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Nonlinear Adaptive Control
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Week - 5,
Lecture 26
Persistence of Excitation (PE): Introduction

Hello everyone, welcome to yet another session of our NPTEL on Nonlinear and Adaptive Control. I am Srikant Sukumar from Systems and Control IIT, Bombay. We are now in the fifth week of this course which I hope all of you are finding rather interesting and rather useful and starting this week we have a new motivating image which is that of a spacecraft from Spacex specifically that is hovering around the earth. And the algorithms that we seek to design are also driving systems such as these so that they can reorient and move autonomously in space.

So, now starting today we will look at a new set of course notes and this is just a sort of preliminary, not a preliminary but something that precedes adaptive control, adaptive control literature was preceded by identification literature and it was rather well known that for any parameter identification to happen successfully there was rather a certain requirement for richness of signals. And this richness of signals was regularly codified in terms of persistency of excitation.

So, until last week we sort of completed our discussion on, well I mean until the last lecture we completed our discussion on and I mean the Lyapunov theorems and the different variants. And now today we are ready to delve into topics on persistency of excitation. So, like I said this precedes what we did in, what we will do in adaptive control so chronologically it in fact came before the adaptive control literature.

Of course, you will also see that there is a healthy connection to stability because we are talking about parameter convergence which is a sort of stability problem. So, there is a healthy connection to stability and so we look at some alternate exponential stability theorems and so on. And so, this is what we will discuss for the rest of the week.

So, let me warn you we are already starting to get more and more mathematical so the topics are getting more and more mathematically involved. So, I would expect you to spend more time trying to understand the material, going through the material and be comfortable working with the material and use it in different problems and application contexts, great. So, let us begin.

So, the first thing we sort of talk about is what is the definition of persistency of excitation. So, vector signal phi of t is said to be persistently exciting, so this is a more how would I put it a definition in words and then of course we will define it more mathematically. So, the notion of persistency of excitation is that if the signal wobbles around enough in every window of length t.

So, if I have a sliding window of time, so I keep sliding the window of time, I keep moving forward forward forward forward forward forward forward in time over each this, each of the, each such window of length t, I want the signal to wobble sufficiently or move around sufficiently enough.

And it is made a little bit more formal saying that the integral of the dyad associated with it is positive definite. Again, we will look at a more mathematical, mathematically precise version just down below, so we do not have to worry too much about understanding this. But dyad is an operator which is a combination of two vectors. So, here in this case it is basically referring to an inner product, in this case it is referring to an inner product.

So, what are we looking at, that what are the elements here in the definition? There is a sliding window of time of this constant width T, capital T so we have kept a cap T window of time and on each such window we are looking, expecting the signal to sort of wobble sufficiently, so that some kind of positive definiteness of a dyad is achieved.

So, without, again, all elements of this are not very clear just by this definition in words. So, we actually look at the more formal mathematical definition for our purposes that is when the signals are in, when we are looking at vector signals in Rn, so that is if there is an integrable signal phi from R plus to Rn then it is called persistently exciting if there exists three constants mu 1, mu 2 and capital T all of them positive such that this kind of integral inequality holds.

What is this integral inequality? And this has to hold for all time small t. So, this is what it means to have a sliding window because as I change t my windows start to slide. So, changing t means I am sliding the window because this capital T that is the range is the of this integration remains the same, it remains capital T but it is just that I am changing the initial time from which I begin.

So, what does this inequality sort of mean let us just try to interpret it. So, the first thing to remember is that this matrix phi phi transpose is belongs to Rn cross n, so it is an n cross n matrix that is the first thing to remember. The second thing is this is in fact positive semi definite because it is a symmetric, first of all it is symmetric because it is just product of phi phi transpose.

So, if I take the transpose it is still phi phi transpose, therefore it is a symmetric matrix, and it is like a quadratic so because it is phi and phi transpose therefore it can never be negative, it can never have negative eigenvalues we already know if the matrix is symmetric then the eigenvalues are real. So, we can talk about positive and negative eigenvalues.

But we also know that this matrix cannot have negative eigenvalues because it is a, it is like a square, if you have any confusion in understanding this you can just think of it like if I multiply both sides by some constant vector say v transpose phi tau phi transpose tau v, suppose I do this because I can move this in and out of the integral, if I do this then you know that this is actually a square, this is actually t, t plus cap T phi trans, phi transpose tau times v square d tau and this is the norm square, this is the norm square.

And this is like something like an x transpose x, x transpose x which is a norm squared and the two-norm square. And the two norm can never be, the norm can never be negative therefore this entire quantity has to be greater than equal to 0. Now, that if the quadratic form is non-negative then the matrix has to be positive semi-definite at least. So, that is really the argument, it is like a square therefore it is positive semi-definite at least.

But we are claiming something more, something more. We are saying that it is in fact lower bounded by mu 1 I which is strictly greater than 0. So, therefore, somehow, we are saying something about the eigenvalue of this integral. Because if you remember for any symmetric matrix we had this nice inequality, what was it? It was something like x transpose Ax less than equal to lambda max A norm x square and lambda min A norm x square, which means that the quadratic form is lower and upward bounded by the largest and smallest eigenvalues.

