

Transcribers Name: Crescendo Transcription Pvt. Ltd.

Nonlinear Adaptive Control

Professor Srikant Sukumar

Systems and Control

Indian Institute of Technology, Bombay

Week- 5

Lecture 28

Exponential Stability of LTV systems, PE and Exponential Stability

Hello everyone, welcome to yet another session of our NPTEL on nonlinear and adaptive control, I am Srikant Sukumar from systems and control, IIT Bombay. We are into the fifth week of this course and I really hope that our journey together has been interesting we have this very nice new background image of this satellite that is orbiting around the Earth. And as we had mentioned before, the algorithms that we derive and analyze in this course, are frequently used to drive systems such as these.

So, what we were we have been doing in this week and in the last class also is basically looking at persistence of excitation and trying to connect it with stability of parameter identification algorithms. So, we have not yet seen any parameter identification algorithms, but I assure you that the dynamics that we sort of see in these lectures appear in such systems very frequently.

So, these are the parameter identification identifier structures are very similar to the sort of systems that we are going to see. So, we of course, established the definition of persistence of excitation, we looked at an alternate exponential stability theorem, then we for the scalar case, we sort of connected persistence excitation to stability, that was a very nice, neat connection.

So, it was very easy to establish in the scalar case, are relatively easy to establish in the scalar case but now, we have moved forward to trying to do something or trying to do something similar for the vector case. And in order to do that, we had started by stating first of all and alternate exponential stability theorem.

Which sort of seemed like a nice variation over Lasalle invariance and Barbalat's lemma which we have been using to analyze stability when we have a non-strict Lyapunov function.. So, here too, it was the case that there was a non-strict Lyapunov function, but then because of this additional integral condition, we could in fact, claim that there is exponential stability.

So, then, we went on to establish more definitions, which we are going to use subsequently, the first is that of uniform complete observability. So, we define what uniform complete observability is and this is again now we have specialized to linear time varying systems. So, these are all only for linear time varying systems, because we use a state transition matrices and such and we define what is the uniform complete observability where we use something that looks like observability grammian but is of course, a more strict requirements issue may then observability itself.

And using this, in fact, we now sort of want to state like an exponential stability theorem for linear time varying system. So, this is where we were until last time, the final thing we talked

about what uniform complete observability is and how it is connected to observability itself. And we of course, want to use this in order to analyze time varying linear systems. So, this is where we are today.

So, we are basically going to start here. So, let me mark this as lecture 5.4. So, we want to now look at an exponential stability theorem for linear time varying systems, assuming that you have a UCO type of condition on the system. So, we again start with the same sort of input sort of output system and where x is the state and y is the outputs and the exponential stability. So, basically, we say that the origin is exponentially stable.

If and only if, for some C such that this A, C pair is UCO that exists a symmetric matrix P such that the two equations 5.4 and 5.5 are satisfied.

What is 5.4? 5.4 is simply saying something about the boundedness and positive definiteness. So, if you remember we had actually connected for symmetric matrices, we had connected x transpose Px , it is basically the positive definiteness of this function. With the positive definiteness of the matrix P . I hope you remember we did this as an example. And this is exactly what is happening here, where we specialize to linear systems. Because all only up around candidates always look like x transpose Px . We no longer talk about the positive definiteness of the function itself.

But we just look at the positive definiteness of this matrix. Because it is sufficient. It is equivalent. For this sort of a construction, positive definiteness of P corresponds to positive definiteness of this function x transpose Px . So, since they are identical, we just start talking about just the matrix P itself. So, of course, the equation 5.4 like I said, this gives us positive definiteness. This talks about decrescence. Now, equation 5.5 says something about the derivative. And why does it say something about the derivative because just look at this carefully.

If I take v dot of x , in fact, it is the $v \cdot \dot{x}$ because t is a function of time. So, I will get x transpose Px dot plus x dot transpose Px plus x transpose P dot x . And I just substitute for the dynamics and I will get x transpose PAx , notice that I have sort of excluded the time arguments, but they are there plus x transpose A transpose Px plus x transpose P dot x . So, this is just obtained by substituting the dynamics here, here.

