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Nonlinear Adaptive Control
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Week 6
Lecture No: 36
Overcoming the Detectability Obstacle: Ortega Construction

Hello everyone, welcome to yet another session of our NPTEL on Nonlinear and Adaptive control. I am Srikant Sukumar from Systems and Control, IIT Bombay. So, we are almost nearing the end of the sixth week. And we have already looked at adaptive control design for a first order scalar system. And we are well into the discussion of looking at adaptive control design for second order scalar systems.

And the rather cool thing is that a lot of mechanical systems, including the spacecraft by SpaceX that you see in the background orbiting the Earth fall into the category of systems that can be modeled by some kind of nonlinear second order systems. And so, what we are already looking at, in the sixth week of this course, will very much be useful to design algorithms and develop our controllers in order to, sort of control autonomously or drive autonomously systems such as what you see in the background.

So, this is where we were last time. And what we had realized was that for the second order system, we sort of hit detectability obstacle, because we started with a non-strict Lyapunov function for the known system case. And what it meant was that we could not even prove that the position error in fact, converges to 0. So, one of the solutions that was one of the obvious solutions is to obviously construct a strict Lyapunov function, which may not always be easy. However, that is definitely one viable choice.

However, one of the, in fact more simple solution was this Ortega construction, which was proposed by Romeo Ortega in the 90s. And this is, he in fact proposed a function, which was not e1 Lyapunov function. So it was Lyapunov like function. And we are starting to look at it from within the context of this simple spring mass damper system. It is not the e, the error system, but the spring mass damper system. And there is no parameter, either there is no parameter error either here. And the aim is to drive x1, x2 to 0 as t goes to infinity.

And the choice that he posed was basically something like this half x2 plus alpha x1 square. And we know that this is only positive semi definite, because we have V of k at k comma minus alpha k is 0 for all values of k. Therefore, in fact, along the straight line, yeah, along the particular straight line, this function is 0 arbitrarily close to the origin also, and therefore, this is not a positive definite function, as per our definition, and hence not a Lyapunov candidate either.

So how do we use this? So this is where we are today. So this is say, lecture 6.6. This is where we are. So, how do we do? How do we use? So, we take the derivative of this V, and that is x2 plus alpha x1 times x2 dot plus alpha x1 dot, and I simply substitute for the derivatives from our dynamics. So x1 dot is of course, x2 so it goes like this. And x2 dot was minus k1 x1 minus k2 x2, which is the standard spring mass damper sort of dynamics.

What do we do after that? We sort of club the extra terms, which is minus k2 minus alpha. And we pull that out here. So, what are we left with this is retained as it is, since minus k2 minus alpha is pulled out, the scaling on x2 is the identity now. And the scaling on this guy becomes k1 over k2 minus alpha. This is what we have. Now, if we make a choice of k2 such that it is greater than alpha. And I am allowed to because it is a spring mass damper, or if I am trying to stabilize I can choose k2 to be anything I want.

So if I choose k2 to be greater than alpha, and alpha is also something that I choose. So I have a lot of things I can tweak. And also I choose alpha such that that is exactly equal to this quantity, exactly equal to this quantity. Which of course, give me some sort, because this becomes a quadratic equation in alpha, so I get a viable solution here. So, if this quantity becomes alpha, you will see that this whole quantity becomes x2 plus alpha x1 whole squared. And that is what we write here, x2 plus alpha x1 whole square.

On top of that, I have already required put in the requirement that k2 is greater than alpha. So this quantity is strictly positive. Therefore, this entire quantity is negative semi definite. The other cool thing to see is that this exactly looks like the V, in fact, this as twice of V and twice of the V function that we chose. So that is what I do that I substitute x2 plus alpha x1 square is twice V.

And I can see that V is of course, a scalar value quantity. So I can see that this is a nice exponential decay it V there is a nice exponential decay. In fact, I can even solve this I can even solve this to say that Vt is V0, e to the power minus 2 k2 minus alpha. And in terms of x2 plus alpha x1 also I can say that, so, two things happen one is that this is a V is actually half squared of this and V0 is half squared of this, so the half's cancel out and then I take a square root. So therefore, the 2 goes away from here.

So, that is essentially the exponential decay of V and therefore the exponential decay of x2 plus alpha x1. And, of course, alpha is also positive one. So, this is what we have this is what we have.

