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**Non-Linear Adaptive Control**
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**Week-6**
**Lecture No: 33**
**Barbalat's Lemma and Signal Chasing Analysis**

Hello, welcome to yet another session of our NPTEL on non-linear and adaptive control. I am Srikant Sukumar from Systems and Control, IIT Bombay. So, we are into the sixth week of our lectures on non-linear adaptive control and in this week we have already started looking at our first adaptive control problem. So before we move on of course we want to remind ourselves that the algorithms that we developed and the systems that we analyze with the tools we have learned are meant to develop and design algorithms that drive systems such as the spacecraft that you see in the background and I mean autonomously. So that is what we are targeting to do with the tools that we learn in this course.

So a little bit more detail on what we have been looking at, so we started with our first sort of adaptive control problem and this was a first order scalar system, so it is a first order differential equation describing the system and of course it is a scalar valued states and therefore we are dealing with the first order scalar system where the unknowns appear linearly and are assumed to be constant.

So how we went about this is that we wanted to sort of the control objective was basically to track a desired trajectory r of t and therefore we designed an error variable, so this is the for standard step we design an error variable e of t, which is essentially just x minus r and we also then load the dynamics for this error variable. Since we know the dynamics for the state writing the dynamics for the error variable is rather easy.

Then the first step was to do a control design for the case when the parameter theta star is assumed to be more and we did that by choosing some kind of a nice target system we talked about the properties of our target system that it should be viable, that is it has to have some kind of a matching with the original system and secondly it has to ensure that the error has nice asymptotically stable properties.

So this is the target system we choose in this case and it satisfies both our requirements in order to get to this target system we require a control which looks like this and it is evident that the parameter of course appears in this control because we essentially cancel the non-linearity and once we are able to do this in the known parameter case we choose a radiant bounded Lyapunov function and we compute the derivative which turns out to be minus k e square which is asymptotically stable and all the nice things and so on and so forth. Now beyond this it was obvious that just by using the Lyapunov theorems error goes to 0 all the nice jazz.

Now how do we deal with the uncertain or unknown parameter case is the application of the certainty equivalence principle which essentially said that you retain the same structure of the controller but you change the you replace the true parameter value which is unknown to you by its estimate. This estimate is going to be designed subsequently so with this altered controller we get a altered error dynamics which contains now the parameter error which is denoted as theta tilde and so what is the idea? The idea is that we use a Lyapunov function to come up with the theta hat dot. We use the Lyapunov candidate to come up with the theta hat dot dynamics. How do we do that, we take the earlier Lyapunov candidate we had and we add to simply a quadratic term in the parameter error.

It is scaled by this gamma factor which is essentially called the adaptation gain, you will see why because it appears in the adaptation law and we just then start simply taking the derivative. We of course do not know what is theta hat dot, we are yet to prescribe it, but we know what is e dot that is substituted in here and it so happens magically like I said but actually not magically but by construction that you have theta tilde appearing in both these terms and therefore theta tilde can be taken out common and so you have this an equation and what you do you simply ensure that this quantity is zero. Why do you have to ensure this quantity is 0, because it is a mixed term like this term is a nice negative quadratic term, there is no sign definiteness in this term which means I cannot guarantee it is positive or negative definite in fact we want negative definite things in the V dot.

Therefore, the best thing we can do is to cancel this and make this 0 go away then so this is what we choose you choose theta hat dot from here so that just goes away and this is essentially your adaptation law and you see that this gamma appears here which is essentially what we call the adaptation gain, because changing gamma will change how fast the adaptation happens. So if I make gamma really small then the adaptation is really fast and so on and so on.

The important thing to remember and a mistake that a lot of folks make is that your update law or your control law cannot depend on theta tilde enough, because theta tilde is unknown because it contains theta star. Therefore, theta tilde cannot appear in the control law or the update law because if it does then you control or your update law is not implementable and that is of course no longer adaptive control it is actually not something that can be used at all.

