Nonlinear Adaptive Control
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Indian Institute of Technology, Bombay
Week 7
Lecture No: 42
Setup of Model Reference Adaptive Control (MRAC) Problem

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Hello everyone, welcome to yet another session of our NPTEL on nonlinear and adaptive control. I Am Srikant Sukumar from systems and control IIT Bombay. So, we are almost at the end of the week number 7 of this NPTEL course, and we are well underway into designing algorithms that can drive autonomous systems such as the satellite orbiting the Earth that we see in our background.

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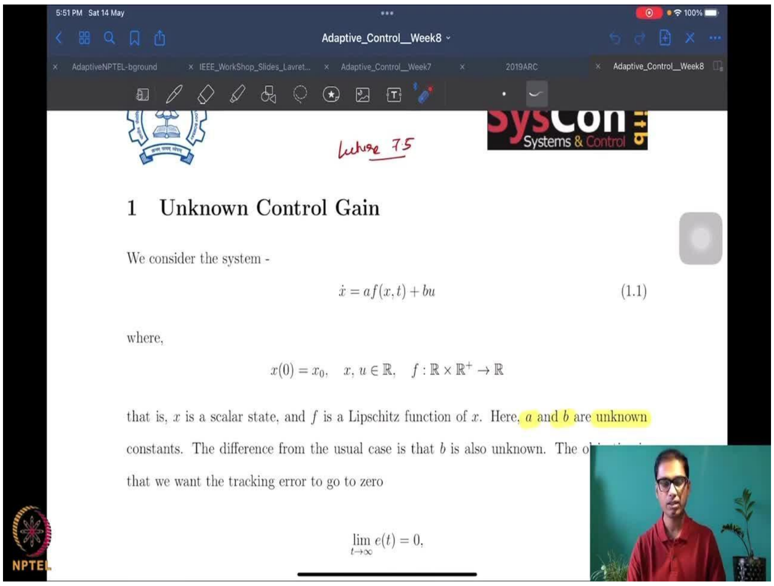

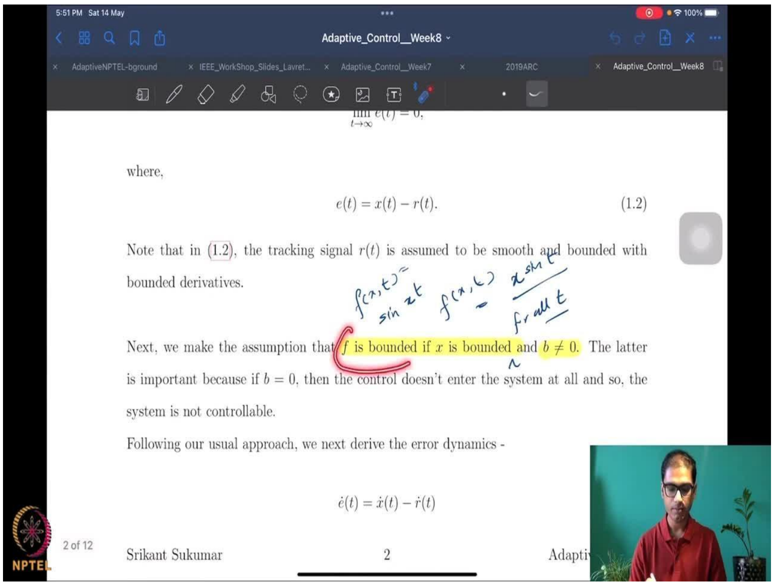

So, what we were looking at last time was a new sort of topic on design of adaptive controller in the case of unknown control gains. So, the design procedure was more or less similar, like the only difference being that there was a gain attached to the controller, which was not the case in all the adaptive designs we saw earlier. And we of course, had the usual tracking objective that we take and then we made some pretty reasonable assumptions on this boundedness of the function f and b being non zero which is essentially translates to controllability of the system.

