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Nonlinear Adaptive Control
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Week-1
Lecture 3
Preliminaries – Part 2

Hello all, welcome again to our NPTEL on Nonlinear and Adaptive Control, I am Srikant Sukumar from systems and control, IIT Bombay. So, as always, we want to be motivated by this very, very exciting background image that we have, we want to eventually be able to figure out how to design algorithms that drive systems for autonomous motion such as this. So, without delaying any further, we continue with our lectures.

Alright So, the last time we looked at a couple of myths and temptations, which sort of plague typical asymptotic analysis. And, after that, we moved on to some vector and matrix norms, basically giving us the ability to sort of deal with objects that evolve on normed linear spaces. So, we defined norms, on vector spaces and therefore, gave them a structure which is that of a normed linear space. We also defined, what search norms are, we looked at few examples of these norms. So, after that we also spoke about the matrix induced norm. Basically, a norm that can be obtained by using vector norm in order to construct a matrix norm, and this is called the matrix induced norm, fine.

So, finally, we of course saw what is this normed linear space structure. And, what is rather comforting for us is to sort of see that whatever spaces that we are usually, going to be dealing with in this course, are in fact, going to be normed linear spaces. So, now we continue, with our lectures, so, we are on lecture number 3.

And, what we want to do is, we want to look at a little bit more detail into the norms that we have constructed until now. So, it is very important that we get a better feel for the norms that we have designed. 1 of the key things in this direction is to actually prove that the norms we have defined are in fact, are indeed norms, valid norms as per the definition.

So, if you see the definition content four aspects, 1 is the non-negativity, then we had the notion of the norm being 0 only when the vector itself is 0, then we had the scalar multiplication property and then finally, the very key and very important triangle inequality property. So, we want to prove this the fact that this infinity norm, and also a particular p-norm are valid norms. Just to see how such proofs would go.

So, let us see, first we have this infinity norm, which is essentially the largest component of any vector that is given to us. So, if I have any vector in R n, so, if I say in this case, we are assuming that x is in R n. And, we are computing the infinity norm by taking the largest component of this largest element of this vector x. So, since it is a finite dimensional vector, this is very easy, Remember, we talked about the notion of supremums, and maximums also last time. So, if the set is finite, computing a maximum is very easy. Just have to compare.

We also saw a couple of examples in this direction in fact. So, let us look at the proof. So, the first is the fact that the infinity norm is greater than equal to 0 is very obvious, why? Because, we are comparing absolute values of the components, we take every component, we compute its absolute value, and notice that this quantity is always positive. So, since this quantity is always positive, the maximum of this quantity, also has to be positive. Well, non-negative. So, if there is element 0, then obviously, the absolute value is also 0. However, this is always non-negative. So that is rather critical.

And, since we are taking a maximum over non-negative numbers, the output is also non-negative. So, the first requirement is very easily satisfied. The second requirement is that, it has to be 0, if and only if the vector itself is 0. And, that is what I have written here. I have just written the statement. But it should be obvious that if I sort of tried to expand a little bit, so this if I have this guy, if I know that the infinity norm is 0, then what am I saying? I am saying that, max of every element is 0. If the max of every element is 0, remember, the absolute value is in fact, non-negative.

So, if, if the maximum of all absolute value of x i is 0, then the only way this is possible is that x i is 0. On the other hand, if I start from here, and I am given that x i itself is 0, then it is obvious that absolute value of x i is 0. And, this means that max of absolute x i is 0. And, that is it, this is the infinity norm. So, we have very easily proven that the infinity norm being 0 is equivalent to each component being 0, that is the vector itself being 0.

The third property that is a scalar multiplication property is rather obvious. It is nothing to do. Because, again, you have an absolute value. So, the alpha will come out here. And alpha can be pulled here. So, the scalar multiplication property is very, very straightforward. I do not, I am not really elaborating on it. But now, we come to the rather critical triangle inequality property. And, as always, whichever is the critical property is what is difficult to prove. So, always more challenging. So, let us look at this. And, let us try to do this.

So, in order to prove that you have a triangle inequality property, I need to show that norm of x plus y, the infinity norm of x plus y, is less than equal to infinity norm of x, plus infinity norm of y. So, that is the first step. That is the last step. Now, I am just writing the definition here. The infinity norm of x plus y is simply max, over 1 to n, absolute value of x i plus y i. Now, it should be obvious to you that, since I am computing the max of x i plus y i, for some k, this is the maximum. So, for some k, this is in fact not less than equal to but you can even say this is exactly equal to this is I can even say this is exactly equal to.