So then what do we do we that was the final stage we got this V dot is minus k e squared what we notice is that the V dot we get in the adaptive case that is in the unknown parameter case is the same as the V dot we end up with in the known parameter case, but there are two difference well there is one major difference is that in the known case there is no theta tilde square which is obvious because there is no error parameter was assumed to be known but in the unknown case there is in fact a theta tilde square term in the V which means in a known case V was sorry V dot was negative definite because there was in fact only one stage but in this case we have actually introduced an additional state theta tilde because we have to otherwise we do not know how to update our parameter. .

So the fact that we get the same V dot does not mean we also get negative definiteness in this case we get only negative semi-definiteness why because we had emphasized this many times during the Lyapanov analysis sections and also now that V dot if you are if any function does not contain some of the states of the system then it cannot be definite it can only be semi different, and that is what this is only negative semi definite and therefore by using the Lyapanov theorems you can only claim that the e and theta tilde states are uniformly stable at 0. So you can only claim uniform stability and nothing more, so this is critical. So this is where we where until last time and then we will see now today how we go forward from here and in fact gain all the properties we need because we need asymptotic stability not uniform stability.

Uniform stability is not enough because it just says that if you start small you remain small if you start with small errors remain with small errors and things like that, but that is not enough we want the error to actually go to zero, we want to do actual tracking and that is what is the whole point of this adaptive control theory, so how do we do that so that is what we see in today's lecture, let me mark it, it is lecture 6.3. So the first thing to remember is that what we do with typically mixed terms that is the terms that we cannot cancel sorry that the terms that cannot remain negative definite is we try to cancel them and that is what we did when we chose our adaptation law as simple as time.

So now what we do we only get uniform stability so we carry out the signal chasing in Barbalat’s Lemma analysis. Now remember that we already saw a sample of signal chasing in Barbalat’s Lemma when you are looking at the analysis part of the syllabus and so we should already have a little bit of an idea of what is about to come and so we start.

We have if you remember I had spoken at that time extensively about there being these very very standard steps these are steps that I had mentioned at that time that all of you just need to master and in fact memorize so that you never change the sequence of the steps say once you just follow these steps the same set of steps work in almost all cases in fact there are a lot of these new research articles on adaptive control, the authors do not even mention these steps anymore because it is so standard that once you have a V dot they pretty much conclude whatever is expected.

So they expect that the reader already can follow these steps on their own, great. So this is the signal chasing analysis so remember where we are I will again write out some key points so V was one half of e squared plus one half of gamma and so this is of course positive definite radial unbounded and all the nice things and V dot came out to be minus k e squared which is negative semi-definite, so this is what V and V dot were.

So this is where we start, so the first thing we claim is that V is lower bounded and non increasing so that is obvious so V is lower boundary because it is strictly positive definite and it is non-increasing because V dot is less than equal to zero; the derivative is less than equal to 0 of a function of time then of course the function cannot increase over time it can do anything but not increase over time, so it is a non-increasing function and we know from these two properties then there exists the limit as t goes to infinity for V of t. So limit t goes to infinity V of t exists and is finite and what do we do, we denote this with V infinity we are going to use this V infinity in our subsequent calculations, so that is step one.

Remember you should just compare these steps with what we did I believe we did this for the spring mass damper example using the Barbalat’s Lemma; should compare these steps and you should be able to see that these steps are rather standard like they are the same almost the same steps that we did even there and in the same sequence also. So now the second point is since V is finite, now why is V finite; so V0 is of course finite, does not make sense if you already started an infinite state or infinite parameter error or something like that just not realistic so V0 is finite and what do I know I know that V of t less than equal to V0. So notice that I am using this funny notation where now I am earlier V was a function of e and theta tilde dot I mean although I have not written it, it was something like V is a function of e theta tilde, but now suddenly I am writing V as a function of time.

So please do not get confused whenever I say V of t it is actually defined as V e of t n theta tilde t and what is e of t theta tilde t these are solutions. If there is a solution trajectory so remember there is a lot of things that are happening here that are implicit which I am not saying but that should occur to you and that should be very clear to me. The first thing is when I write V of t that is V as a function of time I mean that I am now plugging in the value of e that I am plugging in the solutions here not just e and theta tilde as some variables but the solutions. Now remember though I have written this as e of t and theta tilde of t, this implicitly depends on e0 and theta tilde 0. So e0 is here and theta tilde 0 is here, infact both are in both of them so these are actually also dependent on e0 and theta tilde 0, please do not forget this.