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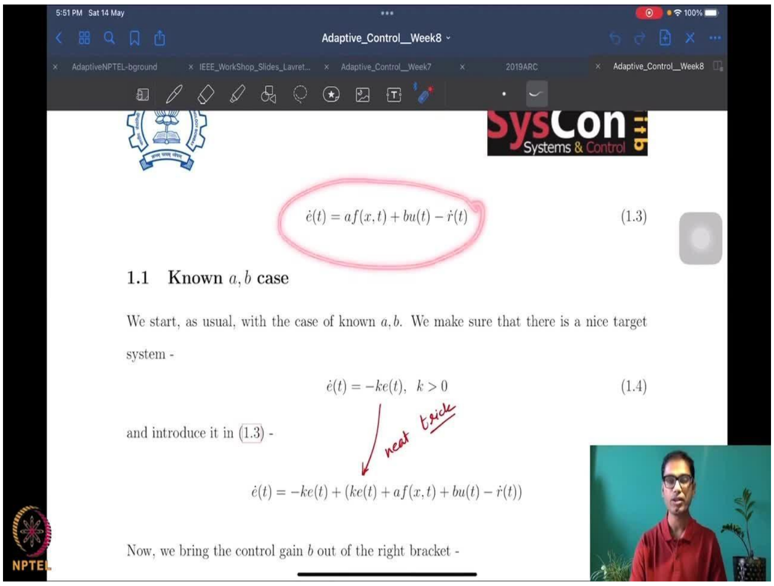
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SysCon
Systems & Control
$\dot{e}(t) = af(x,t) + bu(t) - \dot{r}(t)$ (1.3)
1.1 Known a, b case
We start, as usual, with the case of known a, b. We make sure that there is a nice target system -
$\dot{e}(t) = -ke(t), \; k > 0$ (1.4)
and introduce it in (1.3) -
neat trick
$\dot{e}(t) = -ke(t) + (ke(t) + af(x,t) + bu(t) - \dot{r}(t))$
Now, we bring the control gain b out of the right bracket -
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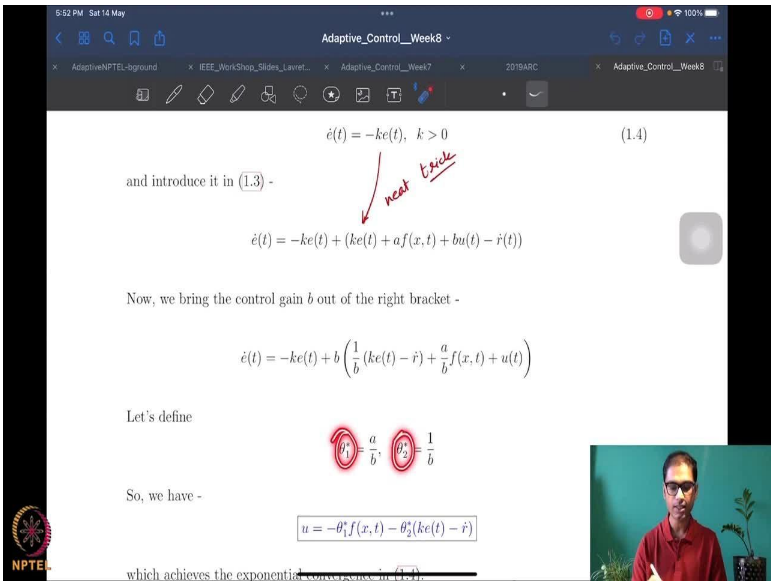
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$\dot{e}(t) = -ke(t) + b\left(\frac{1}{b}(ke(t) - \dot{r}) + \frac{a}{b}f(x,t) + u(t)\right)$
Let's define
$\theta_1^* = \frac{a}{b}, \quad \theta_2^* = \frac{1}{b}$
So, we have -
$\boxed{u = -\theta_1^* f(x,t) - \theta_2^*(ke(t) - \dot{r})}$
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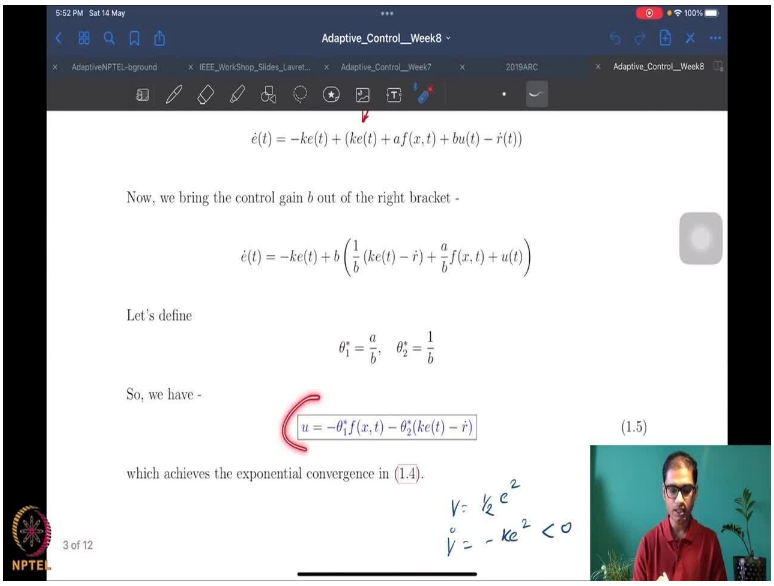
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So, we have -
$\boxed{u = -\theta_1^* f(x,t) - \theta_2^*(ke(t) - \dot{r})}$ (1.5)
which achieves the exponential convergence in (1.4).
$V = \frac{1}{2}e^2$
$\dot{V} = -ke^2 < 0$
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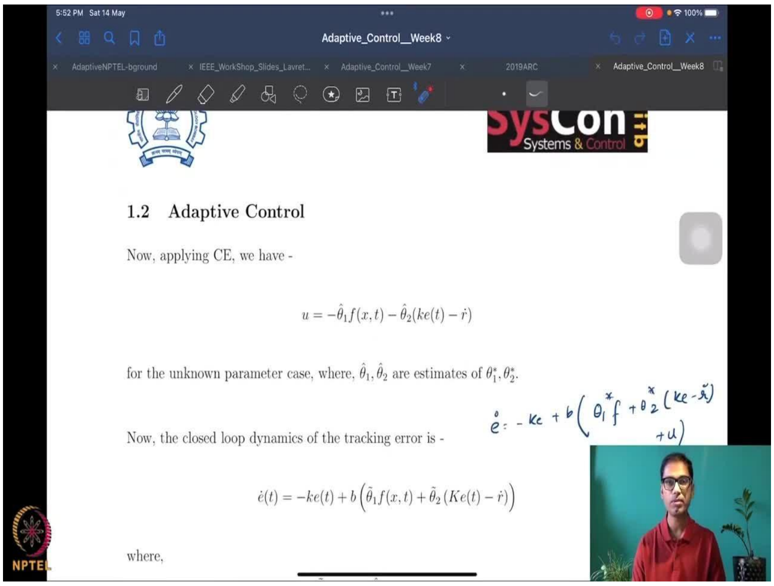

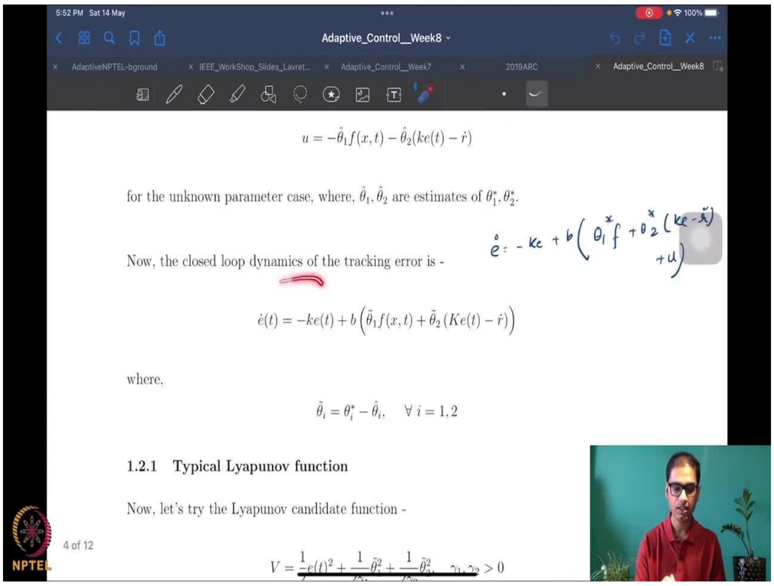

With the aerodynamics in place in the form of equation 1.3 here, we started to do our design by actually applying some nice little tricks, so, that we redefined the parameters of the system into theta 1 star and theta 2 star starting from a and b, and we of course designed the known controller. From this we actually move to the unknown controllers simply using the certainty equivalent principle of replacing the unknowns with the estimates. And we started with our Lyapunov analysis with this new control design.

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$$\dot{e}(t) = -ke(t) + b\left(\tilde{\theta}_1 f(x,t) + \tilde{\theta}_2\left(Ke(t) - \dot{r}\right)\right)$$

where,

$$\tilde{\theta}_i = \theta_i^* - \hat{\theta}_i, \quad \forall\, i = 1,2$$

1.2.1 Typical Lyapunov function

Now, let's try the Lyapunov candidate function -

$$V = \frac{1}{2}e(t)^2 + \frac{1}{2\gamma_1}\tilde{\theta}_1^2 + \frac{1}{2\gamma_2}\tilde{\theta}_2^2, \quad \gamma_1, \gamma_2 > 0 \tag{1.6}$$

where γ_1, γ_2 regulate the speed of parameter estimation.

Now, the $\dot{V}$ is -

$$= -ke^2 + \frac{\tilde{\theta}_1}{\gamma_1}\left(\gamma_1 b e f(x,t) - \dot{\hat{\theta}}_1\right) + \tilde{\theta}_2/\gamma_2\left(\gamma_2 b (k\right.$$

To cancel the parameter error terms, we need to choose -

$$\dot{\hat{\theta}}_1 = \gamma_1 b e(t) f(x,t)$$
$$\dot{\hat{\theta}}_2 = \gamma_2 b e(t)\left(ke(t) - \dot{r}(t)\right)$$

This is a problem as the RHS contains b, which is unknown. If the unknown shows up in the adaptive law, we have achieved nothing. So, we need to take a different approach.