And this is simply just x transpose PA plus A transpose P plus P dot. So, this is exactly what it is, this is exactly what it is. This is basically saying V dot. So, what we are essentially saying is that we are now talking about definiteness of v dot but not by looking at the entire function.

Because we do not need to, we only look at this matrix in between that is what this is. That is what this is. So, what the second relation says this left-hand side is basically coming from the derivative of V dot like you can imagine the right-hand side is just the output matrix inner product, it is minus C transpose C , notice that typically y is in R^p .

So, this is a p by n matrix. So, when p is less than n , so typically C transpose C , which is basically an n by n matrix, is going to be a singular matrix, it is at best going to be negative

semi definite. So, the negative of that is negative semi definite. So, using this result in 5.5, what we have claimed is $V \dot{x}$ is less than equal to minus $x^T C^T C x$, which is at most negative semi definite. Which is at best negative semi definite, not negative definite, because C is at most rank p , p is usually less than n . And so, the product can be at best at p . But it is over dimension n . Again, p is less than n . So, it is a singular inner product.

And but because it is like, like an inner product, still it is like quadratic. So, therefore, it has to be negative with the negative sign. And so, it is at best negative semi definite. So, what we are claiming is that just with this negative semi definite sort of structure, we can claim exponential stability.

And how do we claim this, we invoke our alternate exponential stability theorem, because you notice that all you need is negative semi definiteness. But of course, you need some kind of nice integral term happening. So, we of course, have required that $V \dot{x}$ maybe negative semi definite, but we want that integral of $V \dot{x}$ some say some t to $t + \delta$ should be positive definite.

So, that is what this this alternate exponential stability theorem sorry, sorry has to be negative definite, I apologise has to be not positive definite and negative definite. So, it has to be less than some minus α^3 norm x squared. Has to be less than minus α^3 times norm x squared. But that is not too difficult. Why? Because let us say I do integrate this, let me try to do this.

So, let us actually write this out properly. So, I had to save some space, we want this we want it to be we want the integral to be negative definite. So, how do I claim this I simply integrate both sides of $v \dot{x}^T x^T dt$ well I need to use a different variable $v \dot{x}^T x^T ds$ to $t + I$ do not know some τ .

And this is actually equal to integral from t to $t + \tau$ minus $x^T C^T C x$ ds . Now, using the state transition matrix, I can write this as minus $x^T C^T C x$ ds , again $\phi^T C^T C \phi$ ds times x at t . Now, if you look at this integral here this is exactly the grammian in UCO is exactly the grammian in UCO.

You can verify by looking here exactly this grammian in UCO. And what do I know by the UCO condition that is in fact greater than some constant greater than equal to some β I that is this is positive definite.

So, I am going to simply use that sum greater than some β I . So, this is then less than equal to because there is a negative sign. So, the greater than becomes less than equal to minus β $x^T x$ which is basically minus β norm x squared. And this is exactly what we wanted, is exactly what we wanted for proving exponential stability that even though \dot{x} is negative semi definite only.

So, which is a weak thing it is not negative definite, but the integral over some window is negative definite that it is bounded upper bounded by some minus β norm x square. This is rather cool result. So, this was possible only because, we assumed that the AC pair is UCO.

So, we could actually compare with this definition here and use this thing. And of course, because of the alternate exponential stability theorem. So, I hope you sort of appreciate that, we are actually using every piece that we are introducing in order to come up with the result that we want. So, great. So, with this you can have this exponential stability which is slightly weakened sort of requirement like in a usual linear system Lyapunov equation, you would want the right-hand side to be negative definite, but here it is only negative semi definite.

So, the next sort of ingredient that we want to introduce, which is again something that we will use subsequently is that of UCO under output injection. So, the idea is that if I. So, the basic result says that if I inject a function of the output a linear function of the output into the state then it is not going to change the UCO property.

So, the UCO property is invariant under output injection. So, this is what we claim that is what is this theorem. It says that, if you have some matrix K time varying matrix of course, because we are looking at a time varying linear system, of course, all of this will work for constant linear time invariant systems also, because, all these bounds are obviously very easy to obtain in those cases. So, if you look at a time varying gain K of t .

