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Nonlinear Adaptive Control
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Week-1
Lecture 4
Preliminaries – Part 3

Welcome to another session of the NPTEL on Nonlinear Adaptive Control. I am Srikant Sukumar, from systems and control, IIT Bombay. As always, we are looking at our very nice representative background of a rover on Mars. We hope that the algorithms that we sort of learn, how to design in this course will eventually help us drive autonomous systems, such as the rover on planets, such as Mars, or the moon, and so on. So, without delaying any further, let us go into our lectures.

So, where we were last time is that we had sort of looked at the notion of these normed linear spaces, we are already defined different kinds of norms. And we are also defined the structure, which is a normed linear space. And the idea that we sort of pursued last time was to prove that some of the norms that we have defined are in fact, valid norms.

What does the conditions, the conditions for a function from the vector space to reals to be a valid norm is that, it is non-negative, is 0 only when the vector itself is 0, satisfies the scalar multiplication property, and the triangle inequality. So, as we saw, usually, the first three properties are not significantly hard to show. And the triangle inequality property is what is a little bit more critical and difficult to in fact proof.

Let us go back. So, we did prove all the conditions for the infinity norm the way we had defined it. So, the infinity norm was, in fact proven to be a valid norm on R n. So, we wanted to mimic the same for the two-norm, we in fact started it, we prove the first three properties, which is that the two-norm is non-negative. The fact that the two-norm is zero only if the vector itself is zero, and the scalar multiplication property.

So, we were actually sort of there was a minor error, there was a slight error in how we were trying to prove the triangle inequality. And so that is what we want to first complete today, we want to actually complete the proof that the two-norm as defined that is the Euclidean distance norm, as we said, satisfies the triangle inequality property of norms.

So, let us look at this. So, we started with the two-norm of x plus y squared. And this is nothing but the summation from 1 to n, absolute value of x i plus y i squared. So, if I expand this, I get something like this, that is summation from over i equals 1 to n, I get x i squared, plus y squared, plus twice x i y i. Now, notice that even before I go from here to here, this is actually because I have squared it, so this is the same as taking the square of the quantity x i plus y i itself without the absolute. So, I can actually get rid of this absolute value here. And that is what I do. So, this is still all equalities.

So, the x i plus y i absolute value square is sort of irrelevant here, because I already took a square. So, this is also irrelevant. So, I get rid of this in this expression. And now it should be obvious that I can also plug back in the absolute values, and this quantity is just the square of the two-norm of x. And again, if I look at, I am sorry.

Look at this quantity. That is the square of the two-norm of y. And I am left with 2 summation i equal to 1 to n, x i y i. So, now this time, we do not do what we were trying to do last time. We will do it in a rather simple way from the knowledge of scalar dot products that all of you would know, holds true in the Euclidean space.

So, I know that this expression, right here is nothing but x dot product with y. This is just the scalar dot product of x and y. And what do we know about the scalar dot product, we know a couple of things. One is that the scalar dot product, where it acts like a projection of one vector on the other, if I divide by the unit vector, the other thing that we know is that it is a scalar value. That is why it is called scalar dot product. So, when I take a dot product of two vectors, it is a scalar value, and that should be obvious. This is also a scalar value. So, this is in fact, a scalar quantity.

And we also know that the scalar dot product is evaluated as this formula, norm of x times norm of y times cosine theta, what is theta? So, theta is simply the angle between the vectors, x and y. So, whatever is the angle between these two vectors, theta is that angle. And so, you will have x dot y is simply norm x times norm y times cosine theta.

Now, what do I know about the cosine, we know that the cosine lies between minus 1 and 1 for any value of theta, therefore, the cosine theta is always less than equal to 1. So, using that exploiting the fact that cosine theta is always less than equal to 1, I can immediately create this inequality that is this is less than equal to norm x times norm y.

So, we have very, very effortlessly proved that summation x i y i is less than equal to norm x times norm y. And this is exactly what we wanted it because I will plug it back in here in place of this, so it was, if you notice, it was all equalities until this point. But then it becomes an inequality because of this guy.

