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Nonlinear Adaptive Control
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Week – 5
Lecture 30
Analysis of Parameter Varying Systems
Using General Integral Lemma - Part 1

Hello everyone, welcome to yet another session of our NPTEL on Nonlinear and Adaptive Control, I am Srikant Sukumar from Systems and Control IIT, Bombay. We are again in front of this nice motivational image of the Space X satellite orbiting the Earth. And we are already well on the way of being able to analyze and soon being able to design algorithms that will drive systems such as these to perform autonomously without intervention from the earth.

So, we are almost at the end of our discussions in the week 5 and we have been looking at the topic of persistence of excitation and how it is connected to stability of time varying linear systems specifically. The context of these time varying linear systems is as I mentioned from parameter identification algorithms, we are of course, yet to see these parameter identification algorithms themselves, but will do so rather soon. So, let us sort of look at what we were doing last time specifically.

So, at the end of the last session, we had already true for the vector case that is this problem we had proof for this dynamical system that if phi is persistently exciting, then this is in fact uniformly exponential sorry globally exponentially stables uniformity is anything included. So, we got this rather nice result for these dynamics, where phi is now a vector in Rn.

And, we also just stated we did not actually complete the proof of it, I encourage you to complete a proof of it, but we stated sort of nice exponential stability result for a system of this kind, which is very, very standard structure in model reference adaptive control, where e was the tracking error and theta tilde is the parameter estimation error, we usually get something a system of this kind in the model reference adaptive control context.

And we of course, under certain assumptions, we claim that this system is in fact uniformly globally exponentially stable if and only if this signal phi which is this signal is persistently exciting.

So, this is sort of where we were until last time. So, today, we want to state general integral lemma or nonlinear parameter varying systems. So, that is what that is where we are going to begin today. So, let me sort of mark this as lecture 5.6. So, what is this integral lemma? So, if all of you remember you would already knew we have already seen an alternate version of exponential stability theorem here. Now, this is restricted to systems which have state and time dependence, which is already pretty general. So, one would wonder what more do you require.

But if you are looking if you start to look at what is called parameter varying systems, nonlinear parameter varying systems, that is basically it systems which have a structure like this not here, these are called nonlinear parameter varying systems. Why because you can see that there is this lambda which is a parameter.

So, when we say what is a parameter, a parameter is just some constant value. So, you keep plugging in different values of the parameter lambda, and you get a sort of slightly different or slightly modified dynamical system, this parameter could be anything for example, when we look at the Van der Pol oscillator, if you remember there was a parameter which affected the behavior of the van der pol oscillator. So, similarly, there are many many parameter varying systems especially in adaptive control.

And so, it is rather critical to be able to talk about stability of these parameter varying systems when for a certain range of parameters, I mean, I do not want to claim stability of a parameter varying system for one particular value of the parameter, but I want to be able to claim that for a certain range of parameter values, I am guaranteed to have nice stable performance for example, so, this integral lemma what it actually says is something like that a stability result of parameter varying systems and this is along the lines of this alternate exponential stability theorem. Why do we say that? So, let's look at this integral lemma first.

It says that if there exists constants r, c, p positive. Such that you have you have that this max of infinity in the L infinity norm and the Lp norm for some p and some x 0 in Br is upper bounded by c norm x0. Then x equal to 0 is to be lambda uniformly locally exponentially stable.

And further if this r is infinity that is if c exists for all x0 in Rn, then it is lambda uniformly globally exponentially stable.

So, what is when do you say that lambda is you have lambda ULES, so, what is the purpose of the lambda the point is that it is uniform local exponential stability holds for all lambda in some domain D. So, for a certain range of values of lambda uniform local exponential stability holds, so, to sort of lambda independent in some sense therefore, it is called lambda ULES lambda UGES.

So, now why do I say this is similar to our alternate exponential stability theorem? All you need to think about all you need to do is recall what was the L infinity norm or sorry all the Lp norm and L infinity is just the bound in some sense. So, what is the Lp norm it is basically something like a 0 to infinity norm of xp it is you have to put the time argument here because this is just the vector norm.

So, this is what is the Lp nom for some value of p and you remember that this is sort of an imposing a bound on this some sort of an integral condition on the signal therefore, it is called the integral lemma. And similarly, here to when we look at the exponential stability it is alternates exponentially stability theorem here too, you have something like an integral requirement some kind of a bound on the integral of V dot, which is sort of a function of the state in time.