1.2.2 A different Lyapunov function

Consider the following Lyapunov function -

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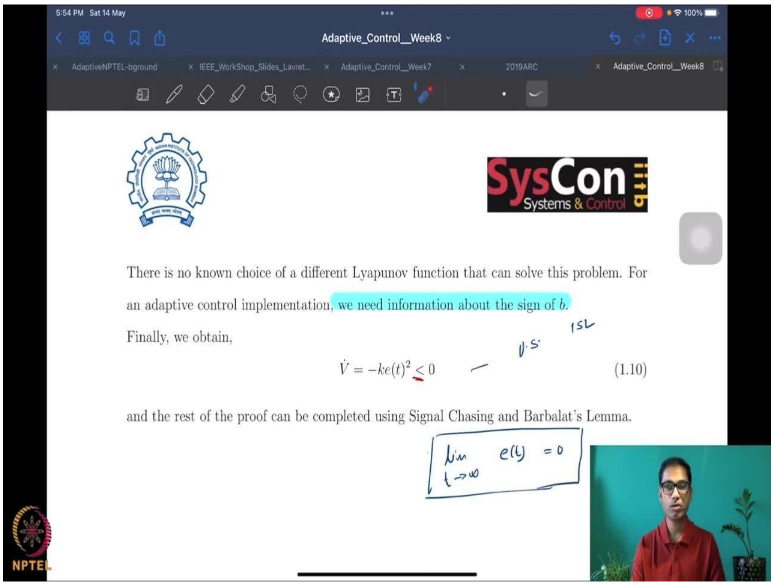
1.2.2 A different Lyapunov function

Consider the following Lyapunov function -

$$V = \frac{1}{2}e(t)^2 + |b|\left(\frac{1}{2\gamma_1}\tilde{\theta}_1^2 + \frac{1}{2\gamma_2}\tilde{\theta}_2^2\right), \quad \gamma_1, \gamma_2 > 0 \tag{1.8}$$

where γ_1, γ_2 regulate the speed of parameter estimation.

The only difference in (1.8) as compared to (1.6) is the $|b|$. However, now, $\dot{V}$ is -



Now, we started with a standard Lyapunov function, which involves taking the known case Lyapunov function and adding quadratic terms corresponding to the unknown parameter, but we quickly realized that this leads us to the non implementability issue, where the unknown parameter itself starts to appear in the update law. So, which is absolutely not allowed, because that is essentially what we are trying to compensate for. So, we tried a different Lyapunov function, because of course, well known in literature, and we simply added an absolute value of b in the Lyapunov function.

And what we saw was instead of the parameter itself appearing in the update law, the sign of the parameter appears in the update law, which leads us to an additional requirement for adaptive control for unknown control gain systems and this requirement is that we need information of… I apologize we need information of the sign of b. So, this is sort of a critical requirement which we cannot do without when we are looking at unknown control gain problems. So, as I mentioned, if there is a solution that one of you can find, which does not require us to use the signum or the sign of b, then we would have done something exceptional. And well, as of now it seems impossible, is what I would say, is how I would put it.

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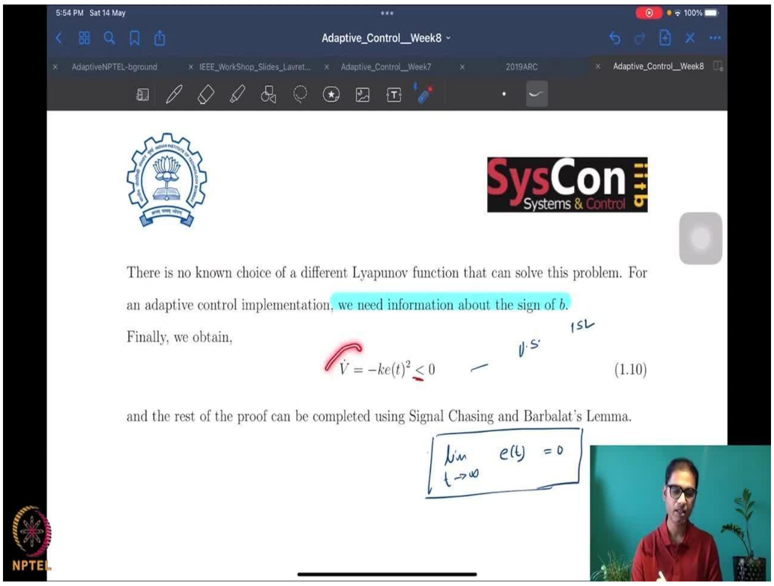

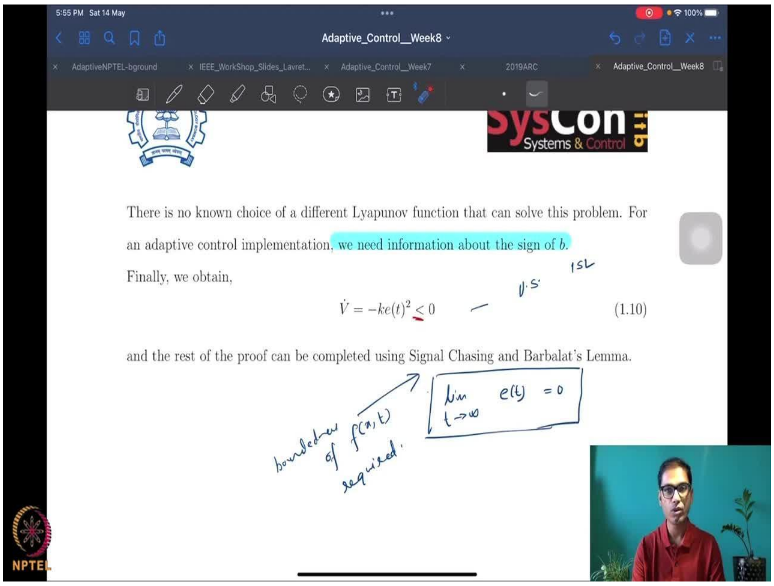

Now, we of course, can we get a negative semi definite V dot, and we can complete the proof using standard signal chasing and Barbalat's lemma corollary type arguments. Of course, remember that in order to complete the signal chasing arguments, we need boundedness of various quantities. And this is where the boundedness of f required. So, the boundedness of f will be required in order to complete the signal chasing arguments.

So, this is where it gets applied, although, we stated the assumption, and we solve the whole problem without seeming required, but that is not true, when you try to do the signal chasing or argument with complete diligence specifically when you try to prove that e dot is in fact, a bounded then you will need the boundedness of this function. So, it is not like we made a frivolous assumption, we will need to use the assumption in order to complete this proof, so, I do recommend that you complete this proof, I do recommend strongly that you complete this proof so that you can understand sort of know what kind of assumptions go into the completing the proof.