And what do I have here, this remains the same, this remains the same, but here are in place 2 norm x times norm y. And it is easy to see that this is nothing but norm of x plus norm of y whole squared. So now, if I sort of cancel the squares on both sides, I have exactly the triangle inequality that I wanted to prove. So, this is essentially complete the proof of triangle inequality for the two-norm. Like I said, we are not really trying to, we are not really going to see proofs of for the 1-norm, 3-norm, 5-norm for any other p-norm.

But typically, proofs will follow in the similar way. So, one of the important, very critical things to note is that this piece of the proof that you saw here, I am actually going to highlight it, because a rather critical piece of the proof. So, this piece of the proof that you see here is in fact, a proof of the Cauchy-Schwarz inequality in Euclidean space. If you notice, the left-hand side is actually how we write the inner product, we will look at this notation a little bit later. x dot y is actually xy, the inner product of x, y. So, this is actually the inner product.

So, you have essentially proven that the inner product of two vectors is less than equal to norm x, norm y, which is exactly the Cauchy-Schwarz inequality. So, we have proven a particular case of the Cauchy-Schwarz inequality right here. So, please remember this. So, this is in by the way, this Cauchy-Schwarz inequality is in fact, a rather key inequality, which is satisfied by all rooms, so the Cauchy-Schwarz inequality is in fact satisfied by all norms. So, you can also do a general proof, but for now, we have a pretty good specific proof. We are quite happy with it. Let us move on. We have now proved that, the norm that we have chosen as vector norms are, in fact, valid vector norms. Great.

Let us look at the next set of ideas, that we are sort of very keen on. We already spoke about the notion of convergence, very loosely. We kept saying limit of a function, or we said if the function converges to a constant, the derivative does not converge, function converges to a constant derivative does not converge to 0. Or the derivative converges to 0, the function does not converge to a constant.

So, we use this word convergence a few times already in the lectures preceding, but we have been using it loosely. In general, whenever, even in the past mathematics courses that you would have attended, whenever you would have spoken of convergence. It is always, always associated with the notion of limits.

So, there is, of course, they are very, very closely connected. Makes sense that whenever we talk about convergence, the notion of limits also shows up. It is very natural. However, let us look at a more formal way of defining convergence. So, once we have a normed linear space, all these notions can be very easily defined without a norm, because the entire idea of convergence is for terms to get close to a particular point. And there is no way to define closeness without the notion of a norm. So, it was very important for us to have a normed linear space.

So, suppose I have a sequence, this is the notation for a sequence. I hope you folks have seen this before, if not just look at, I mean this is very simple notation. So, each term is basically indexed by a i, and i goes from 1 to infinity, so a sequence inherently has to be infinite, there is no such thing as a finite sequence, as soon as I say sequence, infinitely many terms should come to your mind. So, the terms are all indexed. So, you have term x 1, x 2, x 3, x 4 and so on and so forth. So, sequence in a normed linear space is set to converge to a point in this space x.

If for all positive epsilon, there exist a positive integer, not just any integer, I would say Z plus there exists a positive integer such that x i minus x 0, the norm of x i minus x 0 is less than this epsilon that was given to us. For all i greater than equal to this integer n. So, let us sort of revisit this, we want to look at this again. So, things are very clear in our mind. So, what am I saying? I am saying that, any sequence in this normed linear space is set to converge to a point. If I am talking about convergence, I have to qualify it with a point to which we are converging, otherwise, it does not make sense.

So, I have qualified it to the point x naught. So, sequence x i, in x in this normed linear space is set to converge to this point x 0, if for all epsilon, so, if you give me an epsilon, so, the user has to give me any epsilon, and correspondingly I should be able to find an integer, a positive integer n, such that all my terms beyond the n th term, are at least epsilon close. So, this is an illustrative picture, this thing on the left here, these are illustrative picture. And this 1, 2, 3 is the term number, I am basically writing i here. So, i equal to 1, i equal to 2, i equal to 4, 3, and so on.

So, you note that, as i increases, my terms are getting closer and closer to 0, x 0. And this is exactly what is meant by this definition. It is not difficult to look at some examples. So, let us, let us actually look at some examples. So, let us see. Suppose I have x i equal to say 1 over i. So, what does this? So, x i converges to this is the notation 0. And you can immediately see how it is connected to limits. Why can I say that it is going close to 0. Because if given any epsilon positive, and I look at x i minus 0, so in this case, my x 0 is 0. So, if I look at x i minus 0. And I want this less than epsilon, that is what is the requirement.