If it so, happens that over a moving window so, if you take a moving window average or moving window integral, it is it remains bounded overall such windows, then A, C, UCO is equivalent to saying $A + K C C$ is UCO. So, you will see that this is actually an output injection and because if you look at equation 6.2. This is what it is.

The dynamics now contains an injection of the output. So, what are we saying, if we have the A, C system to be that is only this system to be UCO then if K has this nice boundedness property then this entire system is also UCO. So, this is rather nice.

And of course, there is I mean we just use different notation for the bounds the bounds for A, C were denoted β_1 and β_2 in equation 5.1. But here we denote it with $\bar{\beta}_1$ and $\bar{\beta}_2$ just to differentiate because the bounds are going to be different of course, the bounds are not the same, they are both still UCO.

And of course, here we are assuming that A and C matrices are piecewise continuous and so on and so forth. So, some basic regularity assumptions of course, Great. So, this is sort of, I would say all the mechanism that you need in order to look at exponential stability of parameter identification systems. At least the sort of linear parameter identification systems that we are used to that we will be used to dealing with soon enough.

So, what are the ingredients that we have one is we have this nice exponential stability theorem for the linear system, which is like a weakened version of the typical exponential stability theorem, which will have some negative definite right-hand side and so on. And then we also have this notion of uniform complete observability under output injection. So, what it says is that, you if you have like a nice bounded input bounded in this bounded gain K , which is essentially a borrowed in the sense of a moving average, then injecting an output with the gain in the dynamics is not going to alter your UCO property. So, this is what is rather nice.

So, we will try to do a little bit of or maybe one piece of this proof today and then subsequently, we will try to wrap it up in the next session. Because we may not have enough time today to complete the entire proof but anyway so, let us first begin by stating the result what do we say? We say that, you look at let ϕ be a vector signal now, that it is it is taking time and mapping to some \mathbb{R}^n . So, let ϕ and in this case \mathbb{R}^n is exactly the dimension of the dynamics because of how the structure of equation 7.1 is. So, suppose ϕ is mapping time to \mathbb{R}^n , which is the dimension of the state space.

And is piecewise continuous basic. If ϕ is persistently exciting as per our definition, then the origin is globally exponentially stable for this system, where α is just some positive constant gain.

And there is some initial condition x naught. So, the first thing that you want to sort of notice is that equation 7.1, that you see here is not too different with from the scalar example. Here, we had just a minus a squared because it was a scalar. But now you are looking at minus $\alpha \phi$, ϕ transpose, just because it is a vector case, just because it is a vector case. So, of course, you notice that this ϕ , ϕ transpose is an n by n matrix now.

And also, you notice that instantaneously it is only rank 1. So, $\phi \phi$ transpose, in fact, this is the small ϕ I apologize $\phi \phi$ transpose instantaneously is at most rank 1, why? Because ϕ itself is a vector. So, rank 1 at most. And therefore, $\phi \phi$ transpose the product cannot have rank more than the constituent matrices. So, that $\phi \phi$ transpose is an n by n matrix imagine I have 100 states and I, but $\phi \phi$ transpose is just rank 1, just rank 1.

So, remember this, this is why PE is such a nice condition because it sort of says that, if you remember the PE condition is going to say that you if you integrate this from t to t plus cap T and you have $\phi(\tau) \phi(\tau)^T d\tau$, then it is bounded on both sides by some constants μ_1 and μ_2 . So, basically, although instantaneously, it is obvious that it is only rank 1 therefore, there are many, possibly many Eigen values at 0. But if I integrate it on this moving window, then we are saying that the rank becomes full. Great.

So, now we understand the setup of the problem. So, the next thing that we want to do is to obviously, just analyze the stability. And how do we do this in this course, we take a Lyapunov candidate function or candidate Lyapunov function. So, of course, this is a candidate, not a Lyapunov function, but a candidate Lyapunov function.