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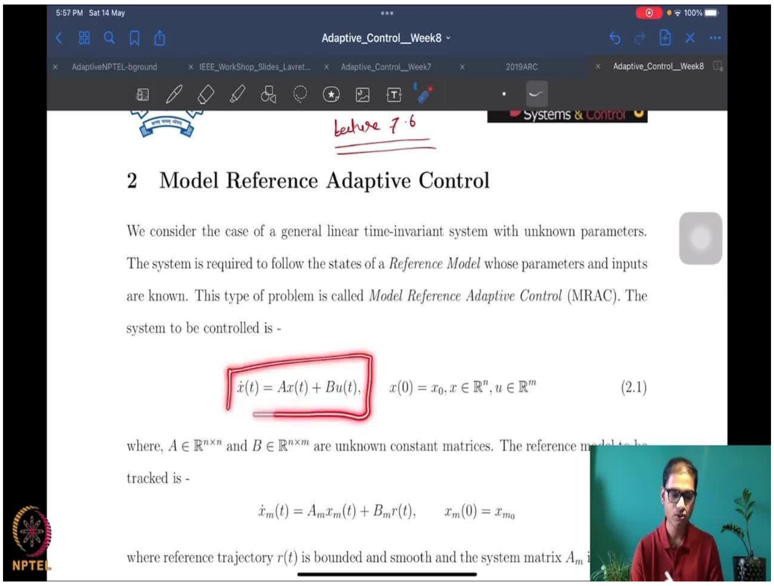

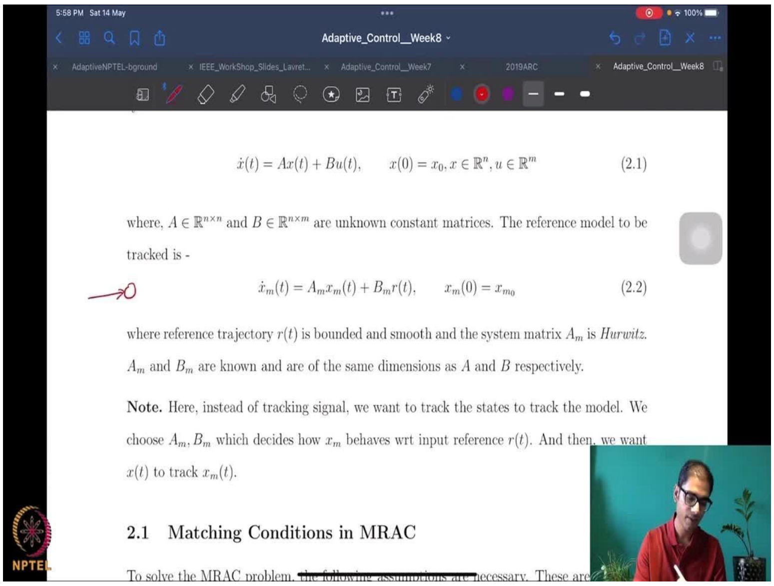

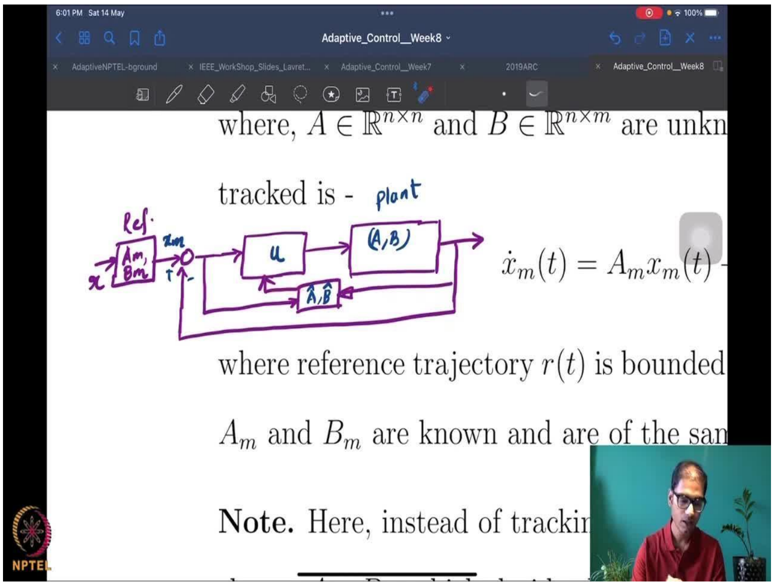

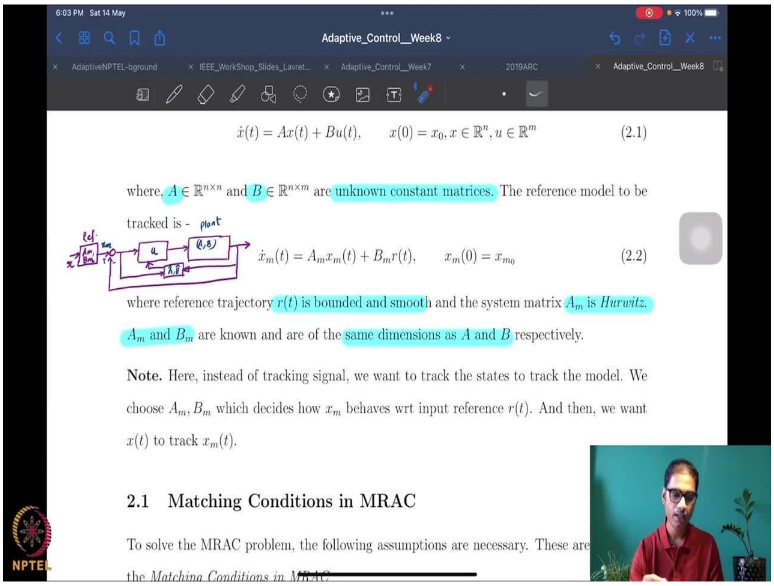

So, now that we have the completed this discussion on the unknown control gain, and we have already done the problem of unknowns in the drift vector field or the unknowns that are connected to the states, we are ready to look at this generic model reference adaptive control problem for linear systems. So, this is where we will begin today. So, I will mark it as lecture. 7.6. So, what is model reference adaptive control? This is a paradigm if you may, in adaptive control, and rather important and occupies a very large place in adaptive control literature.

So, the idea is, I started the linear system, which is x dot is A x plus B u. So, I hope all of you have seen linear system models such as these. And, of course, now we are dealing with vectors, we no longer looking at simplifying or not necessarily simplifying, but scalar systems and double integrators and think we are looking at a linear system first of all, we are not looking at non linearity, there is no f x t, but we are looking at a vector state x in R n control is in R m. So, the dimension of the control is possibly smaller than the dimension of the state. And of course, you have this x dot is A x plus B u, so not necessarily double integrator or anything like that.

So, here instead of x trying to follow some r, so, that the typical thing, x following r or x goes to r as time goes to infinity is what we have been looking at until now. So, instead of that, what we say is x has to follow xm and xm is generated as an output of the reference model, xm is coming out of a differential equation or as an output of a reference model. So physically, you can think of there being a reference model in which there is an input signal and there is an output and your system output is sort of tracking this.

So, if I let me see if I can, if we want to think of it as a block diagram, let me see how this will be. So usually, you have a reference here, so this will be let me make this bigger first. So usually, I will have a reference here, then I will have some kind of control blocks here, then I will have a plant block here, then I will have something like an output, I suppose. And this output of course, feeds back well actually not this output but typically you will have this going here.

And of course, because there is an adaptive controller, there will be an adaptation block, which will also take this input, and possibly also this input and the output from this goes here. So, and here something more complex happens you have an input system and there is a r. So, this is basically the reference plant, this is something like a reference plant I will probably write it here governed by Am, Bm.