And so, therefore, this integral lemma is somehow to be seen as a extension of this alternate exponential stability characterization for parameter varying systems, so, this is what is this result there is a very nice exercise that I want you guys to attempt.

If you remember, we already looked at this simple harmonic oscillator and we in fact did to the stability of that in this class in this previous one of these lectures, you see, we start use this simple harmonic oscillator and we constructed a an overlay upon a function in order to be able to prove an exponential stability. In fact, we could prove exponential stability for this case. So, now, what we are saying is, let us modify the system and add a parameter.

In fact, two parameters lambda one and lambda two, notice lambda here can be a dimension, so, it is not necessarily scalar or something like that. So, here it is lambda one and lambda two. So, lambda is basically a vector in r2.

And further we are saying that we have a domain for lambda, which is that it is strictly positive that each component that is lambda one and lambda two are both strictly positive quantities. So, what we are asking you to do is to use the integral lemma to prove that this is in fact UGES lambda UGES and lambda ULES, so, one of the two in fact, whichever one you can.

So, this is rather nice, I mean, intuitively again this all makes sense that if these gains are any take any positive value you are fine, but what we want you to do is to prove this for all possible values of lambda one lambda two positive and therefore, we want you to invoke the integral lemma. So, this is rather nice.

Now, associated with this integral lemma, there are also another couple of results. And we are stating these results here, and then I will work a problem, so, not in exact same sequence. But anyway, we are stating these results here, because we want to show the utility of this new integral lemma also to prove convergence of parameter identification systems.

Also, as always, this entire discussion on persistence, this entire week has been on using persistence to claim some kind of stability for parameter identification systems. So, we want to do the same for parameter varying systems also. So, of course, we have very good motivation for it, we will talk about it soon. But first I want to define another notion and also talk about one more lemma which is for parameter varying systems.

So, let us look at this. So, this is lambda uniform persistency of excitation. So, we have already defined persistence of excitation, but now for parameter varying systems, we are defining a new version of persistence of excitation, and that is the lambda uniform persistence of excitation. And it is also shortened as lambda uPE. What is it, it is pretty straightforward, you have the same kind of function. The only thing is now the function also depends on a parameter because your system depends on the parameter. So, you can hope that this function also depends on some parameter.

And we have this outer product of this phi, which is now lower bounded by mu I. Notice we do not use the upper bound here, so I think we had I had already mentioned, when I did make the definition of persistence of excitation, that a lot of people do not use the upper bound, because that is just really codifying the boundedness of the signal, which is sort of okay, so, we are not really using the upper bound here.

But the lower bound is the critical one, which cannot be avoided, that is what is most critical for persistence. So, persistence of excitation, this expression look exactly the same barring this lambda parameter appearing here. So, what are we saying, we are saying that we satisfy a similar condition for all t. So, over a sliding window of time capital T and for all lambda in the domain. So, this is the additional requirement here.

So, that this happens for all possible values of lambda in the domain, therefore it is a lambda uniform property, it is a lambda uniform property notice that mu does not depend on small t as always the usual things. So, this is what is lambda uniform persistency of excitation, just an extension of persistence of excitation property to parameter varying systems.

Now, once we have this property of lambda uPE, there is something called a measure lemma, which says that if the function phi is also upper bounded, then the length of time for which it exceeds a certain magnitude is lower bounded by a positive quantity, so, this is sort of intuitive again, so if you think about it, so if you think a persistent signal, so we make some axis, you think a persistent signal.

Whatever to be signs, sinusoids or something like that. So, you when if you say you have something like this (())(14:53), so if I have something yes I am not sure why this has happening, this is a sort of persistent signal. The point is if I make take any window of persistence say whatever I mean I think this will be like a persistent window.

So, if I keep sliding over this time T say the period is T here, T capital to be larger than this time period, then it becomes a persistent window. So, if I slide over this time T there is of course, some kind of a persistence of excitation that is this sort of a condition will be satisfied.

Now, the important thing to remember is that, if a signal is persistent and it is also bounded and this is a bounded signal, as you can see, then we are saying that the signal cannot remain very, very small for the entire length of time. Otherwise, you cannot have a uniform persistence type property, so, that is what we are saying. So, there exists some finite lower bound on the time, this is the length of time I mu t is actually the length of time you we actually denoted in terms of measures of the set and so on, but, for the purpose of the discussion that we are having, I mu t is just the length of time.