So, what I can do is, I can simply erase this, the infinity norm, that is the max of x i plus y i is in fact, exactly equal to. Absolute value of x k plus y k for some k. And, because after all, I am taking a max. So, it has to be true for some i equal to k, that is what this is. Now, by the triangle inequality property of the absolute value. So, the absolute value actually satisfies the triangle inequality. This is very well known. I am not going to prove it, easily verifiable also. So, from this triangle inequality on the absolute value property, I get that absolute value of x k plus y k is less than equal to absolute value of x k plus absolute value of y k. Again, for some k, in 1 to n.

All this why? these quantities in the left by the way, so I started with an equality, again an inequality now, I have an inequality but all the while, this quantity remains on the left-hand side. So, this is how we do a lot of inequality proving. So, get used to it, this is how we do a lot of inequality, we start with a quantity and keeping this on the left-hand side, we keep doing inequalities, equalities and inequalities.

And, so on and so forth. Usually, one directional. If you are doing less than equal, it will always be less than equal to, if it is greater than equal, to it will always be greater than equal to on every step, or equal to. It will never switch between less than, and greater than. Because, then you cannot actually prove anything.

So, this is a very standard method of proving things. So, this is x k plus y k for some k in 1 to n. And, this should be obvious that this is less than equal to max from 1 to n x i absolute value. And max from 1 to n, y i in absolute value. Because, if I take any x k arbitrary, for some arbitrary k if 1 to n, and I take its absolute value, it has to be smaller than if I take the max over all possible such case, which is what this is.

And, it is now evident, that this quantity is x infinity. And, this quantity is y infinity. So, we started with this, we did a lot of inequalities, we have proven that x plus y infinity is in fact less than x infinity plus y infinity. So, that is sort of the end of the proof. So, we have proven that the infinity norm as defined satisfies all the norm properties, and therefore is a valid norm. Alright, great.

So, now, we want to do something very similar. We want to do something very, very similar for a two-norm. So, let us see, if we can in fact, do something like this. So, how is the two-norm defined? So, we are not of course dealing, I mean, similar proofs can be done for all p-norms, but we are focusing on these infinity norm and two-norm. Because these are rather important norms. So, the two-norm is the, like I mentioned last time, is a Euclidean distance, the way we know it, the way we know how to measure distance. And, the infinity norm is of course, rather important norm.

So, we, I mean, of course, you can do the same exercise for all other norms. So, I am not going to do it for all possible norms. So, is defined as summation over 1 to n, absolute value of x i square and then I take a square root. So, immediately, very, very in passing, I say that the first three properties are obviously satisfied. So, I mean, I will still write a little bit here now, I am still going to write a little bit. So, if you look at non-negativity. Non-negativity that is.

So, this should be sort of evident to you, because, again, if you look at you are taking square of absolute values. So, it is a positive quantity. Then, I am taking summation of this positive quantity, then I am taking a square root of that, again a positive quantity.

So, great. So, this is fine, this is good. Then the next one, is that the two-norm of x is 0, if and only if x itself is the 0 vector, that is again something that is not too difficult to verify. So, if I sort of try to compute, if I sort of try to write what is the square of a two-norm, then I will get something like x 1 square, plus x 2 square, plus x n square.

And, suppose I say that this is 0. And the only way this is possible, is that each component is 0. It is very easy to argue this. Because, if any of them is non-zero, then the outcome is non-zero, why? Simply because we are only adding things, we are never subtracting anything here, and nothing gets subtracted here, only things get added here. If anything is non-zero is definitely going to add something, and this will be not 0. So, the only way that norm x 2 square is 0, is if x i itself is 0, which means that the vector x is 0.

And similarly, if x is 0, if the vector x is 0, the fact that I mean that is going from this way to this that is, given my vector being 0 vector, then the norm is 0 is very, very obvious. So, the first two properties are this, then you have the scalar multiplication property, which is that the two-norm of alpha x has to come out like this, any scalar multiplication has to come out like this.

Now, it should be obvious that this guy will expand to square root of summation alpha square x i squared. And, this is simply that. And so that is it, we have proved this property. Very straightforward. which is why I said that they are obviously satisfied. But here you go. I mean, just to be clear, we have in fact proved it.

So, now the triangle inequality as always our key difficult property, if you may, so we want to talk about the triangle inequality property. So, if you look at the square of the two-norm for x plus y, again, I have taken the quantity that I want on the left right here, it is going to stay like this.

Then, this can be expanded, or this can be actually written as summation over from 1 to n, i equal to 1 to n x i plus y i squared. Now, this quantity actually evaluates to this, the quantity inside the summation evaluates to this. So, let me sort of verify if this is the case. So, the quantity inside the summation will actually evaluate to this. And now, what do I know? I know that this right quantity here is less than this guy, not difficult to verify. Because the 2x i, y i is only going to reduce x i squared plus y i squared. And, square makes the signs positive.