So, this is the reference, this is the xm with a plus minus here. And then you have something like a control here u, then you have the plant here which is given by A B is the plant itself. You have the adaptive block, which is A hat, B hat, if I may, it’s crude, it is a crude form, but this is how it will look. So, this is like how this adaptive control block will look because the plant matrices A, B are also not known.

So, I will create an adaptive law for that that will also feed into the control, then there is a reference system which takes in a reference r and outputs what the system needs to track, which is this x of m. And then all of it comes together to give you the control and the control then, of course, feeds into this guy. So, that is how this model reference adaptive control block diagram will look like. So, that is the big difference here.

And of course, as usual, we assume that this reference input is bounded and smooth and all that stuff. We also assume that are of several assumptions maybe I should mark them. So, bounded and smooth, since Am is assumed to be Hurwitz. So basically, for the reference system, it has to be a stable reference system. So, the matrix Am is a Hurwitz matrix, Am and Bm are assumed to be known and the same dimensions as A and B.

So, and of course, A, B are unknown constant matrices. So, A and B are assumed to be unknown. So these are the quantities that are not known. So, this linear system is essentially unknown and you are trying to figure out what this linear system is, or at least you are trying to compensate for the fact that you do not know this linear system.

So, you literally do not know this linear system. But of course, there are a few other assumptions, I mean, we will come to the assumptions. So, but the, but these are the assumptions on the references system. So, anything that you want to track has to take as input a nice bounded smooth signal. So, this is just standard assumption, again, to ensure that the reference signal that or the xm that you are trying to match comes up to be a nice enough signal.

And then of course, we assume that Am is a Hurwitz matrix, which ensures again, stability of the whole system. So, there is a bounded input Hurwitz matrix, so output will also be bounded. This is standard and well known resulting linear systems, that if you have a stable system, which is perturbed by some bounded input, then the output is also going to be bounded. So, that is what we get.

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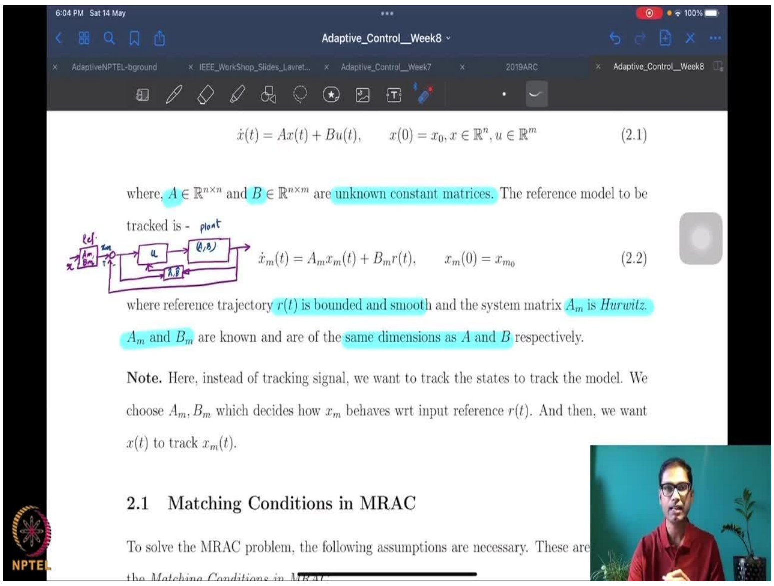


$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{2.1}$$

where, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are unknown constant matrices. The reference model to be tracked is - plant

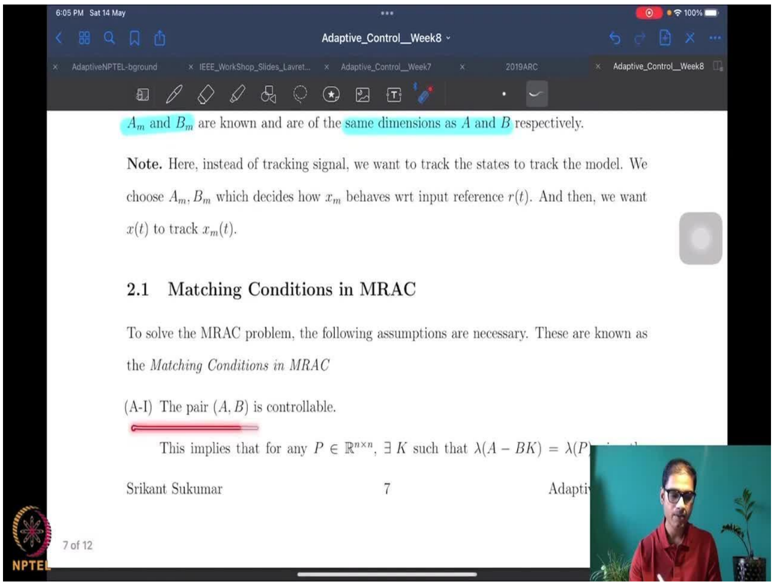
$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t), \qquad x_m(0) = x_{m_0} \tag{2.2}$$

where reference trajectory $r(t)$ is bounded and smooth and the system matrix A_m is *Hurwitz*. A_m and B_m are known and are of the same dimensions as A and B respectively.

Note. Here, instead of tracking signal, we want to track the states to track the model. We choose A_m, B_m which decides how x_m behaves wrt input reference $r(t)$. And then, we want $x(t)$ to track $x_m(t)$.

2.1 Matching Conditions in MRAC

To solve the MRAC problem, the following assumptions are necessary. These are



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2.1 Matching Conditions in MRAC

To solve the MRAC problem, the following assumptions are necessary. These are known as the *Matching Conditions in MRAC*

(A-I) The pair (A, B) is controllable.

This implies that for any $P \in \mathbb{R}^{n\times n}$, $\exists\, K$ such that $\lambda(A - BK) = \lambda(P$

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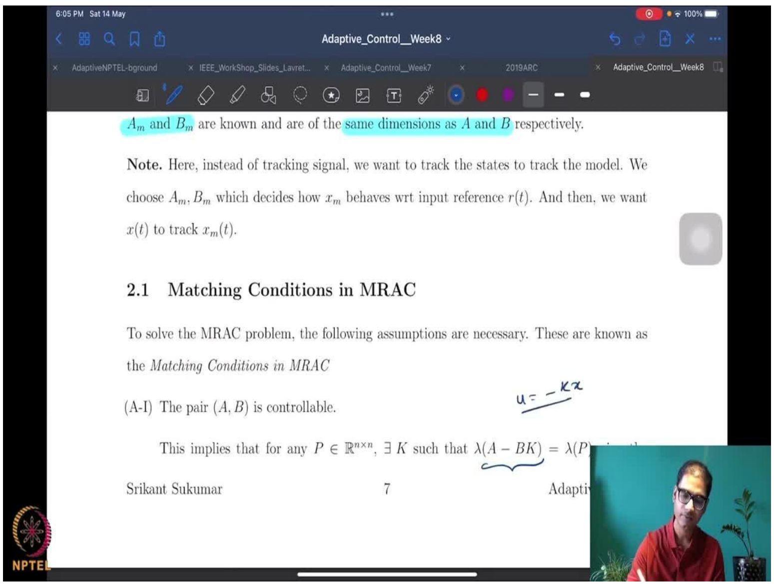