I want to find an n such that beyond the n th term, everything, every term is epsilon away from the equilibrium, or epsilon away from the point of convergence x 0. So, I want to be epsilon away. So, what should I choose my i? Very easy. So, what should I choose my i as. So, it is very easy. I mean, it is all I have to do is that i should be greater than ceiling function of 1 over epsilon. What is the ceiling function? Is basically a function which gives me the closest integer bigger than a number. So, if I give you .7, ceiling is 1. If I give you .5, it is still 1. So, let us not worry about the halfs, but the ceiling function essentially gives you the, closest integer larger than the number.

So, if I give you 5.9, 6, 5.2, 6. So, the closest integer larger than the number. So, if I take i larger than ceiling of 1 over epsilon, then what do I know? I know x i is less than epsilon. So, I am done. As simple as that, you take an example x i is, let me repeat x i is 1 over i. I know just by my limit ideas that this is going to go to 0.

So, the point of convergence x 0 is in fact 0 in this case. So, how do I prove it? How do I find the corresponding n because in order to prove that any series sequence converges, I need to actually be able to give an n. So, suppose I start with any epsilon, epsilon is arbitrary notice that epsilon is very much arbitrary.

And I want to satisfy this sort of inequality for i greater than equal to N. So, what is it? So, this guy here I choose as my N, just using the ceiling function basically, I know that I want to have 1 by epsilon, anything larger than 1 by epsilon, but 1 by epsilon may not be an integer. So, I just choose the integer larger than 1 by epsilon.

So, I take the ceiling function. So, if I choose this kind of any i greater than equal to this N. Then, I know that x i is less than epsilon, and I am done. Because this, and this are the same, because all terms are positive in this case. So, the absolute value is really not much, the absolute value function does not play any role, great. So, that is how you sort of look at convergence. Of course, this is a very simple example, little more to do when the example is slightly more complicated, but not significantly more to be honest.

What is a Cauchy sequence? So, we have seen convergence, so Cauchy sequence is a slightly different notion, it says that if I have a sequence x i, again in a normed linear space, then it is said to be Cauchy, if successive terms start getting closer, that is essentially what is quantified, or qualified by these epsilons and Ns. So, the successive terms get closer. What does it mean? That if I am given any epsilon positive, again I can find an N. Notice all in in both these cases the N depends on epsilon. I mean, you can see N depends on epsilon.

So, sequences are to be Cauchy if for all positive epsilon, there exists a positive integer, let me again do this, there exists a positive integer such that successive points are epsilon distance away for i, j greater than equal to N. So, as you go, as your terms become, as your i becomes larger and larger, successive terms are close, that is essentially what this is. So, you can see this picture. In fact, the series that we just proposed, x i equal to 1 by i is also a Cauchy sequence. This is also a Cauchy sequence. I am not going to prove this you can actually think about.

So, I will actually say this find N, given epsilon. Please treat this as an exercise and do this. So, if I am giving an epsilon, try to find the N corresponding to this. So, you want the successive terms to be small. So, the N will turn out to be slightly different looking than what you have here. But the fact is, this is also a Cauchy sequence. So, an important thing to remember is that let us see. I am going to write this in red. Note, convergence implies Cauchy. But Cauchy does not imply convergence.

So, sequences is convergent, which is why the example I give you works, I know. Because the sequence is convergent. I know that it is Cauchy. In fact, from this definition, right from this definition, you can prove that it is a Cauchy sequence. I am not actually showing that proof. But you can do that.

But for this specific case, I encourage you to find N, given an epsilon to prove that this sequence is in fact a Cauchy sequence. The other way around is not true, if a sequence is Cauchy, it is not necessarily convergent. So, I can always construct funny example, and I have constructed one such funny example for you folks here.

So, my normed linear space is the open set 0, 1. what is the open set 0, 1? I really you know hope what is an open set, open set 0, 1 is everything inside the 0, 1 except for the endpoints, the endpoint that is 0 and 1 are not part of the set x, because I have essentially constructed this, if you may ridicule a set to prove my point that Cauchy does not imply convergence.