So, the 2x i y i, is always going to bring down x i square plus y i square, it is I mean, it is you cannot guarantee the 2x i y i is sign definite. That is 2x i y i cannot be guaranteed to be positive, therefore it can be positive, which would be great. Let us see if this is correct. Let us sort of think about this a little bit more carefully. I think I said that in sort of passing. So, if I evaluate this, this is x i squared plus y i squared, plus twice x i y i. So, I think I should try to populate some of the intermediate steps, which would be better. So, this guy is can I say that this guy is I do not think this would be quite okay. In fact, how I would say it.

This is a I would like to sort of rearrange how I say things here. So, this is not, sort of completely right I feel. So, what you want to say here is that this is actually less than equal to x i square plus y i square plus twice x i y i, this is what makes sense. So, this is not completely right. So, this is there is an additional term here. I hope that is sort of obvious to you, I hope that is obvious to you.

Now, the rest of the steps I have to sort of rethink, this logic will not work anymore. So, this is now, I would like to work on this term, and what I would want to claim is that this guy is less than or equal to twice norm x, norm y, I would like to claim that this is what is happening. This is what I would like to claim, does that make sense?

Because if that happens if this is true. So, if so, let us see, if true then this right-hand side entire thing becomes actually equal to, or less than equal to. So, if you see this quantity is the two-norm of x square, and this quantity is two-norm of y square. So, this is actually something like two-norm of x squared, plus the two-norm of y square, plus twice the two-norm of x, and the two-norm of y.

So, now and this is basically equal to norm of x 2, plus norm of y 2 whole square, and my proof is done, because if you see on the left-hand side, I had two-norm of x plus y square, and here I have two-norm of x plus two-norm of y whole squared. So, if I remove the squares on both sides, my proof is done.

So now, what am I left to prove? I am left to prove this guy. Let us see if we can we will try to do it. If I cannot. I will probably show it to you next time. But let us give it a shot right now. So, I have twice summation over i x i and y i on the left-hand side. So, which is in the right-hand side. So, this is the LHS, the RHS is twice summation over i x i squared multiplied by summation over i y i square, and I think I am missing a square root here. So, I am missing a square root.

So, this does this look very easy, obvious. This is something that I will have to probably get back to you on. I will have to think how we will do this proof in a there is some smart use of Cauchy–Schwarz that I can directly do, but I do not want to do that. I want to do it on this particular case, rather than try to employ a general formula. Let us see.

So, you can see that that even in this simple proof, we sort of get into an interesting sort of turn. It is not that it simply goes through, like we want. The terms look rather similar the 2x i y i, and 2xy look very similar. But actually, they are very, very different looking terms, when you expand them.

So, LHS actually looks something very simple like this, the right-hand side is this. So, it is like a, you have square root, and so on. So, I mean, how I would think of approaching this is that I will take a square of this. And then I will have terms like some x i square and summation of y i squared.

And then I would like to see what happens, what happens to each of these? So, I want to see that. How to deal with each of these terms? So, because then I will also have to take a square here, in fact, a square outside. And if I take a square outside here, then I am left with a square of sum of products. And then I have to sort of correlate these. Then I will have to correlate this. So, and my aim is to sort of just prove that one is smaller than the other.

So, let me actually think this, this piece of proof, I will actually cover again next time, let us not worry about it. But the point is, you will eventually see that the two-norm also is a valid norm, we will actually complete this proof, we have already verified the first three properties, which are rather easy. The triangle inequality property of course, is a little bit more complicated to prove. So, we will look at this proof.

So, after that, we want to look at notions of convergence. So, that is what we want to do next time. So, that is the plan for next time. So, once we understand the notions of norms, how norms work, we want to talk about convergence, that is because in order to actually define convergence, you need the notion of a norm, because otherwise you cannot actually, convergence means basically going close to something. So, like we discussed, the entire idea of going close to a point cannot be easily defined in vectors sense, if you do not have a norm. And so, once we have a norm, and a norm linear space it is easy to prove convergence. So that is what we are going to do next time.

So, what did we see today? So, what we did today is rather basic proofs of which norms are actually satisfying these norm properties. So, we get to prove the norm property for one of them, but we looked at how to prove that the infinity norm is actually norm. We are still to see the triangle inequality property for the two-norm. And of course, this is all leading up to actually defining convergence, Cauchy sequence and, inner product spaces and so on and so forth. So, convergence is a rather key property for us, which is what we will look at next, thank you.