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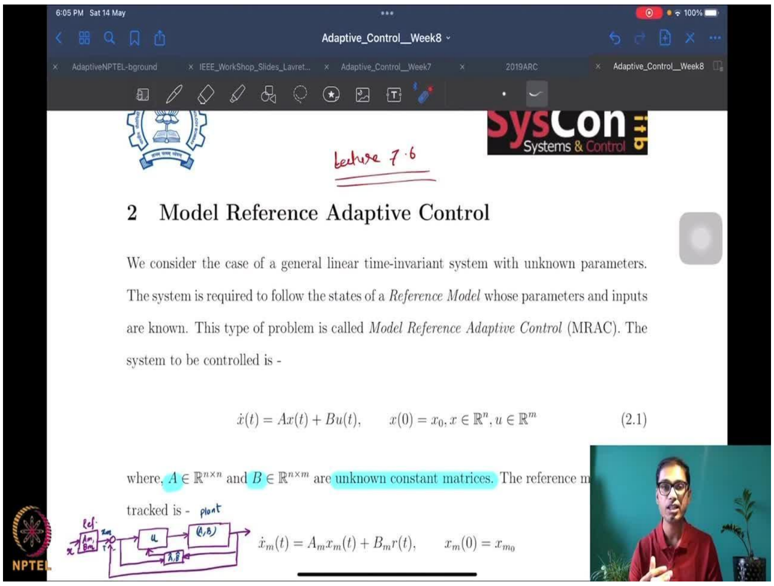

Lecture 7.6

2 Model Reference Adaptive Control

We consider the case of a general linear time-invariant system with unknown parameters. The system is required to follow the states of a *Reference Model* whose parameters and inputs are known. This type of problem is called *Model Reference Adaptive Control* (MRAC). The system to be controlled is -

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{2.1}$$

where, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are unknown constant matrices. The reference m

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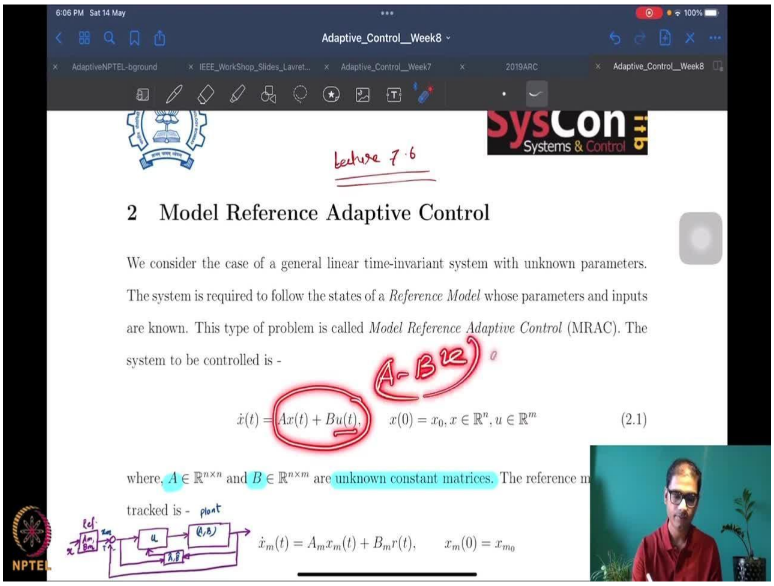

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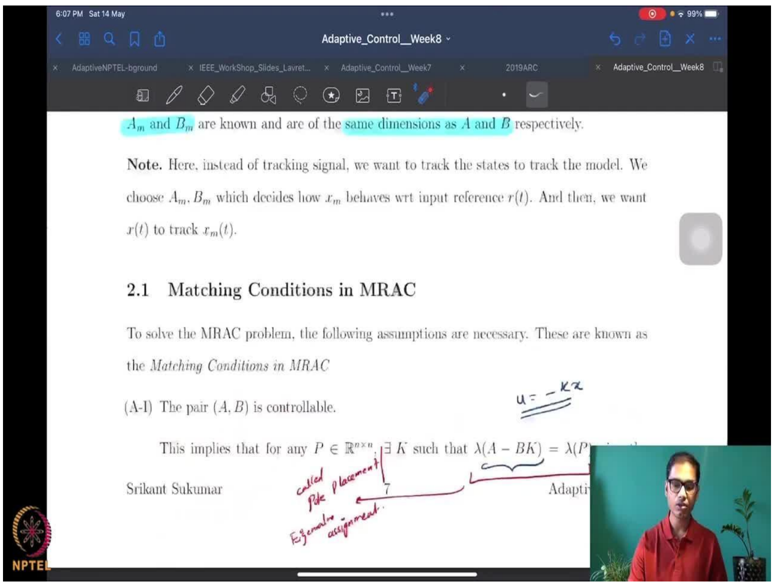

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$u = -Kx$

This implies that for any $P \in \mathbb{R}^{n\times n}$, $\exists\, K$ such that $\lambda(A - BK) = \lambda(P$

called Pole placement

Eigenvalue assignment

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So, now, what we want to do is drive x to xm. And we do not know matrices A and B. Now, the question is, well, actually, the first question that we ask is, what kind of assumptions are required? So, we have already put in several assumptions on the reference, but we also need assumptions on A and B, all the original system for that matter. And so, what are these assumptions? These assumptions are of course, called the matching conditions in model reference adaptive control, we are already used to matching conditions that is when you want x1 to track r, you want x2 to track r dot in a double integrator system, so this is also a matching condition.

So, we have such similar matching conditions here. The first is that the pair A, B has to be controllable, this is not exactly a matching condition, but we are clubbing it loosely under matching conditions because of the pair A, B is not controllable, there is no scope of designing controllers for the system. So, this is more like a necessary condition for us to be able to do any control problem or solve any control problem. So, the first is A, B has to be controllable pair, this implies that if I am given any matrix P in n by n or n by n matrix P there must exist the matrix K such that A minus BK equals P.

So, what does it mean? So, A minus BK is coming how? So, this is when u is chosen to be minus Kx, which means that, so controllability as you would already know I hope, if you do not I really urge you to go back and revise what is controllability for a linear state space system. So, what it means is that if A, B is a controllable pair then given any matrix P such that I want the system to look like x dot equals P x, then I can choose I should be able to find K in K such that u equals minus K x does the job. So, if I put u equals minus K x, this becomes A minus BK x and so, I have my desired system.

And now, notice that what we try to match are not the matrices themselves. So, notice very carefully that this assumption does not guarantee that the matrices A minus BK and P match, No, they just ensure that the Eigen values of the 2 matrices match. So, this is called standard pole placement if you may, I mean this is I would say called pole placement or in time domain it is called Eigen value assignment, this is called pole placement or eigenvalue assignment. What it means is? We are not really going to be able to match 2 matrices, but what so that not every entry of the matrix might match but what matches is the eigenvalues of the matrices. So, this is called pole placement or eigenvalue assignment. So, this is possible whenever A and B are a controllable pair, if you do not know what is controllable pair, please go and revise, that is the first condition.

