar notion for matrices.

So, now I know by matching condition that B is equal to B m L star inverse and this L star inverse can be written as sigma L star times gamma again sigma L star is a scalar. So, I can pull it out to the left but gamma remains where it is because we are working with matrices remember we have to be very careful about the order I cannot move things back and forth and all that. So, remember I if you look at how we have been doing the analysis.

We have not changed the order of things that is really what you need to be careful about when working with matrices, that is it. Now when we substitute this in the dynamics this is relationship that B equals to sigma sigma of L star times B m gamma. So, that you substitute in the dynamics and that is it I mean you just replace B by this quantity.

And we will see why? You just replace B with this quantity. So, like I said earlier because A m is a Hurwitz matrix it satisfy the Lyapunov equation which means that given a Q positive definite there exists a P positive definite such that this is satisfied. So, now we choose the Lyapunov function remember we already had this choice which was positive definite.

Now we add to it terms in the, in terms of the unknowns now in the scalar case we would have just picked things like theta 1 tilde squared by 2 gamma 1 plus theta 2 tilde squared by 2 gamma 2 and so on and so forth. In this case that is not possible, in this case that is not possible because we are not dealing with scalars one possibility is you convert these K tildes to vectors and do the same but an equivalent and a nicer and more elegant ways to use trace of matrices.

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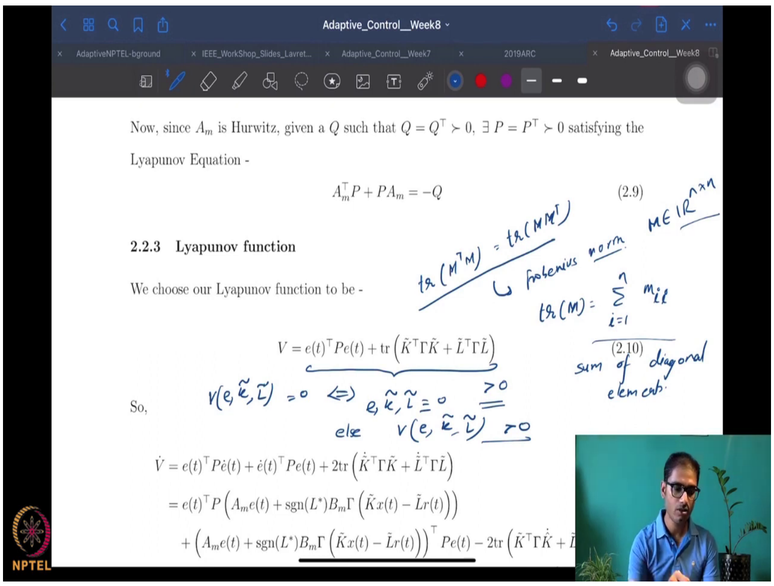

So, the so trace of something like M transpose M is actually equal to the trace of M M transpose is actually connected to the Frobenius norm. So, it is a norm and norms always form good candidates for Lyapunov functions because if you remember we take norm x squared as a Lyapunov candidate.

So, norms always form a good candidate for Lyapunov function. So, what is the trace? Trace is simply the trace of M is of course the sum i equal to 1 to n M subscript ii which is sum diagonal elements and trace enjoys a lot of cool nice neat little properties which we use.

So, we did not we have not used exactly the norm but it is like a weighted norm that we have used it is K tilde transpose gamma K tilde and this is why we defined this gamma with very with some important particular reason. So, this gamma appears here. So, this is K tilde K tilde transpose K delta L tilde transpose gamma L tilde. Now this remember that this is also positive definite.

So, well I mean I will actually say the whole thing is positive different because if I weight it weight by a positive definite matrix like gamma this is still a norm. So, and trace will of course distribute. So, this is trace of this plus trace of this. So, each of this is in fact a norm. It is a weighted norm but it is still a norm and therefore this is positive definite any norm is positive definite that is it is zero only at zero value of the state and non-zero outside.

So, it is easy to show that V e K tilde L tilde equal to 0 if and only if e K tilde L tilde equal to the 0 matrices else V e K tilde L tilde is strictly positive. Therefore, it is a positive definite function these are the conditions for a function to be positive definite. So, the rest of the analysis is a little bit long. So, we will continue it in our next session.

So, just to summarize what did we do today, we continued our discussion on the model reference adaptive control, we have started with the control design, we looked at the known case, tried to identify the similarities with the scalar case but it was some rather nice interesting construction of a target system which gave us the structure of the control actually.

And then we also identified a Lyapunov a candidate function using the linear system Lyapunov equation. And then we started looking at the unknown case, and how we get we just propose a certainty equivalence controller and now we finally looked at the structure of the Lyapunov equation.

Sorry, the Lyapunov function for this system which is rather different and unusual and it uses trace functions, which is the common choice of Lyapunov functions when matrices are involved because this is a Frobenius norm. So, this is where we stop and I hope you followed what we did today well, if not please look at the video very carefully because we will continue this next time. Thank you.