So whatever we get as an outcome of this analysis is not uniform with respect to initial conditions, because once you choose an initial condition you get some outcome. The good thing is we do not have to actually specify what initial condition because for any initial condition all these entire analysis will go through we are not that worried but still it should be there at the back of your mind that this expression of V of t that I write here the so nonchalantly actually contains the initial condition because V of t is obtained by writing the solution of e and theta tilde as a function of time that is how you get V of t, and that is how you compute V dot otherwise there is no question of computing the time derivative if there is not time. This is how I compute a time derivative and so this when we use this notation you remember that this is basically just the directional derivative along the dynamics and this is exactly consistent with V being a function of time.

If I took dV by dt this is exactly how I would take dV by dt, I would take dV by de theta tilde and theta tilde dot, is exactly how I do and therefore the two definitions are consistent so therefore when I take derivative with respect to time this is actually a valid sort of operation. So therefore V of t has a lot of hidden meaning, so please remember this. So now that we know that V dot is less than equal to 0 therefore V of t is less than equal to V0. What does it mean; it means that V of t is also finite because V0 is finite and if V of t is finite it means that e of t and theta tilde of t are also finite and what do we know about signals that are finite we know that they belong to class L infinity.

If a signal is finite for all time then it belongs to class L infinity because the infinity norm is finite because it is just the supremum, if e of t and theta tilde of t are finite for all t then the supremum norm that is the L infinity norm also has to be finite. Now the next thing that we do is we integrate both sides of this equation so and it is claimed here that e is L2 from integrating both sides of the V dot equation now how do we get that here it is not sort of explained how do we get that let us try to integrate this so if I integrate say V dot dt 0 to t and this is less than equal to minus k integral 0 to t e squared tau d tau. Now this is actually equal to dV, so this will be V0 to Vt and this is dV and this is less than equal to minus k again 0 to t e squared tau d tau.

So what can I say now I can say from here if I take limit as t goes to infinity on both sides what do I get, I get that V infinity minus V0 is equal to minus k 0 to t e squared sorry not this is 0 to infinity e squared tau d tau, which means that 0 to infinity e square tau d tau is equal to V0, V0 minus V infinity divided by k. And what is this? This is just the square of the infinity norm. So this is basically just equal to the two norms squared and we have just proved that it is finite so that is what we are saying that by integrating both sides I can get that the two norm is of e is finite which means that e is an L2 signal that is what we have proved that the two norms squared is a finite quantity therefore two norm itself is a finite L2 norm is a finite quantity and if a signal has a finite L2 norm then the signal is set to belong to L2 that is what we say.

Now we can immediately well I mean are we done? No we are not yet done we are not yet done there is a step in between here and what is that, this in between step I am going to mark it as 3.5 and what is it I can also claim that e dot belongs to L infinity, how, what is e dot, let us see let us try to look at what is e dot; e dot is just let us say e dot is this; so it is minus k e plus theta tilde f so let us just copy it here so here what is minus k e plus theta tilde f x t. So if I assume f x t bounded for bounded x and for all t, if I make this assumption I already know that e theta tilde are bounded. So I know that e is bounded therefore k e is bounded, I know theta tilde is bounded, also if e is bounded then x has to be bounded, because so e infinity also implies x infinity why because e is just x minus r and r is a bounded smooth signal.

Therefore if e is bounded x is also bounded, if x is bounded I am assuming that this function is bounded for bounded x and then for all time, so therefore this entire right hand side is bounded that is what is e dot is an infinity and then we can directly use the corollary of the Barbalat's Lemma I really hope all of you remember if not you should revise it go back and revise it I am not going to state; well I will state it essentially the corollary says that if a signal is f L infinity and Lp for some p and if the derivative of the signal that is e dot is L infinity then the signal goes to 0 as t goes to 0 and that is what we have for e we can claim that e is both L2 and L infinity. So p is 2 and I can also claim that e dot is infinity and therefore I have that e goes to 0 as t goes to 0.