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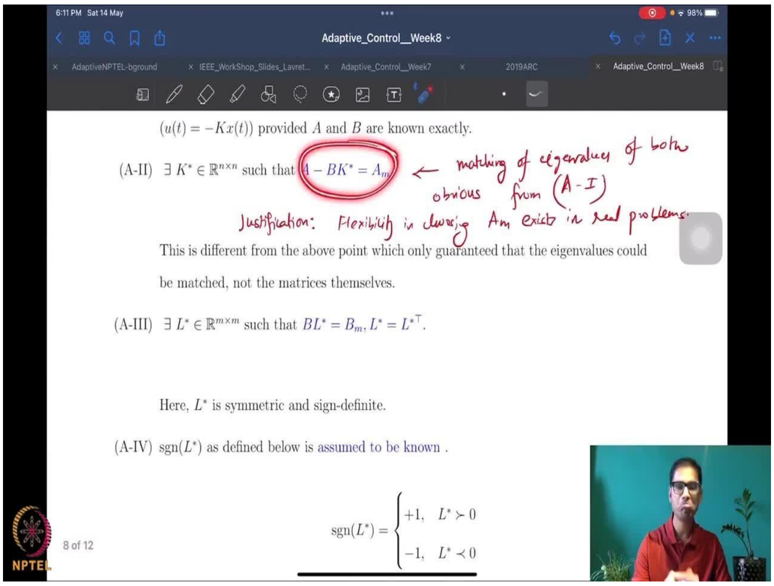

So, now, the next assumption actually makes this sort of more stringent it says that there exists a K star, again an n by n matrix, let us see. There exists a K star an n by n matrix such that A minus BK star is exactly equal to Am. So, now we have made it stringent. So, this is where it is different from point because point A1 because we are not just saying that the Eigen values are matching, No, we are actually saying that the 2 matrices match.

Now, there is much to be said about the assumptions that we are making, they are restrictive in several scenarios, and one might ask, why should it makes it is pretty reasonable to say that A, B is a controllable pair because without that control is not possible and therefore, that Eigen values of these 2 match, very reasonable, because you already said Am in pole which I can do any pole placement if A, B is a controllable pair therefore, matching the Eigen values of these is obvious.

So, matching of Eigen values of both is obvious from assumption A1, it is obvious from assumption A1, but we are talking about something more. So now, why a lot of times we are okay with this assumption A2, is we some how say that we have flexibility in choosing Am a lot of times, because we are not always specifying Am very stringently, the user or the designer sorry the user may not specify Am that strictly because in most cases we are interested in pole placement only and not this Am matrix itself. So, it is very possible that you will be able to choose Am such that the Eigen values are consistent with what the user wants, but it also ensures this kind of a matching happens.

So, sort of a justification for this assumption, we are not saying that this is without any flaws or anything, these assumptions have a lot of flaws, but again, these are the only conditions under which solutions exist. So, what is the justification? It is that flexibility in choosing Am exists in real problems, more often than not there is flexibility in choosing this Am. And that is why such a condition may be digested is what I would say, otherwise, it is not.

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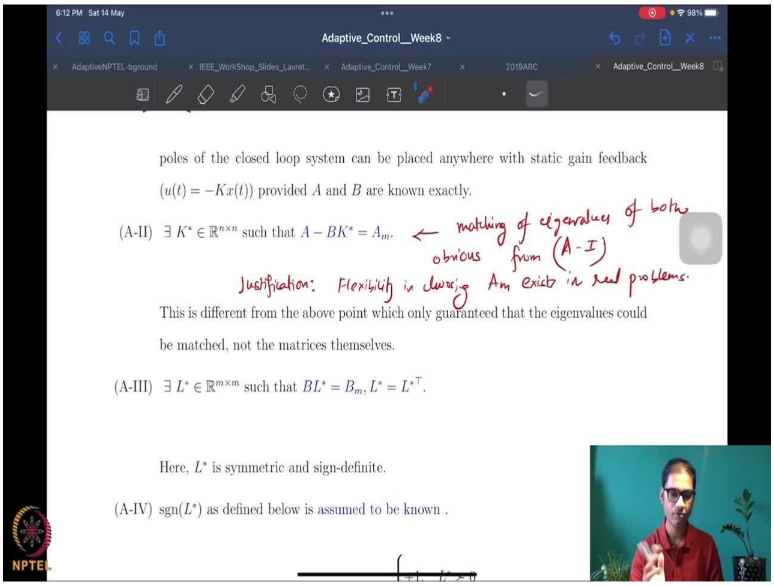