So, now, we do a little bit of signal chasing type analysis, the steps are slightly different here. But they still work out fine. So, the first thing is V is still lower bounded because V is greater than equal to 0 and it is non increasing because V dot is less than equal to 0. I mean, even though V is not a Lyapunov candidate, I still have both of these that it is lower bound and it is non increasing. Therefore, V infinity exists and is finite. V exists plus finite and V infinity is what we use to denote this limit us limit of V as V goes to infinity.

So, further, because we is x2 plus alpha x1 square, we know that this is a bounded quantity, we can also show that x2 plus alpha e1 is in L2, and x2 dot plus alpha x1 dot is in Lp.

In fact, this step is I would say not required because, we have already proved that x2 plus alpha x1 goes to 0 as t goes to infinity. So, we do not need this do not need this step. This can already be claimed from, this can already be claimed from your exponential decay of V, we already did this. Because you can see, this is the formula for x2 plus alpha x1 at time t. So, if I take limit as t goes to infinity on both sides, this guy is going to 0.

So, x2 plus alpha x1 is going to 0. So, what do we know? We know two things, x2 plus alpha x1 is a bounded signal. So, let me call this some phi of t defined as that and I know that the limit as t goes to infinity phi of t is 0, sorry, not phi of t, I have already called it p of t later on. So, p of t is defined as this, limit as t goes to infinity p of t is 0, so I know these two things. I know these two things.

I know further that x2 is equal to x1 dot, so I can rewrite p as also x1 dot plus alpha x1. So what do we do? We now apply the Laplace transform on both sides and do a little bit of rearranging of terms. So how do I do that Laplace transform here gives me t s, and here I get as x1 minus x1 0, and, and alpha X1 s it goes to the other side to become minus alpha X1 s.

Now, in order to compute X1 s, I combine these two and I get X1 s, X1 0 or s plus alpha and P s over X s. Just by, rearranging and manipulating terms. Now, if I apply the final value chain in order to apply the final value theorem, what is it, I know that final value theorem says that the limit as t goes to infinity, x of t is basically equal to the limit as s goes to 0 as X s. And that is what we do. So, I multiply s on both sides, and take the limit as s goes to 0. So I get from the left hand side, I get limit as s goes to 0 s X1 s, which is actually this quantity. And further, and of course I can, on the right hand side, I have this guy.

Now what do I know about this? I also know that p goes to 0 as t goes to infinity. So from here, I immediately have that again by the final value theorem. limit as s goes to 0, s times P of s is 0.

Now notice, this is what is there here in the numerator. So when I take the limit here, as s goes to 0, I can distribute it in the numerator in the denominator. So the denominator as s goes to 0 is alpha. And the numerator limit as s goes to 0 is 0. Therefore, the outcome is in fact seen.

And here it is just 0 because there is an s here is some constant, and this becomes alpha. So basically, the limit here is 0. So what have we shown? Is that x1, let us see. What we have shown is that x1, in fact, goes to 0, as t goes to infinity. And that is what by Final Value Theorem.

Further, we already know p goes to 0 as t goes to infinity, and p is x2 plus alpha x1 dot. Now, if I take a limit on both sides of this. So if I take the limit as t goes to infinity on both sides of this. What do I know? I know that this guy is going to 0 I know this guy is going to 0 therefore, x2 has to go to 0, as t goes to infinity. There is no two ways about that.

So, what have we been able to show? We have been able to show that both x1 and x2, there are a couple of things. One is we have been able to show that both x1 and x2 go to 0. Now, what about boundedness? It is not mentioned here, but it is also important for our trajectories, our state to remain bounded.

Now, we can do this in separate, two separate ways, I mean, you can choose another Lyapunov function which is say V bar equals half the same thing as before k1 x1 square plus x2 square and we have V bar dot is minus k2 x2 square from this dynamic, just by substituting for the dynamics of the spring mass damper from the previous example, I know that this is exactly what they get, and this is less than or equal to 0. And this will immediately imply that x1 x2 uniformly stable at the origin. So, stability can be obtained if you want by this earlier Lyapunov candidate which is actually a Lyapunov. So, I do not have to worry about stability that was already given to me.

But, but this particular Lyapunov like construction actually gave me convergence, I can claim that x1 x2 go to 0, as t goes to infinity. So I even got my nice asymptotic convergence. So, what we want to do is? We want to use the same idea for the adaptive control problems.