Why is it a valid candidate, we should remind ourselves like every time you see a function V , that it is a valid candidate function, one, it is C^1 , this is once continuously differentiable, no problem, just a quadratic. We just have to use vector principles to take the derivative that is all. And two it is positive definite. And in fact, in fact, it is radially unbounded. So, this is in fact radially unbounded and because again, it is just a square, what it plays the role of x square, whatever x square is signifying in scalar systems, $X^T X$ is signifying the same in vector systems when x is \mathbb{R}^n .

So, then we, of course, carefully take the derivative along trajectories. And it is very simple. It basically is just $x^T \dot{x}$. So, basically, I get something like $2x^T \dot{x}$. And if you substitute for \dot{x} which is $-\alpha x$ you start to get something like this. And this is, of course, the square of $x^T x$.

And because it is like, again, like an inner product of x^T with it, or $x^T x$ with itself. So, this is just that. So, I can of course write it in this way. Nice compact form. Now, of course, we write the integral because of course, we want to use this alternate stability theorems. So, we want to, of course, compute the integral of this guy. Notice that we are still in the regime of linear systems.

So, this is still a linear system, although it is nonlinear in t , but it is just a function of time and it is linear in the state. So, whenever we are talking about linearity, in a state space system we are talking about linearity with respect to the states, so with respect to the state, it is linear. So, it is a linear system.

Maybe nonlinear in time, but we do not care about that, great. So, now, we just compute the integral, because remember that this alternate exponential stability theorem required us to do so, that is compute the integral and that is what we are trying to do here. So, if I integrate from t to $t + \bar{T}$, I think that is the bracket missing here of \dot{V} , we just write the expression right here.

And now, all our subsequent analysis is essentially going to be trying to bound the scale subsequent analysis is to be bounding this particular quantity. And why are we doing that again, the alternate exponential stability theorem requires that we have like a nice upper bound on this quantity on the left.

So, that is exactly why we are going to try to bound this quantity, how do we begin? We begin from the PE assumption because ϕ is PE. Therefore, the system $\dot{x} = 0$ comma $\phi^T x$ is UCO. What is this system? Whenever I write this kind of notation, this system is the first piece is indicating the A matrix and the second piece is indicating the C matrix. So, the system is $\dot{x} = 0$ and $y = \phi^T x$ this is the system. Now, we are saying that this system is uniformly completely observable, why? So, we know that this is the PE condition on ϕ . So, this is the PE on ϕ . Yes, I am going to nicely mark it.

So, this is the P condition on ϕ and what is the UCO of this? So, UCO condition will be I have to take the integral from t to $t + \bar{T}$ I have to do $\phi^T(s, t)$ with the transpose. So, ϕ is the state transition matrix then I have to take C^T . So, which is basically $\phi^T \tau$ and then C which is $\phi^T \tau$ and the state transition matrix again, please do not unfortunately, we are using the similar looking notation for the state transition matrix and this function but please do not get confused this is the small ϕ and this is the capital ϕ . So, this is a bar here this denotes the state transition matrix it is standard notation.

And unfortunately, we have chosen this notation for the P functions and this is also sort of standard because we could have gone with something else but it is done now. So, bear with it do not get confused between these two this is just the signal that we are saying as PE.

And this is the state transition matrix. Now, the question is what is $\phi(s, t)$ for this system? \dot{x} is 0. So, what do I know there is $x(t)$ is equal to $x(0)$ for all t greater than equal to $t = 0$. So, therefore, the state transition matrix is in fact identity. So, I hope all of you understand this if not please revise what is the state transition matrix.

So, the state transition matrix is in fact identity and if I plug it here I will simply get t plus $\text{cap } T \phi \tau \phi^T \tau d \tau$. So, this is exactly the matrix involved in the UCO or basically the UCO grammian matrix and this is exactly identical to this guy. So, by virtue of persistence I have UCO of this system.

So, what are we looked at today? So, what we looked at is basically now we looked at a few more additional concepts which is UCO under output injection and of course one exponential stability theorem for linear time varying systems and now we are starting to connect it with our proof of exponential stability of parameter identifier systems under persistence. So, we have just started the proof and of course, we are going to subsequently finish it up also, in the upcoming sessions. Alright, that is where we stop. Thank you.