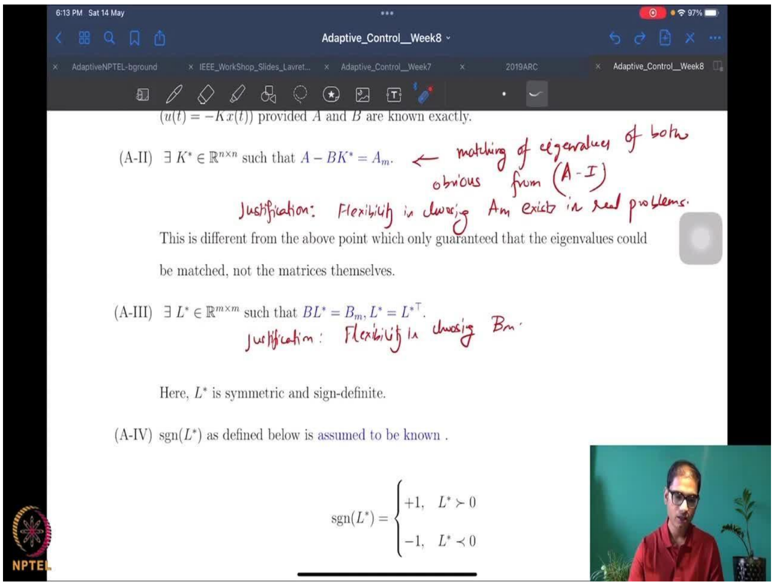

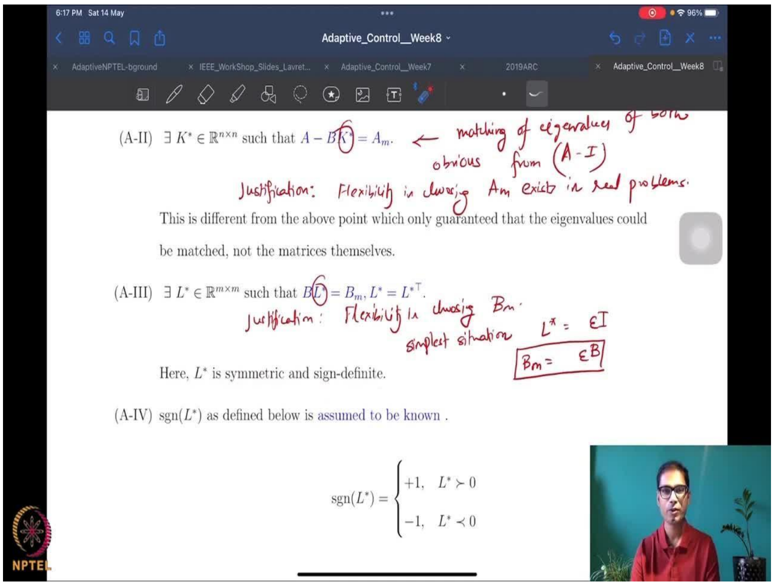

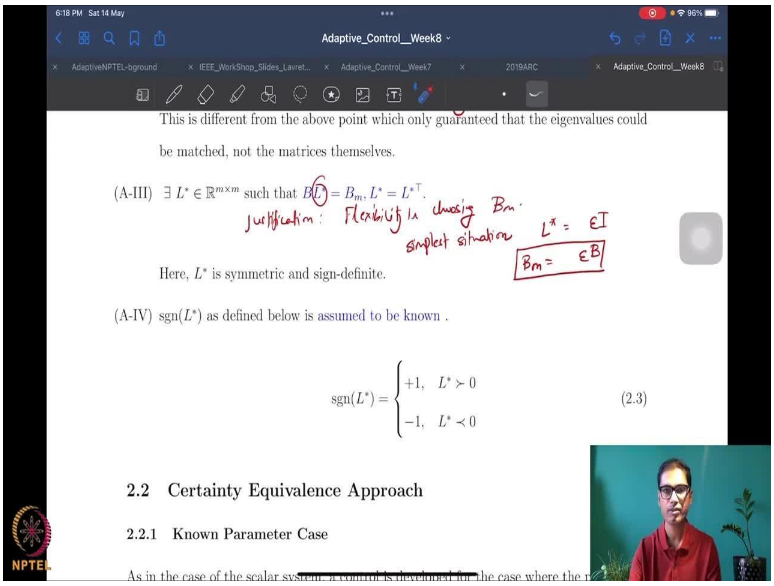

So, let us look at the next one write the next assumption A3 requires the existence of an L star such that BL star is equal to Bm. Now, again, remember that because I do not know A, I cannot have a knowledge of K star, I hope that is evident. So, what we are doing by this trick is actually again, redesign or reformulating our parameters, we are going to move from parameter A to parameter K star, so that is the idea. So, we do not know A or B, in fact, so we do not know K star actually. So, let us sort of I hope that is evident.

So, now, this next assumption is something again, similar along same lines, it says B L star has to be equal to Bm, so there should exist a L star such that B L star equals Bm. So, again the justification is similar and it is that there is a flexibility in choosing Bm there is some flexibility in choosing Bm. So, yes, these assumptions are to be taken with a pinch of salt, yes, I agree that these assumptions are very restrictive, but these are the sorts of justifications under which we operate. In fact, without this again solving model reference adaptive control problems is impossible, not more, without these assumptions. So, this is sort of what are the restrictive or the restrictions of the solution that we can provide.

So, the other, so, another part of assumption 3 is also that L star to be symmetric and signed definite. So, another part of the assumption is like L star is to be symmetric and signed definite. So, the way to think about this on top of talking about flexibility in choosing Bm is that somehow we have seen that Bm is the has similar definiteness properties as B. So, if I multiply B by for example, a positive definite matrix L star on the right hand side and I get a Bm, which has no specific properties or anything, because Bm is not required to have any specific properties.

Then, if you think about it, then this is almost might be identical to doing some kind of a nice group operation on B itself. So, there is if I just think of very simple or the simplest situation, let us think of a simpler situation, where let us see the simplest situation. Let us think of L star as some epsilon identity, some epsilon scaled with identity very simple situation. So, what is Bm going to be? It is going to be epsilon B. So, this is the sort of again the argument which gives us some hope that this kind of inequality can also be justified. So, if L star is very close to the identity matrix, then Bm is almost similar to B itself, but again remember Bm and Am are known, while B and A are unknowns.

And also remember that in typical adaptive control, we do not claim anything about being able to identify the parameters exactly or unique, we only claim to be able to compensate it and guarantee tracking. So, we have created we have made these assumptions, which are suitable for us to do this model reference adaptive control design. And we will go on to do a design and of course, we do some kind of a tracking of x with xm. So, x goes to xm is what we prove asymmtotically.

But of course, we will not be able to claim that K goes to get K star or L goes to L star, because our new parameters become this K star and L star so, these become our new parameters. And we sort of compensate for the effects of this of not knowing this case K star and L star. So, that is essentially what will happen and of course, we assume the L star is symmetric and sign definite also.

So, more assumption on this L star, so, several assumptions. And on top of this, just like the scalar case, in assumption 4, we also claim that we know signum L star. Remember, in the scalar case, we required the sign of B to be known. So, L star here actually serves like sign I mean serves to quantify the same thing. So, we required the signum of L star and what is signum of L star? Because L star is a matrix. So, what is the signum of a matrix, let us define here, it is defined as plus 1, if the matrix is positive definite and it is defined as negative 1, if the matrix is negative definite.

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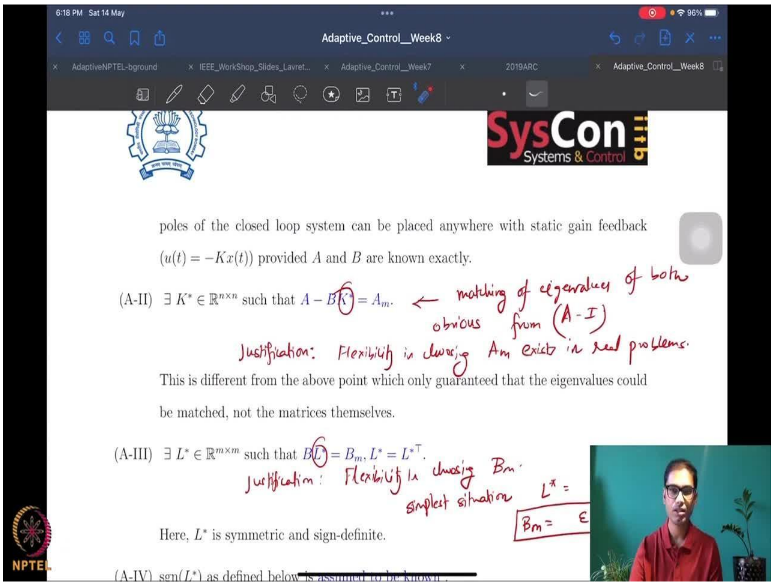

So, these are the sorts of assumptions we have I mean, I just wanted to start off this session with trying to understand the assumptions. First we have the controllability assumption, then we have the assumption on A and Am matching via this condition then we have a B and Bm matching via this condition not just that the matching matrix has to be symmetric and sign definite and further we also require to know the sign of this matching matrix. So, that is it.

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So, what we saw today is essentially a sort of setup of the model reference adaptive control problem, which is one of the most popular and famous and occupies a large space in adaptive control literature, a lot of adaptive control literature has developed around model reference adaptive control.

And we just saw the setup, we just saw the assumptions today in this session, and what we plan to do in subsequent sessions is to analyze and design adaptive controllers for the system. We also understand we take everything with a pinch of salt we understand that these assumptions may not always hold, but we also understand that adaptive control is not going to actually identify the parameters in several situations, but simply try to compensate for these unknowns.

So, in spite of the fact that these assumptions may not always be validated in certain circumstances, they these kinds of model reference adaptive controllers work very well. In fact, they have been flight tested on a fighter jets, so they have been functioning rather well for several decades now. And they actually give good real performance in one of the few nonlinear controllers which have actually been implemented in the industry. So, this is where we stop today and we will continue again in the subsequent session. Thank you.