So, we already had this sort of a control. And we already had the original dynamics. And of course, we had the error dynamics, which was exactly like this. And now instead of choosing k1, k1 e1 squared plus e2 squared, we choose this as Lyapunov function for it, I will not call it Lyapunov function, but the Lyapunov like function, it is the Lyapunov like function does not a Lyapunov function, use this one, because this is not a positive definite quantity.

And now we again take the time derivative just like before, so I get e2, e2 plus alpha e1, e2 dot plus alpha e1 dot and this quantity just moves. So this guy just remains as it is all through. Because we are yet to choose the theta hat dot, so we choose the theta hat dot every time by taking a Lyapunov candidate or a Lyapunov like function. And from the derivative, we try to choose the theta hat dot, so, that V is at least negative semi definite. So, V dot is at least negative semi definite that is really the idea.

So, again, we substitute for e2 dot and e1 dot, e1 dot is just e2 and e2 dot is this entire thing. So, wait a second, this is not correct, e2 dot is in fact plus theta tilde, this is what is e2 dot. And then what do we do we again, if you notice, I still get minus k2 minus alpha here. So, I take this minus k2 minus alpha common and I get t2 plus alpha e1 as it is and the second term, we choose by appropriate choice of alpha.

And with k2 greater than alpha, you will get this sort of a term that if you are confused, you can simply write this as minus k2 minus alpha e2 plus alpha e1 is the same calculation as before e2 plus k1 over k2 minus alpha e1, that that is what becomes of all these terms. Except for this theta tilde term. So, everything except the theta tilde term becomes this.

So, therefore, if we choose k2 greater than alpha and alpha equal to this guy, we will get here, so, these terms become this. And the theta tilde term we club, the two theta tilde terms together, so, I take the theta tilde common down here, and I get e2 plus alpha e1 times f and here I get theta hat dot over sigma. So, now, notice that we have to choose a different update law, update law here is now, this guy.

Earlier if you notice the previous, Lyapunov candidate, the theta hat dot only contained e2. But now, it has e2 and alpha e1 both has e2 and alpha e1 both. So, that is what and of course, I mean then use the gamma for that adaptation gain here you use the sigma that is fine. So, sigma is also some positive adaptation game. So that shows up here. So, once we choose that this term essentially vanishes and we are left with this gain, this is also negative semi definite.

Now remember, we did not do any new magic we did not so. So, what happens is that V dot is again the same as what you have in the non adaptive case even though the V may have started different. So this is still the same negative semi definite V dot.

But, what do we know now? So, here, the steps are slightly different. Whatever I had, I had said here that this step is not required and this step is not required, but now we will require the steps. We will. Why? Because this is not V anymore, the right hand side is not V anymore, V will contain also the theta tilde squared. So, I cannot use directly the expression for V. So, that is what we say hence we would not be able to write V dot as minus gamma V as in the known parameter case. So we use signal chasing arguments.

So what is it? So this is where we use the earlier arguments, we know that I am going to write those here to be honest, and because we said we do not need that. So, I will write these steps a, b and c. So, what is step a? V infinity, so V greater than equal to 0 V dot less than equal to 0 implies V infinity exists plus finite.

Then the second step is that the e2 plus alpha e1 and theta tilde are both bounded because that is up that is in the V and V is non increasing. Therefore, both e2 plus alpha 1 and theta tilde are bounded.

The third step is e2 plus alpha e1 is L2. Why? That can be obtained by integrating this thing from 0 to infinity and I can, I can do this because V infinity exists and is finite. And we just showed that.

And further finally, we can write that e2 plus alpha e1 the derivative of that is also bounded. How? Let us compute the derivative, e2 dot plus alpha e1 dot, in fact, it is exactly this guy here inside the bracket, this is e2 dot plus alpha e1 dot and everything here e1, well I mean after taking this k2 minus alpha common you will get end with this sort of a choice essentially you will have e2 plus alpha e1 the derivative will be from here it will be that e2, wait a second, it is minus k2 minus alpha times e2 plus alpha e1 plus one over k2 minus, plus one over k2 minus alpha theta tilde f.

So, this is what you will have and so, now, I know that e2 plus alpha e1 is bounded theta tilde is bounded if I assume boundedness on f, I have also the derivative.

Therefore, the derivative is bounded. So, simply by the Barbalat's corollary to the Barbalat's lemma I can claim this e2 plus alpha e1 goes to 0 as t goes to infinity. And I also know that e2 plus alpha e1 is bounded. So, I have the exact same thing.