Therefore, if I write it in this nice form I am just, if I write in this form all I am saying is that the smallest eigenvalue of this matrix with the integral of course is has to be greater than equal to this mu 1. So, the smallest eigenvalue of a symmetric matrix being greater than, being greater than equal to a positive constant. Means what? It means that this is a positive definite matrix, this is a positive definite matrix.

The other side is rather simple, it just says it is some kind of a bounded matrix, so there is an upper bound on the eigenvalue of this integral, so it is like a, it is a simpler condition. In fact, many textbooks define this without, define persistency of excitation without this right-hand side, so I would not put too much of a stress on this side, so it is more of a question of how you want to do the math, that is about it.

So, what is the idea, what is the idea? So, we are saying, so the first thing we know that the integrand is greater than equal to 0. What else? We know that phi phi transpose singular for each t. Why is that? Why is phi phi transpose singular for each t? See phi itself is just a vector. What is the maximum rank of a vector? 1.

Now, if I take product of two matrices then the rank of the product is less than or equal to the rank of the each of the matrices. So, now what am I doing? I am taking two matrices vector is also a matrix and I take their product. Therefore, the rank has to be less than equal to the rank of each one of them, which means the rank of this whole thing is less than equal to 1, can be at most 1 is what I am saying.

So, the rank of this product is at most 1, so this is rather interesting, I am saying the rank of this product inside the integral is at most 1. But I am saying that, when I integrate this quantity over this window of time then the rank in fact becomes n because if the rank does not become n this cannot become positive definite which is what is indicated by this inequality, because or even or an even simpler idea this is less than equal to the smallest eigenvalue which is positive.

So, we are saying that the smallest eigenvalue is greater than or equal to mu 1. Which means what? Which means that all eigenvalues have to be strictly positive. Which means I am somehow saying that although the integrand has rank at most 1, the integral over this time window of capital T has rank n, has maximum rank. So, this is the rather cool property that we are looking for and this is the property of persistence of excitation.

So, we therefore, which is what is mentioned in this note, the matrix itself phi phi transpose is singular for all time. So, the p condition somehow requires that the phi rotates sufficiently such that phi phi transpose is uniformly positive definite. Why do we say uniform? Because these bonds do not depend on the small t, mu 1 and mu 2 are independent of the small t, you remember that in all our definitions uniformity has always got to do with the time argument.

So, in this case that time argument is the small t, if mu 1, mu 2 are independent of it, so you keep sliding it does not matter where you are, your bound remain the same. Because if they do not, then you are sort of it is rather troublesome, if you do not have a uniform bound, then you will not be able to complete the sort of analysis that we try to do, excellent.

So, we start with a singular, a vector, we make an outer product phi phi transpose is an outer product, we know that it is symmetric, we know it is positive semi-definite, we also know it is singular with at most rank at most 1. But the cool feature that we are looking for is that when I integrate it over a window of time capital T, I want it to become positive definite. Of course, I also expect it to remain bounded, this is fine, not able to, not asking for 1's there, great.

So, that is what is sort of we are saying, we create an outer product, this condition has to be valid for all small t, and we slide window of size capital T and phi phi transpose symmetric positive semi definite. But if when we do a moving window average it is positive definite. So, if we do moving window average, it is always positive definite, so this is rather strong.

So, let us look at some examples of what kind of systems are in fact persistently exciting or what kind of signals are in fact persistently exciting. So, the first example obviously is the scalar signal, well, we almost a lot of our examples are scalar signals but whatever you can construct the vector counterparts without too much of a trouble.

The first very very easy example is a constant signal, the constant signal is of course persistently exciting, obviously. And its lower and upper bound are exactly this value itself and the lower and upper bound will be exactly c and therefore it is trivially persistently exciting.

If you integrate this over C d tau t to t plus cap tau you will always get C T irrespective of what is your small t. And therefore, the lower and upper bounds are can be exactly capital T, C times capital T. So, one of the things you can remember, so I mean, so I can actually mark this to be my mu 1 and mu 2.

So, one of the things to remember is that these mu ones and mu twos can depend on capital T. So, a lot of times again one of the things that is done is this definition is again taken with a 1 over capital T to do sort of an averaging out, to do sort of an averaging out, so the t does not appear in these constants.

So, this is another thing that is standard, definitely not non-standard. So, that you average it over this time capital T and without this this is not an average it is just a summation, I mean if you think of breaking the integral of course. But with this one over capital T it is a sort of average. So, that is why we call it this this a word moving window average is being used.

So, here also if I do 1 over capital T instead of just, so I am sorry, this is C C, this is C square. So, this is just this guy. So, mu 1 and mu 2 are in fact just C squared, C squared because it is phi phi transpose. So, in this case it is C squared, great. So, this is a like we said this is trivially persistently exciting, no problem. So, let us look at another example, of course this is sort of example where the signal can possibly hit 0, this was trivially persistent because this never hit 0 anyway, so we are not worried about this signal so much.