Now notice we already have the uniform stability result, that the fact that you know e and theta tilde are uniformly stable at the origin because of the standard Lyapanov theorem, but now we also can claim that e converges to 0 as t goes to infinity. Then this is very good very good for us, why, because we had an unknown parameter and what did we do we designed an adaptive control, why because we had the control, we had a controller and we also had an update law and with these two elements with these two elements I can claim that the tracking error actually goes to 0 as t goes to infinity and also remains uniformly stable which is rather nice, I cannot claim asymptotic stability of the entire system like I would have loved to but I still can claim something rather nice that the tracking error remains stable and converges to 0 of the origin as t goes to infinity which means that I get my exact tracking in spite of an unknown parameter, so this is the power of adaptive control I want you to sort of absorb it.

So I have not said anything about theta tilde, so look at what happens let us look at what happens. So but the important thing to note is that the tracking error goes to zero, in spite of there being unknown parameters. Now we want to know if the unknown parameters converge to the true value or not. How do we do that; so we know that e dot is integrable since e goes to 0 as t goes to infinity. So how do we claim that e dot integrability means that if I do this they should be finite and this is because I will essentially get limit as t goes to infinity e of t minus e 0 here which is just minus 0 because this limit is going to 0, you just proved that; so e dot is an integrable signal, so that is the first claim here. Second is you can compute e double dot to verify that e double dot is also bounded, how do we do that it is not too difficult and what is e double dot you already so what is e dot.

So I have to take the derivative of that so this is e double dot is equal to minus k e dot I am not writing k e dot explicitly yet plus theta tilde dot plus f plus theta tilde f dot, I have removed the arguments from f. So I am not going to write k e dot so this is minus because I know e dot is already bounded, so minus k e dot plus theta tilde dot, what is theta tilde dot? Theta tilde dot is 1 over gamma e f, theta tilde dot is in fact negative of 1 over gamma sorry 1 over gamma e f times f so it is e f square plus theta tilde f dot. So now what do I know, I know that e dot is bounded we already proved it then we know that e and f are bounded, f is bounded by our assumption e is bounded by the Lyapanov analysis. Now if I further assume again something on f dot, assume f dot bounded for bounded x and all time.

See we make a similar assumption on f dot as we did on f then what can I claim, I can claim this entire quantity e double dot is also bounded. So that is what we we can verify very easily that e double dot is also bounded. Now one of the results that we had looked at said that if e dot is uniformly if e double dot is bounded that is the derivative of a signal is bounded then the signal itself is uniformly continuous that is what we claim here and so then by the original Barabalat's Lemma, what does the original Barabalat's Lemma say, it says if a signal is integrable and if it is uniformly continuous then it goes to 0 as t goes to infinity. So from that we know that e dot was integrable and we know that e dot was uniformly continuous therefore e dot goes to 0 as t goes to infinity.

So we have not just proved that e goes to 0 as t goes to infinity we also prove that it is derivative goes to 0 as t goes to infinity, which we did very carefully without resorting to our usual mistakes or the pitfalls in adaptive control or pitfalls in non-linear analysis, where if a function goes to 0 a function goes to constant we assume its derivative goes to 0 and things like that. No, we did this very carefully and formally, so e dot goes to 0 as t goes to infinity. Now what was e dot, e dot was this guy so if I if I know that the left hand side goes to 0 as t goes to infinity I know this guy goes to the 0 as t goes to infinity then what am I left with I know that this product also has to go to 0 as t equals infinity and that is what I write here, but this does not mean anything about the convergence on theta time and that is what is standard in adaptive.

So what did we look at today we sort of tried to complete our analysis for this first order scalar system and we are sort of at the end of this analysis essentially we required the use of Barbalat's Lemma and Signal chasing in order to complete this analysis and what we are able to show is that inspite of a parameter error, the tracking error goes to 0. So therefore we can do exact tracking so if it is a robot can exactly track a trajectory but we cannot guarantee that the parameter converges to its true value we can only show that something like a theta tilde times f goes to 0 which does not mean necessarily the parameters goes to 0. So this is where we stopped today and we look at more details of this kind of a problems in the subsequence sessions, thank you.