So, this is like the length of time, why we do not call it length of time is because I mu t could be split in several pieces. So, the idea is that this signal phi if you try to draw a horizontal line of this size, say this, so, this is the line, so, this is phi of course, the y axis is phi. So, if this is the line and you try to look at when this is greater than this value, it is this, this, this, this and so on. So, what the claim is, is that this length of time over which it is positive, that is this time, you can use a different color this time plus this time plus this time, in the x axis is in fact, lower bounded by strictly positive one in fact, given to be this value.

So, basically, we are saying that if you make a sort of horizontal line above which you want your curve to lie, and if it is persistently exciting, if it is upper bounded, then you are saying that the length of time over which it exceeds this value has to be strictly positive, otherwise, the signal cannot be persistently excited. So, this is like a if it is like a one way result of course, if it is lambda uPE, then there is a strictly positive length of time over which it exceeds a certain value.

So, if that value is given by this, I mean, you can choose a different value and you will get a different answer here, but the point is as long as you choose a value here, it will be a strictly positive quantity here, this is important.

So, this is the property that we can actually use to prove the asymptotic convergence of a system like this. Notice, this is our standard scalar system that we have been considering until now. But now, the important difference is the presence of this parameter lambda, one might ask again, why do we care? Why do we have a parameter?

So, again in parameter identification systems, it is very common to have this kind of a setup false, where you are again with this way your term connected to the x or basically the theta tilde that is a parameter error in fact, would also have a parameter dependent term attached to it, not just a time dependent.

And therefore, we and we would like some kind of stability property which is uniform with this parameter, it cannot depend on this parameter. And this is the usual requirement. I mean you we will try to point it out again when we get to it. And therefore, it is rather important for us to also look at stability of these systems. So, how do we go about it, as usual, we are saying this is greater than equal to 0, you are also saying that at lambda is lambda uniformly, persistently exciting, now it just P, but lambda uniformly.

And then we, earlier we actually integrated this in order to try to prove that this is like this has some nice properties. Now we do not integrate, we take a Lyapunov function, Lyapunov like function if you may, as one-half x squared. And if you take a V dot, it is pretty simple, you get minus a squared x squared, which I know is negative semi definite. And now I want to start invoking the integral lemma.

What is the integral lemma, the integral lemma simply requires that the max of the infinity and a p norm should be upper bounded by some scaling of initial condition norm. And this is the vector norm, by the way, just the vector norm on the right-hand side of x0 of the initial condition.

On the left-hand side are both sigma norms. So, if you have such a norm, you have uniformly local or uniform global exponential stability.

So, now what do we have, we have that Vt is less than equal V0 because this is less than or equal to 0, therefore, it is not increasing. From that I can simply write x squared t is less than equal to x squared 0, therefore, absolute value of x t is less than equal to absolute value of x0. And what is the Infinity non-infinity norm is just the largest absolute value.

So, we already know that for all t, this happens, xt is less than or equal to x0, which means the largest possible xt, which is the infinity norm or the supremum norm if you may is also less than equal to x0.

So, done, so one is done, so, I have already bounded the infinity norm by some scaling of the initial condition. I mean, I can write it as absolute value (())(22:17), you can write as a norm it is a scalar quantity, so it does not matter. Absolute value is the norm. So, the next thing that I want to do is I want to prove that some p norm is also upper bounded by the initial condition, how do I do that? So, I know that we start using our Bartlett's lemma type arguments, we know that V is lower bounded, and it is non-increasing, therefore, the infinity V Infinity exists and is finite.

So, if I integrate both sides of this equation, now, just like one of the steps in our signal chasing analysis, I hope all of you remember it. If not, I would ask you to go revise. If I integrate both sides of this from 0 to infinity, so the left-hand side is V infinity minus V0, I can do this only because of this guy here. And otherwise, I cannot actually integrate. And so, this, I can do this. And this is V infinity minus V0.

And the right-hand side here is simply I have just reproduced it here as 0 to infinity a squared t lambda x squared tdt.

Then, I am going to invoke all these measure lemmas and the integral gammas and so on. So, what do I know? So, I just look at this quantity here. And I reproduce it here. What do I know? I know that I can split this time into intervals of time capital T, I can keep splitting this time into intervals of time capital T. That is what I do. I just take summation from 1 to infinity, integral from k minus 1 t to kT of the same thing, why do I do that?

Because I know that my persistence is in the center. And k of course, goes to infinity, so summation all the way to infinite tie. Now, let me be careful here.