So, I have the exact same thing as before. So, now, I do the same I can do the same steps and I can I can say the same steps. If I, what do I know Suppose I define p of t as e2 t plus alpha e1 t. I know that p of t belongs to L infinity and I know that limit as t goes to infinity p of t is identically 0. And I also know that p of t can also be written as e1 dot t plus alpha e1 t, then I can use the same final value theorem arguments by taking Laplace transform here. So, this is Ps s E1 s minus e1 0 plus alpha E1 s and so, I will get E1 s as s e1 as sE1 s as s e1 0 over s plus alpha plus s times P s divided by s plus alpha.

So if I take the limit as S goes to 0 on both sides, I know that this is going to 0, I know that this is going to 0. So basically I have this is equal to 0. So, which means that implies that e1 is going to 0 as t goes to infinity. And because p is also going to 0, and p is e2 plus alpha e1. So, I also have that e2 is going to 0 as t goes to infinity. And that is exactly what we want. That is exactly what we want, for the adaptive case. That is exactly what we want for the adaptive case.

So notice that there was a change in how I chose the V, V function, which is now Lyapunov like function. And there was of course, a change in how we update theta hat dot, because we change the V, as soon as you change the V. Remember that the update law was always obtained by computing the V dot and trying to make it negative semi definite. And so it is obvious that if I change the V, the choice of V, the choice of the update law, of course, changes.

So it is not like I am getting convergence of the tracking errors to 0 for free. Because this is just analysis thing, after all, choosing a V and doing derivative and substituting dynamics is the analysis, one might ask, what did I actually change in the controller. So I did, the controller remained exactly the same, but the update for the theta hat changed. And with this change in the update, using the same sort of steps that I did, for the known case, using this, Ortega construction.

I could actually show that both e1, and e2 are now going to go to 0 as t goes to infinity. By the way, this sort of proof that we did that if e1 dot plus alpha e1 goes to 0 as t goes to infinity, this proof can also be done in the time domain, it is not impossible to do this proof in the time domain.

Because this is a nice exponential e k, nice system, it is a nice exponential decaying system with a bounded time or bounded vanishing perturbation. So this sort of analysis can also be done. I mean, I would strongly encourage you to look at how to prove the same thing without the final value theorem and going to the s domain or the Laplace domain.

I would strongly urge you to try to do this simply by directly integrating this system, because this is just a bounded vanishing perturbation on an exponentially stable system.

So, so that is great. So this sort of brings us to the end of this week's lectures. And we have done rather interesting thing we have seen our first set of adaptive control problems, we saw that how to do the first order case start with the unknown case, start with a known case, use certainty equivalence to design a control for the unknown case, and then turn to then update law using the appropriately Lyapunov candidate function.

And we also show how to choose a candidate Lyapunov function. But as soon as you want to do second order system, we saw that it is not very difficult to pick the detectability obstacle if you end up choosing a non-strict Lyapunov function for the original system. It is not that complicated, or that difficult to get to a detectability obstacle.

And then we also saw a means of alleviating the detectability obstacle by choosing what we call the Ortega construction, which is essentially a Lyapunov like function, which still helps you with the signals using analysis for these kind of integrator type systems. So you essentially have an integrator type system if it was not an integrated type system, this type of construction may not work, you might need to use a strict Lyapunov construction.

But then with this type of Ortega construction, we showed that with the second order scalar system, we are able to, in fact, make a nice adaptive control design and show that both the tracking errors go to 0. As always, we do not claim anything on the parameter either we of course show that it is bounded, but we do not claim anything on whether it is converging or not to the true value.

And this, of course brings it notions of persistence of excitation. We looked at that for the first order scalar case, we did not look at it for the second order scalar case. But it is similar. You can do a very similar thing for the second order case also, and probably use persistence results that you have discussed before to claim that under rich enough trajectories, you will be able to identify the parameters.

But, how to find the rich enough trajectories, how many frequencies do you need and the trajectory these are all still heuristics. It is more like trial and error. And like we have already discussed, it may not always be possible in real application to create these arbitrarily oscillating trajectories, because you may not want your robot to follow such trajectories.

So I hope it was a very interesting week 6, interesting excursion into the start of the design process. And so henceforth, we will see more and more of the design. So I am very hopeful that you will be able to pick up too much from here and start applying to these small tiny problems that all of you are probably already trying to solve in your own field of work. Thanks a lot, and I will see you again soon.