So, if I take my sequences, 1 minus 1 by n, what happens as n goes to infinity? One might ask. You can see very easily that as n goes to infinity, your x n actually goes to 1. So, this is where you converge to. But the thing is, that 1 is not part of x. So, this might seem a funny, weird, trivial sort of example, but it is not so trivial. You, the series seems to be tending to a point but the point is not part of the set. And if you notice the definition of convergence, it should be you should this should remind you of the fine points in any definition, the point where you converge has to be part of the set x.

This is rather critical, I mean, I have constructed sort of a fake example if you may, but there are there can be many more realistic examples where the Cauchy sequence will not converge. One good thing is most of the spaces that we concern ourselves with, they are in these spaces Cauchy sequences, in fact converge. And such spaces are called Banach spaces, or complete normed linear spaces.

So, most spaces we consider like the R n, R m, R p, whatever, they are all Banach spaces, that is all Cauchy sequences converge to some point in space. This is rather useful. I mean, a set sort of normed space does not have this property can be very troublesome space, because we are always interested in doing convergence analysis and seeing where different signals converge to.

Now, if you do not have a complete space property, and your sequence or your signal seems to be tending to a point, but then the point is not part of the set, then tending to a point which is not part of the set, then you will land in some trouble. As far as I mean convergence analysis goes. Because you have defined a norm only on this set, x. Outside the set x. You do not know if the norm is satisfied, you do not have the norm works, if the norm exists. So again, do not go just by this example, this is sort of constructed cooked up example. But there can be more realistic, and funny scenarios where you do not have completeness.

But also, as an aside, I must clarify that everything we consider in this course are going to be complete normed linear space, which is essentially also called Banach spaces. So, like we said examples are R n with the infinity norm, or the two-norm, or in fact any norm, there are all Banach spaces. So, you have seen quite a few notions, quite a little bit of structure, once we have the structure of a norm on a vector space, we see that we can talk about convergence, So, which is rather nice, we can talk of Cauchy sequences, completeness, and so on.

Another structure, which is a little bit more general than a norm is the inner product. So, again, we just saw it in the previous slide, I used this notation, where we said that we talked about the dot product, so dot product is the sort of the simplest inner product that we all know, in R n, the dot product is the standard inner product, and then it is also denoted like this.

So, that is what is an inner product space, an inner product space is again, a linear space. Here we call it a special normed linear space, but it is actually any linear space with an inner product operation, what is it? It is a function which takes two elements of the vector space and maps to the field. By field I mean where do each component of the vectors belong?

For example, when I have two vectors in R n, every component is a real number. So, the field is their field of reals, vectors in the field of reals. And then you have a few properties of these inner products. So, what are these, just like we had properties for the norm function. Similarly, for the inner product function also we have certain properties, and what are these properties? The first is that, it is symmetric, again we are talking about as spaces in the reals. So, otherwise there will be some conjugates and so on and so forth.

So, the first property is that the inner product is symmetry inner product of x with y is a same as inner product of y with x. So, sequences does not matter, then it has the distributivity property. So, x comma y plus z inner product is the same as inner product of x, y plus inner product of x,

z. The third is the scalar multiplication property, the alpha, the scalar multiplier just comes out of the inner product, and a final property is that the inner product of x with respect to itself is non-negative, and 0 if and only if x equal to 0.

So, this last statement that you see should already start to remind you of norm, because this is 1 of the norm properties. Here, you took one element x, and you said that this function is now non-negative and 0 only when the vector itself is 0. So, this is also a property of the norms, if you notice.

So, what is an example I already said, mean the mystery is gone. So, I already said the scalar dot product on R n is a valid inner product in the field of reals. So, and like I also said also hinted in fact, the x with this inner product is also normed linear space. As soon as it give you an inner product space, you can always construct a normed linear space by defining the norm as inner product of x with itself. So, given an inner product space, you always have a normed linear space. So, I will stop here.

So, in summary, what we looked at today is the, sort of we completed the proof of the fact that the two-norm the way we define is a valid norm. Then once we have the structure of normed linear space, we were able to talk about convergence, and we did. Then we looked at what is a Cauchy sequence, and how the two are not exactly equivalent always.

And then, if they are equivalent, the fact that there is a Banach space, and then we define an additional structure or a new structure, which is the notion of the inner products. And we also saw that the standard dot product that we have been talking about is in fact the inner product, and this inner product space naturally leads to a normed linear space also, just by virtue of this construction. So, that is where we will stop today. Thank you.