The next one is something like this is a periodic signal with a period capital T itself which is this window size and what is this function, it is a max of sine t and 0. And of course, we take capital T to be 2 pi because that is the period of this signal. So, we take the max of sine t and 0 and this is also.

So, basically the purpose of this is to somehow make sure that you do not go below 0 that is all. I mean not that it matters, not that it matters so much, in fact I could just take instead of this something like f of t is sine t and t 0 to 2 pi. And now if I take t equal to, capital T equal to 2 pi then I will be doing integral from sum t to t plus 2 pi and sine t, sorry sine tau t d tau, sorry sine square tau, d tau, something it is basically sine square tau d tau.

So, what is, let us see. So, now I am, I want to basically use see if I can integrate this, let us see, let us try to do this. So, this is something like this is equal to t t plus 2 pi sine square tau d tau and I do want to see if I can use some kind of a trigonometric formula. Can I do that? Let us see, I think 1 plus cosine 2 tau will be equal to 2 sine square tau, I believe this is an, let us see I believe this is an inequality, first quantity, because cos 2 tau is cos square tau minus sine square tau, let us see.

I think this, I think we can use something like this because cos 2 tau is cos square t minus sine square t so this becomes sine square t plus sine square tau, so this is 1 minus cos 2 tau. So, this is this guy and if I integrate this I will get something like tau minus half sine twice tau and this is t to t plus 2 pi so the first term will just give me 2 pi and the second term will be minus half sine 2t plus 4 pi minus sine 2t.

So, what is sine 2t plus 4 pi? Because this is just periodic with 2 pi, so this is actually going to just cancel. So, I am going to be left with 2 pi and if I actually took a 1 over 2 pi here or a so 1 over or 1 over t here which is 2 by itself, 1 over t here so this is 2 pi by 2 pi that is equal to 1, so this signal is also persistently exciting.

So, pretty straight forward I just integrated the square of the signal because the outer product if the signal itself is a scalar then the outer product is simply the square of the signal and that is what I did and it is not difficult to compute that this is also persistently exciting. Now, if you look at this kind of a signal again I mean I do not need the squared here, if you look at this kind of a signal f of t is e to the power minus t sine t so this is actually we are not going to compute it but what I mean let us see.

So, if you look at how the signal evolves this will really start to decay very fast. So, if you look at this and we make this envelope can e to the power minus t, if you make this envelope then what will happen is that this sinusoid will lie between this and it start to decay very fast, very fast it will start to decay very fast.

Now, this kind of a signal is not persistently exciting, I am not actually computing the interval, the integral itself, I would leave it to you folks to try to compute this. But this is going to be, this is not going to be persistently exciting. Why? Because this is sort of decaying. And now what happens if it decays especially exponentially, what happens is that this integral over this window of capital T, I mean I make a window then I move the window, move the window, move the window and so on and so forth you can see I get smaller and smaller pieces, smaller smaller amplitudes.

And in fact, this amplitude will somehow be scaled by like an e to the power minus t and because I get smaller and smaller amplitudes you will see that my mu 1 and mu 2 for a particular window start to get smaller and smaller. And the thing is if the mu 1 and mu 2 get start to get smaller and smaller I get to choose only 1 mu 1 and mu 2, it has to be uniform in time, mu 1 and mu 2 cannot depend on t.

So, if I keep moving the window and this amplitude becomes smaller the corresponding mu 1 and mu 2s come out to be smaller for each of these windows. Then the only thing I can choose is the smallest mu1, I can only choose the smallest mu 1 and the largest mu 2. So, the largest mu 2 is not a problem because I just choose it from the first one, but the smallest mu 1 if you keep going on and on, on and on and on you can see that this is going to become really really tiny. So, the smallest mu 1 I can pick it is just like that uniform stability argument, the smallest mu 1 that I will be able to pick is in fact 0.

And if the mu 1 that we pick is 0 then this is not persistent because we require this, we require mu 1 to be strictly positive because you already know it is semi-definite, we already know that the outer product is semi-definite, no magic there. What we need is that the outer product be, the integral of the outer product be strict significant, is strictly positive definite that is, if you take an average over this window of time t has to be strictly positive definite. And that is why we need a uniform window, uniform mu 1 independent of the small t.

So, this kind of a signal is not persistently exciting and these are for the purposes of identification these are not good signals. So, if I may these are not good for identification, these are not good for identification.

So, anyway so what, so I will sort of summarize what we did today and what we are going to do subsequently. So, what we started today was a discussion on persistence of excitation and this is a rather rather important notion in parameter identification. And we will sort of try to connect it in basic ways to parameter identification in the subsequent lectures.

But today we saw the definitions of, definition of persistence of excitation and we also saw what kind of signals are in fact persistently exciting. So, and it really just involved computing a simple integral of an outer product and in fact for the more complicated cases it is not difficult to verify this numerically also, so it is not that difficult to do a numerical verification. And so, like I said, subsequently you will start looking at how this is connected to your adaptive identification, great. This is where we stop. Thank you.