Now, let me expand this. So, we can read this better. So, if you look at this, I come here and here I invoke my measure lemma. My measure lemma says that by the persistence property lambda uPE property, I know that this integral, sorry this, this pi will be larger than this for this much time. And that is what I do. So, I actually, because all the quantities here are non-negative. So, I can only do parts of the integral I do not need to do the entire integration. So, what will I do? I will just do part of the integration. And what is that part of the integration?

And I know so basically what do I know, I know that for this part only, I just integrate this piece, so I pull it out, I can pull it out because there is a constant, constant lower bound on this. And that is this guy, so the constant lower bound on this is this guy, I have just taken a square here. And this is the time interval.

Because there is a constant lower bound on this for this much time, I can actually pull this out and have this much time here multiplying. Let me actually think, is this correct?

So, I assume I can basically assume that it is larger than a signal, which is exactly this value here, within this interval, and in fact, zero everywhere else. Like so, basically, what I am saying is, because I am integrating it, I can see that is where it is larger than the square of this for a certain interval.

And beyond that, I do not care. It might as well be 0. It might as well be 0 because I do not really care. (())(26:55) as it be 0, I am just wondering if I can write this as k minus 1 T or into kT or I have to actually consider this sort of integral only. That is sort of what I am wondering.

So, I think you cannot have this sort of an interval anymore, what you will have to do is this integral will be over I mu comma t set, and I do not think there will be this multiplication anymore, because this is already a bound. So, this will be I mu, I would say k capital T, because it is the kth interval.

Or I can even index this with k. And this will be basically just x squared t dt. This is just basically x squared t dt. And this is I am, so, what I am claiming is this is actually t equal to some norm, the two norms squared from 0 to infinity. So, I am claiming that this is alpha x2 the two-signal norm with x whole squared. So, this all of it, so but I think there is a little bit more of a step to be done here.

So, anyway, let me let me continue with what will happen. So, you see that I take this integral only over this much time, I still need to sort of answer whether it should be from k minus 1 T to kT or just this piece. It looks like it will have to be for just this piece. And if this is equal to in the limit, this alpha which is this quantity, this is alpha is just this guy. And this summation can go back here, because k is appearing only here.

Then this if this becomes equal to the two norm to signal norm. Then I can plug this back into this inequality here, because this negative sign goes here to become V0 minus V infinity and this is known to be greater than equal to the alpha times two norm of x.

So, therefore, I have alpha times two normal facts to be less than equal to V0 minus V infinity, which means that the two norm of x will actually this is like a squared, this is a square. So, this is also squared, square is less than equal to two norm less than equal to V0 minus will V infinity. And since V infinity is positive, this is less than equal to V0 by alpha. Again, what is V0 by alpha, this is actually equal to non x0 square by 2 alpha. This is non x0 by 2 alpha, so, I am done. I bounded the two norm the L2 norm by some, I can take square roots on both sides.

So, I bounded the L2 norm by some constant multiplying x0, and I bounded the L infinity norm with the constant multiplying x0. And that is what we need for the integral lemma to be satisfied assumptions of the integral lemma to be satisfied.

Both are upper bounded by some constant multiplying x0, so just take the larger constant and you have this and it is done, we took p equal to 2.

The only thing I need to sort of verify, which I will do at a later stage is how this translates to the two norm, it is not difficult because we are taking sort of limit as k goes to infinity, things going, this is fine. But this is how you can prove that for the parameter varying case. Also, you have lambda uniform global exponential or lambda uniform local exponential stability.

We also have another tiny result on output injection, which is a parameter dependent version. I mean, this is part of the notes, we do not use it directly as of this stage. This is useful when we are trying to prove the vector version of stability for the system, like a phi phi transpose type of a thing. There is like we did for the purely time dependent case. But I am going to leave it be as of now because we do not really use it directly, so if we do need it later, we will invoke it.

So, this is this was sort of the conclusion of our lectures for this week number of 5. And we looked in detail on persistence of excitation and how it is connected to stability of parameter identification systems. We learned a lot of technical tools, alternate exponential stability theorems, uniform observability and you see a wondering output injection, we also looked at a more general integral Lemma and corresponding definitions for parameter varying systems. So, it was a very technical, technically intense week, I would say, I would really urge all of you to carefully look at the lectures from this field and try to understand as much as possible. I will see you folks again soon. Thanks.